

## OBSERVATION OF NONLINEAR VERTICUM-TYPE SYSTEMS APPLIED TO ECOLOGICAL MONITORING

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In this paper the concept of a nonlinear verticum-type observation system is introduced. These systems are composed from several “subsystems” connected sequentially in a particular way: a part of the state variables of each “subsystem” also appears in the next “subsystem” as an “exogenous variable” which can also be interpreted as a control generated by an “exosystem”. Therefore, these “subsystems” are not observation systems, but formally can be considered as control-observation systems. The problem of observability of such systems can be reduced to rank conditions on the “subsystems”. Indeed, under the condition of Lyapunov stability of an equilibrium of the “large”, verticum-type system, it is shown that the Kalman rank condition on the linearization of the “subsystems” implies the observability of the original, nonlinear verticum-type system.

For an illustration of the above linearization result, a stage-structured fishery model with reserve area is considered. Observability for this system is obtained by applying the above linearization and decomposition approach. Furthermore, it is also shown that, applying an appropriate observer design method to each subsystem, from the observation of the biomass densities of the adult (harvested) stage, in both areas, the biomass densities of the pre-recruit stage can be efficiently estimated.

*Keywords:* Verticum-type system; nonlinear system; observability; observer design; fishery with reserve area.

### 1. Introduction

The research line concerning the application of mathematical systems theory to monitoring and control of population systems was initialized in Varga [19, 20], where

sufficient conditions for local controllability and observability of nonlinear systems with invariant manifold have been proved and applied to populations controlled by artificial selection, and to phenotypic observation of the genetic composition of a population under selection. For further results on frequency-dependent population systems, see also Scarelli and Varga [16], López *et al.* [9] and Varga [21].

Later on, for density-dependent multi-species population models, monitoring problems have been formulated in terms of observation systems. In the context of population ecology, observability means that from the observation of the densities of certain species considered as indicators, in principle, the whole state process of the population system can be uniquely recovered. Observability has been analyzed in different multi-species population system models in Varga *et al.* [23–25] and Shamandy [17]. Both Lotka–Volterra and non-Lotka–Volterra type systems have been studied for local observability, and effective state estimation by observer design have been obtained in e.g. López [10, 11], Gámez *et al.* [3, 4]. For a general review on the application of mathematical systems theory in population biology, see Varga [22], a recent update of this survey is Gámez [2].

Verticum-type systems have been introduced by Molnár [12] for modeling certain industrial systems. These systems, in a top-down way, are hierarchically composed of linear “subsystems” such that a part of the state variables of each subsystem influence the state of the next “subsystem”. In Molnár [12], necessary and sufficient conditions for observability and controllability of such systems were obtained, and their further systems-theoretical properties were studied in Molnár [13], Molnár and Szigeti [14].

Recently, verticum-type systems have found an application in population ecology. As a matter of fact, population interactions are typically nonlinear, but in Gámez *et al.* [5] a concrete ecological interaction chain of the type resource — producer — primary user — secondary consumer was found to have a verticum-like structure which admitted to reduce the monitoring problem of observability to a linearized version of the original model. In the present paper, in a general approach, nonlinear verticum-type systems are introduced. In Sec. 2, the basic definitions and a sufficient condition for local observability of such systems is obtained. The proof is based on a natural decomposition and a linearization method of Lee and Markus [8]. In Sec. 3, as an example of a nonlinear verticum-type system, a stage-structured fishery model with reserve area is considered, and its stability properties are discussed.

Section 4 is devoted to a monitoring problem for a fishery system. First local observability is obtained by applying the linearization and decomposition approach of Sec. 2. Furthermore, it is also shown that applying the observer design method of Sundarapandian [18] to each subsystem, from the observation of the biomass densities of the adult (harvested) stage in both areas, the biomass densities of pre-recruit stage can be efficiently estimated.

