On the non-autonomous logistic equation

R. Carreño, I. López and M. Gámez†

Departamento de Estadística y Matemática Aplicada

Universidad de Almería. Spain.

1 Introduction

It can be observed in many papers on periodic differential equations that a lot of hypotheses are stated in terms of the fundamental concept of average [1], [6] and [4].

In this paper we study the logistic equation

$$x' = xF(t,x) \tag{1.1}$$

where $F : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ is a continuous function such that $xF(t,x)$ is locally Lipschitz continuous in x and $F(t,x)$ is decreasing in x. When $F(t,x) = f(x)$ is continuous, system (1.1) has been considered as a model to describe the growth properties of a single population. In the special case, $f(x) = a - bx$ (a, b positive real numbers), the above system is know as the Verhulst equation and its history can be found in [5].

Obviously, the growth properties of every natural population vary thought time, and so the non-autonomous system (1.1) is more realistic than its autonomous counterpart.

The aim of this paper is to improve the results of Vance and Condding-ton [9], in which it was assumed that the long-term average values of F obey certain properties. More precisely, it was assumed that there exists a time period of fixed length such that, no matter when an interval of this length begins, the population will experience a representative sample of all ecologically important events during this time interval that it will ever experience in all time.

*Corresponding author: M. Gámez, fax:(34)950-015167, email: mgamez@ual.es

The main assumptions in this paper involve a sort of average concept which does not depend on a time unit and will be explained in the last section of the article.

The paper is organized as follows. In section 2 we prove that (1.1) is dissipative if $F(t,R) \leq B'(t)$ for some $R > 0$ and some bounded continuously differentiable function $B : \mathbb{R} \to \mathbb{R}$. This condition implies that "the upper average of $F(t,R)$ is less than zero". Analogously, we prove that (1.1) is persistent if it is dissipative and $F(t,\delta) \geq A'(t)$ for some $\delta > 0$ and some bounded continuously differentiable function $A : \mathbb{R} \to \mathbb{R}$.

In section 3 we prove the existence of a solution of (1.1) in the set $\mathcal{C}_+$ of all continuous functions which are bounded above and below by positive constants. Moreover we classify the set of all solutions of (1.1) which are defined in $\mathbb{R}$.

In section 4, we study uniqueness of the solutions of (1.1) in $\mathcal{C}_+$ and we prove, under certain additional restrictions, that any pair of solution of (1.1) are asymptotic as $t \to +\infty$.

In section 5, we prove that the results in section 1 improve the main results in [9]. Finally, in section 6 we explain how our hypotheses in section 1 can be seen as some kind of average restrictions.

2 Dissipativity and persistence

We consider the logistic equation

$$x' = xF(t,x) \tag{2.1}$$

where $F : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ is a continuous function which is locally Lipschitz continuous in x. In this section, we assume that:

H1) $F(t,x)$ is decreasing in x.

H2) $\int_0^\infty [F(t,x) - F(t,y)]dt = +\infty \quad \text{if} \quad 0 \leq x < y$.

H3) There exist $R > 0$ and a bounded and continuously differentiable function $B : \mathbb{R} \to \mathbb{R}$ such that,

$$F(t,R) \leq B'(t) \quad ; \quad t \in \mathbb{R}. \tag{2.2}$$

We also use the following notations. Given a bounded function $D : \mathbb{R} \to \mathbb{R}$ we write $I_D = -inf\{D(t) - D(s) : \ s \leq t\}$, $S_D = sup\{D(t) - D(s) : \ s \leq t\}$, $M_D = I_D + S_D$. Note that $I_D, S_D \geq 0$.

In the following, u denotes a (uncontinuable) solution of (2.1) such that $u(\tau) > 0$ for some $\tau \in$ dom(u). Here, and henceforth, dom(u) denotes the (maximal) domain of u. Note that $u(t) > 0$ for all $t \in$ dom(u).

Proposition 2.1 *Let τ be as above. Then u is defined on $[\tau, \infty)$ and*

$$u(t) \leq max\{u(\tau),\ Rexp(I_B)\}e^{B(t)-B(\tau)}; \quad t \geq \tau. \tag{2.3}$$

In particular, if $u(\tau) \leq Rexp(I_B)$, then

$$u(t) \leq Rexp(M_B),\ \text{for } t \geq \tau.$$

Proof. For $t \geq \tau$, let $v(t)$ be the right-hand side of (2.3). Then,

$$v(t) \geq Rexp(I_B)e^{B(t)-B(\tau)} > R$$

and $v'(t) = B'(t)v(t)$. From this H1) and H3), v is a supersolution of (2.1) such that $v(\tau) \geq u(\tau)$ and hence, $v(t) \geq u(t)$ if $t \in dom(u) \bigcap [\tau, \infty)$. The proof now follows easily. ■

Theorem 2.2 *[Dissipativity] For each positive solution u of (