## 2. Observability of Nonlinear Verticum-Type Systems

Let  $k, n_i, r_i \in N$ ,  $h_i : R^{n_i} \rightarrow R^{r_i}$ , ( $i \in \overline{0, k}$ ),  $f_i \in C^1(R^{n_i \times n_{i-1}}, R^{n_i})$  ( $i \in \overline{1, k}$ ), and consider the nonlinear system

$$\begin{aligned} \dot{x}_0 &= f_0(x_0), \\ y_0 &= h_0(x_0), \end{aligned} \tag{2.1}$$

and for all  $i \in \overline{1, k}$

$$\begin{aligned} \dot{x}_i &= f_i(x_i, x_{i-1}), \\ y_i &= h_i(x_i). \end{aligned} \tag{2.2}$$

Denoting  $n := \sum_{i=0}^k n_i$ ,  $r := \sum_{i=0}^k r_i$ ; let  $x = (x_0, x_1, \dots, x_k) \in R^n$ ,  $f : R^n \rightarrow R^n$  with

$$f(x) := (f_0(x_0), f_1(x_1, x_0), \dots, f_k(x_k, x_{k-1})),$$

and  $h : R^n \rightarrow R^r$  with

$$h(x) := (h_0(x_0), h_1(x_1), \dots, h_k(x_k)).$$

We shall suppose that there exists  $x^* = (x_0^*, x_1^*, \dots, x_k^*) > 0$ , such that

$$f(x^*) = (f_0(x_0^*), f_1(x_1^*, x_0^*), \dots, f_k(x_k^*, x_{k-1}^*)).$$

**Definition 2.1.** Observation system

$$\begin{aligned} \dot{x} &= f(x), \\ y &= h(x), \end{aligned} \tag{2.3}$$

is said to be of verticum type.

**Remark 2.2.** Equations (2.2) ( $i \in \overline{1, k}$ ) do not define a standard observation system in this setting, because of the presence of the “exogenous” variable  $x_{i-1}$  connecting it to system (2.2), with  $i - 1$ .

**Remark 2.3.** It is known that near equilibrium  $x^*$  all solutions of system (2.3) can be defined on the same time interval  $[0, T]$ . In what follows  $T > 0$  will be considered fixed and concerning observability, the reference to  $T$  will be suppressed.

For the analysis of observability of system (2.3), let us linearize system (2.2), at the respective equilibria  $x_i^* \in \overline{0, k}$ , obtaining the linearized systems

$$\begin{aligned} \dot{x}_0 &= A_{00}x_0, \\ y_0 &= C_0x_0, \end{aligned} \tag{2.4}$$

and for all  $i \in \overline{1, k}$

$$\begin{aligned} \dot{x}_i &= A_{ii}x_i + A_{ii-1}x_{i-1}, \\ y_i &= C_i x_i, \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} A_{00} &= f'_0(x_0^*), \quad C_0 = h'_0(x_0^*); \\ A_{ii} &= \frac{\partial f_i}{\partial x_i}(x_i^*, x_{i-1}^*), \quad A_{ii-1} = \frac{\partial f_i}{\partial x_{i-1}}(x_i^*, x_{i-1}^*), \quad C_i = h'_i(x_i^*) \quad (i \in \overline{1, k}). \end{aligned}$$

Define matrices  $A \in R^{n \times n}$ ,  $C \in R^{r \times n}$  as follows:

$$\begin{aligned} A &= \begin{pmatrix} A_{00} & 0 & 0 & \dots & 0 & \dots & 0 \\ A_{10} & A_{11} & 0 & \dots & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot & \dots & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & \cdot & A_{k-1, k-1} & 0 \\ 0 & 0 & 0 & \dots & \cdot & A_{k, k-1} & A_{k, k} \end{pmatrix}, \\ C &= \begin{pmatrix} C_0 & 0 & \dots & 0 \\ 0 & C_1 & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & C_k \end{pmatrix} \end{aligned}$$

obtaining linear observation system

$$\begin{aligned} \dot{x} &= Ax, \\ y &= Cx \end{aligned} \tag{2.6}$$

of verticum type (see [12]). In the latter paper, a Kalman-type necessary and sufficient condition for observability of linear verticum-type systems was obtained. Here we recall only its “sufficient part” to be applied below.

**Theorem 2.4** ([12]). *Suppose that*

$$\text{rank} \begin{bmatrix} C_i \\ C_i A_{ii} \\ \vdots \\ C_i A_{ii}^{n_i-1} \end{bmatrix} = n_i \quad (i \in \overline{0, k}).$$

*Then the linear verticum-type system (2.6) is observable.*

**Remark 2.5.** If  $x^*$  is a Lyapunov stable equilibrium of system

$$\dot{x} = f(x),$$

then (2.2) ( $i \in \overline{1, k}$ ) can be considered as a control-observation system with “small” controls in the following sense. By the Lyapunov stability of  $x^*$ , for all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that  $|x(0) - x^*| < \delta$  implies  $|x(t) - x^*| < \varepsilon$  (for  $t \in [0, T]$ ). In particular,  $|x_{i-1}(t) - x_{i-1}^*| < \varepsilon$  for all  $t \in [0, T]$ .

Considering  $u_i(t) := x_{i-1}(t) - x_{i-1}^*$  as a control for system

$$\dot{x}_i = f_i(x_i, x_{i-1}^* + u_i),$$

(2.2) becomes a control-observation system in the sense of Appendix A. Suppose that for each  $i \in \overline{0, k}$

$$\text{rank} \begin{bmatrix} C_i \\ C_i A_{ii} \\ \vdots \\ C_i A_{ii}^{n_i-1} \end{bmatrix} = n_i,$$

then by Theorem 2.1 the verticum-type system (2.6) is observable.

Hence, the linearization of the observation system (2.3) is observable. Therefore, by Kalman’s theorem on observability of linear systems (see Kalman *et al.* [7]), the rank condition (A.3) of Appendix A is fulfilled, which by Theorem A.1 implies local observability of system (2.3) near equilibrium  $x^*$ .

The above reasoning can be summarized in the following theorem.

**Theorem 2.6.** *If equilibrium  $x^*$  is Lyapunov stable for system  $\dot{x} = f(x)$ , and*

$$\text{rank} \begin{bmatrix} C_i \\ C_i A_{ii} \\ \vdots \\ C_i A_{ii}^{n_i-1} \end{bmatrix} = n_i \quad (i \in \overline{0, k}),$$

*then observation system (2.3) is observable near its equilibrium  $x^*$ .*

### 3. A Stage-Structured Fishery Model with Reserve Area

For the illustration of the above theorem we consider a modification of the stage-structured fishery model of Guiro *et al.* [6], supposing that there is reserve area where fishing is not allowed. In what follows, the first index of the biomass density  $N$  will indicate the area:  $i = 1$  for the reserve and  $i = 2$  for the free area; the second index will refer to the development stage:  $j = 0$  for the pre-recruits, i.e. the eggs, larvae and the juveniles together, and  $j = 1$  the exploited stage of the population.

The dynamics of the system is modeled by the following autonomous system of differential equations

$$\dot{N}_{10} = -m_{10}N_{10} + f_{11}N_{11} - p_{11}N_{10}N_{11} - p_{10}N_{10}^2, \quad (3.1)$$

$$\dot{N}_{11} = \alpha_{11}N_{10} - m_{11}N_{11} - \beta N_{11}, \quad (3.2)$$

$$\dot{N}_{20} = -m_{20}N_{20} + f_{21}N_{21} - p_{21}N_{20}N_{21} - p_{20}N_{20}^2, \quad (3.3)$$

$$\dot{N}_{21} = \alpha_{21}N_{20} - m_{21}N_{21} + \beta N_{11} - qEN_{21}, \quad (3.4)$$

where:

$m_{ij}$  = natural mortality rate of class  $ij$ ,

$\alpha_{ij}$  = linear aging coefficient in areas  $i = 1, 2$ ,

$p_{i0}$  = juvenile competition parameter in areas  $i = 1, 2$ ,

$f_{i1}$  = fecundity rate of adult fish in areas  $i = 1, 2$ ,

$p_{i1}$  = predation rate of class 1 on class 0 in areas  $i = 1, 2$ ,

$q$  = catchability coefficient of class 1 in the unreserved area,

$\beta$  = migration rate of the second class from reserved area to unreserved area,

$E$  = constant fishing effort.

We note that a fishing model with reserve area without stage structure displaying logistic growth was studied in Gámez *et al.* [4].

### 3.1. Existence of positive equilibrium

Let us now find conditions for the existence of a nontrivial (i.e. positive) equilibrium of dynamic system (3.1)–(3.4), where all components are present. Define functions  $f_i : R^2 \rightarrow R^2$ ,  $i = 1, 2$

$$f_1(N_{10}, N_{11}) = \begin{pmatrix} -m_{10}N_{10} + f_{11}N_{11} - p_{11}N_{10}N_{11} - p_{10}N_{10}^2 \\ \alpha_{11}N_{10} - m_{11}N_{11} - \beta N_{11} \end{pmatrix},$$

$$f_2(N_{20}, N_{21}) = \begin{pmatrix} -m_{20}N_{20} + f_{21}N_{21} - p_{21}N_{20}N_{21} - p_{20}N_{20}^2 \\ \alpha_{21}N_{20} - m_{21}N_{21} + \beta N_{11} - qEN_{21} \end{pmatrix}.$$

Then, we can define  $f : R^4 \rightarrow R^4$  given by the right-hand side of this system:

$$f(N) = f(N_{10}, N_{11}, N_{20}, N_{21}) := \begin{bmatrix} f_1(N_{10}, N_{11}) \\ f_2(N_{20}, N_{21}) \end{bmatrix}.$$

Then a vector  $N^*$ ,  $N^* > 0$  is an equilibrium for the considered dynamics if and only if  $f(N^*) = 0$ . From (3.1) we immediately get

$$N_{11}^* = \frac{\alpha_{11}N_{10}^*}{m_{11} + \beta}.$$

Hence, by (3.2) we obtain

$$N_{10}^* = \frac{f_{11}\alpha_{11} - m_{10}(m_{11} + \beta)}{p_{11}\alpha_{11} + p_{11}(m_{11} + \beta)}.$$

In the following we will suppose

$$f_{11}\alpha_{11} - m_{10}(m_{11} + \beta) > 0. \quad (3.5)$$

Analogously, from (3.4) we get

$$N_{21}^* = \frac{\alpha_{21}N_{20}^* + \beta N_{11}^*}{m_{21} + qE}.$$

Finally, from (3.3) we obtain a quadratic equation for  $N_{20}^*$ :

$$a(N_{20}^*)^2 + bN_{20}^* + c = 0$$

where

$$a = (m_{21} + qE)p_{20} + p_{21}\alpha_{21}; \quad b = (m_{21} + qE)\alpha_{20} + p_{21}\beta N_{11}^*; \quad c = -f_{21}\beta N_{11}^*.$$

Since  $a > 0$  and  $c < 0$ , solution

$$N_{20}^{*+} = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

is positive and

$$N_{20}^{*-} = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

is negative.

Summing up, we conclude that under condition (3.5) system (3.1)–(3.4) have a unique positive equilibrium.

### 3.2. Asymptotic stability of the positive equilibrium

In the verticum-type nonlinear model (3.1)–(3.4) we can determine two subsystems according to the definition given in Sec. 2,

$$\begin{aligned} \dot{N}_{10} &= -m_{10}N_{10} + f_{11}N_{11} - p_{11}N_{10}N_{11} - p_{10}N_{10}^2, \\ \dot{N}_{11} &= \alpha_{11}N_{10} - m_{11}N_{11} - \beta N_{11}, \end{aligned} \quad (3.6)$$

$$\begin{aligned} \dot{N}_{20} &= -m_{20}N_{20} + f_{21}N_{21} - p_{21}N_{20}N_{21} - p_{20}N_{20}^2, \\ \dot{N}_{21} &= \alpha_{21}N_{20} - m_{21}N_{21} + \beta N_{11} - qEN_{21}. \end{aligned} \quad (3.7)$$

To study the stability of subsystem (3.6) we calculate the Jacobian

$$A_1 := f'_1(N_1) = \begin{pmatrix} -m_{10} - p_{11}N_{11} - 2p_{10}N_{10} & f_{11} - p_{11}N_{10} \\ \alpha_{11} & -m_{11} - \beta \end{pmatrix},$$

where  $N_1 = (N_{10}, N_{11})$ . Here, there are two possible equilibrium states,  $N_1^{0*} = (0, 0)$  and  $N_1^{1*} = (N_{10}^*, N_{11}^*)$ . At  $N_1^{0*}$ , we have

$$A_1(0, 0) = \begin{pmatrix} -m_{10} & f_{11} \\ \alpha_{11} & -m_{11} - \beta \end{pmatrix}.$$

Then the characteristic equation is  $\lambda^2 + B_1\lambda + B_2 = 0$ , where

$$B_1 = m_{10} + m_{11} + \beta; \quad B_2 = m_{10}(m_{11} + \beta) - \alpha_{11}f_{11}. \quad (3.8)$$

Here  $B_1 > 0$  and by condition (3.5) we have  $B_2 < 0$ . Applying the Routh–Hurwitz criterion (see [1]) we obtain that equilibrium  $N_1^{0*}$  is unstable. In biological terms this means that a release of any stage class at low density in the reserve area would colonize this area.

Proceeding analogously for the positive equilibrium  $N_1^{1*}$ , the Jacobian now becomes

$$A_1(N_{10}^*, N_{11}^*) = \begin{pmatrix} -m_{10} - p_{11}N_{11}^* - 2p_{10}N_{10}^* & f_{11} - p_{11}N_{10}^* \\ \alpha_{11} & -m_{11} - \beta \end{pmatrix}$$

and its characteristic equation is  $\lambda^2 + B_1\lambda + B_2 = 0$  with

$$B_1 = m_{10} + p_{11}N_{11}^* + 2p_{10}N_{10}^* + m_{11} + \beta, \\ B_2 = (m_{10} + p_{11}N_{11}^* + 2p_{10}N_{10}^*)(m_{11} + \beta) - \alpha_{11}(f_{11} - p_{11}N_{10}^*).$$

Now  $f_{11} - p_{11}N_{10}^*$  implies  $B_1 > 0$  and  $B_2 > 0$ , from which by the Routh–Hurwitz criterion we get that equilibrium  $N_1^{1*}$  is (locally) asymptotically stable. The straightforward biological consequence of this is stable coexistence in the reserve area.

On the other hand, set

$$A_2 := f'_2(N_2) = \begin{pmatrix} -m_{20} - p_{21}N_{21} - 2p_{20}N_{20} & f_{21} - p_{21}N_{20} \\ \alpha_{21} & -m_{21} - qE \end{pmatrix},$$

where  $N_2 = (N_{20}, N_{21})$ . Now condition  $f_{21} - p_{21}N_{20}^* < 0$  implies asymptotic stability of the positive equilibrium  $N_2^{1*} = (n_{20}^*, N_{21}^*)$ . Therefore, we have proved the following result.

**Theorem 3.1.** *Suppose that (3.5) holds and*

$$f_{11} - p_{11}N_{10}^* < 0 \quad \text{and} \quad f_{21} - p_{21}N_{20}^* < 0, \quad (3.9)$$

*then equilibrium and are asymptotically stable for systems, respectively.*



A biological interpretation of asymptotic stability is a stable coexistence of both stages in both areas.

Finally, we analyze the stability of the equilibria of the whole system (3.1)–(3.4). To this end we calculate the corresponding Jacobian,

$$A := f'(N_1, N_2) = \begin{pmatrix} & 0 & 0 \\ A_1 & & \\ 0 & 0 & \\ 0 & 0 & A_2 \end{pmatrix}.$$

It is easy to see that the characteristic polynomial of  $A$  has the form

$$p(\lambda) = p_1(\lambda) \cdot p_2(\lambda),$$

where  $p_1(\lambda)$  and  $p_2(\lambda)$  are the characteristic polynomials of  $A_1$  and  $A_2$ , respectively. Hence we obtain the following theorem.

**Theorem 3.2.** *If conditions (3.5) and (3.9) are satisfied, then  $N^{1*} = (N_1^{1*}, N_2^{1*})$  is an asymptotically stable positive equilibrium of system (3.1)–(3.4).*

**Remark 3.3.** Since asymptotic stability implies Lyapunov stability, in the next section we can apply Theorem 2.2 to the corresponding nonlinear verticum-type observation system.

**Remark 3.4.** We note that it is easy to see that the trivial equilibrium of system (3.7) is unstable, if

$$f_{11}\alpha_{11} - m_{10}(m_{11}\beta) < 0 \quad \text{and} \quad f_{21}\alpha_{21} - m_{20}(m_{21} - qE) > 0$$

and asymptotically stable, if the opposite inequalities hold.

## 4. Observability and Observer Design for the Model

### 4.1. Observability

We consider the observation function  $h : R^4 \rightarrow R^2$ , defined by

$$y = h(N_1, N_2) = (N_{11} - N_{11}^*, qE(N_{21} - N_{21}^*)). \quad (4.1)$$

Our aim is to analyze the observability of observation system (3.1)–(3.4), (4.1) using the results of the previous sections.

Given the observation function

$$y_1 = h_1(N_{10}, N_{11}) = N_{11} - N_{11}^*, \quad (4.2)$$

we calculate its linearization

$$C_1 := h'_1(N_1^*) = (0, 1).$$

It is easy to check that  $\text{rank}(C_1|C_1A_1) = 2$ , therefore by Theorem A.1 of Appendix A we can guarantee local observability of system (3.6)–(4.2). Analogously, if we consider the observation function

$$y_2 = h_2(N_{20}, N_{21}) = qE(N_{21} - N_{21}^*) \quad (4.3)$$

for system (3.7) and calculate

$$C_2 := h'_2(N_2^*) = (0, qE).$$

Again we have  $\text{rank}(C_2|C_2A_2) = 2$ , therefore from Theorem A.1 of Appendix A we have local observability of system (3.7)–(4.3).

Since under the appropriate conditions equilibrium  $x^*$  is asymptotically stable and hence also Lyapunov stable, applying Theorem 2.2 we obtain the following theorem.

**Theorem 4.1.** *Suppose that conditions (3.5) and (4.2) hold. Then observation system (3.1)–(3.4), (4.1) is locally observable near equilibrium  $N^* = (N_1^{1*}, N_2^{2*})$ .*

#### 4.2. Construction of an observer system

Given the observation system (3.6)–(4.2), using the corresponding observer design of Sundarapandian [18], it is sufficient to find a matrix  $K^1$  such that  $A_1 - K^1C_1$  is Hurwitz. It is easy to check that with  $k_2^1 > 0$ ,

$$K^1 = \begin{pmatrix} 0 \\ k_2^1 \end{pmatrix}$$

is appropriate.

Analogously, for observation system (3.7)–(4.3), with  $k_2^2 > 0$ ,

$$K^2 = \begin{pmatrix} 0 \\ k_2^2 \end{pmatrix}.$$

$A_2 - K^2C_2$  is Hurwitz, guaranteeing the construction of the observer system.

From these results, for

$$K := \begin{pmatrix} K^1 \\ K^2 \end{pmatrix}, \quad C := (C_1 \ C_2)$$

we can check that  $A - KC$  is Hurwitz, which allows us to construct the observer for system (3.1)–(3.4), (4.1). Moreover, this observer is composed of the observers constructed for the two subsystems.

**Example 4.2.** Use the following model parameters of [15]:

$$\begin{aligned}
 m_{10} &= 0.4; & m_{20} &= 0.4, \\
 f_{11} &= 0.5; & f_{21} &= 0.5, \\
 p_{11} &= 0.1; & p_{21} &= 0.1, \\
 p_{10} &= 0.2; & p_{20} &= 0.1, \\
 \alpha_{11} &= 0.8; & \alpha_{21} &= 0.9, \\
 m_{11} &= 0.05; & m_{21} &= 0.1, \\
 \beta &= 0.65; & q &= 0.07; & E &= 0.5.
 \end{aligned}$$

To construct the observer system for (3.6)–(4.2) we take

$$K^1 = \begin{pmatrix} 0 \\ 10 \end{pmatrix}.$$

Then the observer system is

$$\begin{aligned}
 \dot{Z}_{10} &= -0.4Z_{10} + 0.5Z_{11} - 0.1Z_{10}Z_{11} - 0.2Z_{10}^2, \\
 \dot{Z}_{11} &= 0.8Z_{10} - 0.05Z_{11} - 0.65Z_{11} + 10(N_{11} - Z_{11}).
 \end{aligned} \tag{4.4}$$

Considering  $N_1(0) = (0.1, 0.2)$  as initial value for the system (3.6), and  $Z_1(0) = (0.5, 0.1)$  for the observer (4.4), in Fig. 1 we can see how the solution of the observer system approaches the solution of the original system. To construct the observer of system (3.7)–(4.3) we take

$$K^2 = \begin{pmatrix} 0 \\ 10 \end{pmatrix}.$$

Then the observer system is

$$\begin{aligned}
 \dot{Z}_{20} &= -0.4Z_{20} + 0.5Z_{21} - 0.1Z_{20}Z_{21} - 0.1Z_{20}^2, \\
 \dot{Z}_{21} &= 0.9Z_{20} - 0.1Z_{21} + 0.65Z_{11} - 0.07 \cdot 0.5Z_{21} + 10 \cdot 0.07 \cdot 0.5(N_{21} - Z_{21}).
 \end{aligned} \tag{4.5}$$

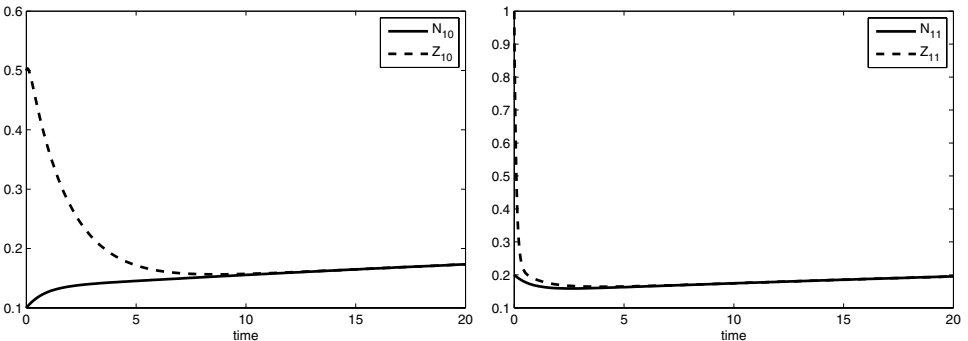


Fig. 1. Solution of the observer (4.4) approaching the solution of the original system (3.6).

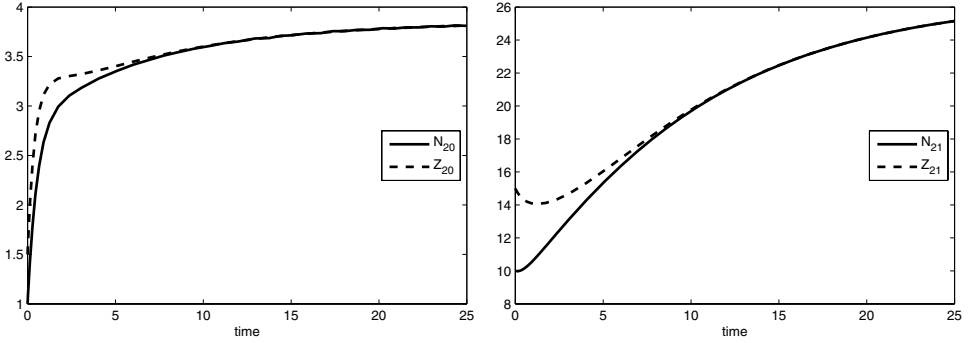


Fig. 2. Solution of the observer (4.5) approaching the solution of the original system (3.7).

If we consider  $N_2(0) = (1, 10)$  as initial value for system (3.7), and  $Z_2(0) = (1.5, 15)$  for the observer (4.5), we obtain the result plotted in Fig. 2. Now the observer for system (3.1)–(3.4), (4.1) can be simply composed from the single observers (4.4) and (4.5). In Fig. 3 we can see how the solution of the observers (4.4)–(4.5) with

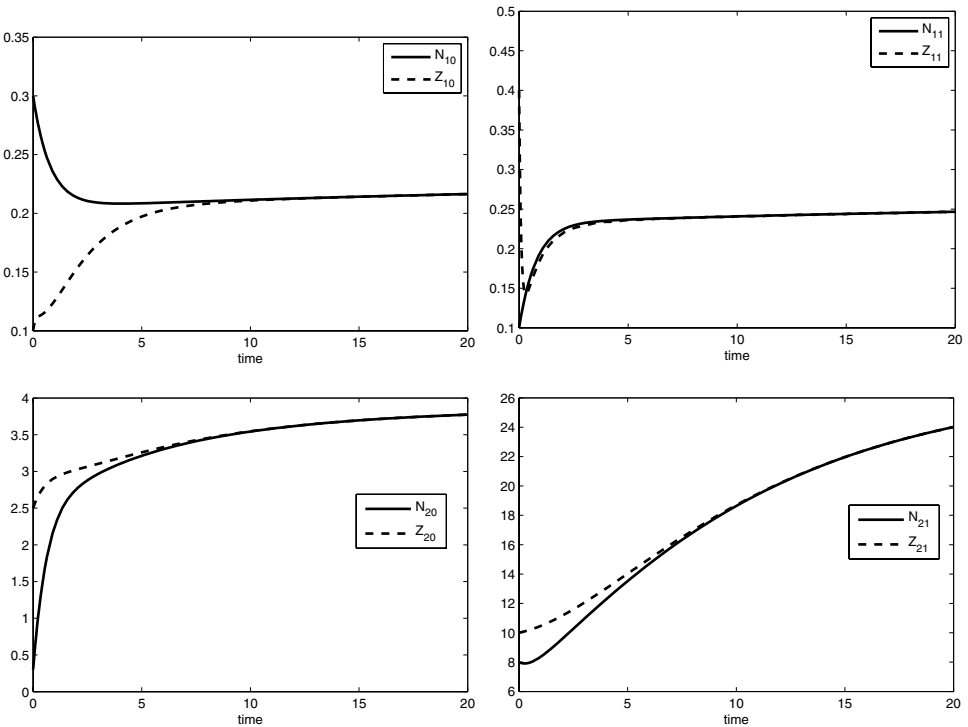


Fig. 3. Solution of the observers (4.4) and (4.5) approaching the solution of the original system (3.1)–(3.4) by coordinates.

initial value  $Z(0) = (0.1, 0.4, 2.5, 10)$ , estimates the solution of system (3.1)–(3.4) with initial value  $N(0) = (0.3, 0.1, 0.3, 8)$ .

## 5. Discussion

The concept of linear verticum-type systems have been introduced for the description of certain complex industrial systems consisting of several units (“subsystems”) where in a “vertical chain” each “subsystem” is unilaterally connected to the next one by a state variable. In industrial systems these connections typically describe a material flow from one production unit to the next one.

In population ecological modeling, verticum-type systems basically occur in two situations: First, when either in an ecological chain there is a unilateral interaction such as commensalism (e.g. a commensalist animal may make use of a plant as part of its habitat without harming it). Second, when there is a unilateral biomass flow from one habitat to another. In this paper the latter case have been studied. To this end it was necessary to generalize the concept of a verticum-type system to the nonlinear case. For a decomposition of the monitoring problem of such systems we also extended the sufficient condition for observability (existing for linear systems), to the nonlinear case.

It is also shown that for an efficient state estimation based on the partial observation of the system, the construction of a Luenberger-type observer system can also be carried out by a decomposition into “subsystems”.

For an illustration of our results we considered the simplest nontrivial case of a stage-structured fishery model with reserved area. We note that our approach can also be extended to more complicated verticum-type population systems, also including some abiotic effects and/or changing environment.

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## Appendix A

Suppose  $m, n, r \in \mathbb{N}$ ,  $F \in C^1(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$ ,  $h \in C^1(\mathbb{R}^n, \mathbb{R}^r)$ ,  $(x^*, u^*) \in \mathbb{R}^n \times \mathbb{R}^m$  such that  $F(x^*, u^*) = 0$  and  $h(x^*) = 0$ .

**Remark A.1.** It is known (see e.g. Lee and Markus [8]) that, given a fixed  $T > 0$ , there exists an  $\varepsilon_0 > 0$  such that for all  $z_0 \in \mathbb{R}^n$  with  $\|z_0 - x^*\| < \varepsilon_0$  there exists a unique continuously differentiable function  $x \in C^1([0, T], \mathbb{R}^n)$  such that  $\dot{x}(t) = F(x(t))$ , for all  $t \in [0, T]$ .

**Definition A.2.** With the above notation, consider the control-observation system in  $R^n$

$$\dot{x} = F(x, u), \tag{A.1}$$

$$y = h(x). \tag{A.2}$$

System (A.1) and (A.2) is said to be locally observable near the equilibrium if there exists  $\varepsilon > 0$  such that

$$|u(t) - u^*| < \varepsilon, \quad |x(0) - x^*| < \varepsilon, \quad |\bar{x}(0) - x^*| < \varepsilon,$$

and

$$h(x(t)) = h(\bar{x}(t)) \quad \text{for all } t \in [0, T]$$

imply that

$$x(0) = \bar{x}(0).$$

**Theorem A.3 (Lee and Markus, [8]).** Consider the control-observation system (A.1) and (A.2) in  $R^n$  with

$$A = \frac{\partial}{\partial x} F(x^*, u^*), \quad C = h'(x^*).$$

Assume

$$\text{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n. \tag{A.3}$$

Then system is locally observable near the equilibrium.

**Remark A.4.** The theorem similar to the previous one is also valid for function  $F$  not depending on control.

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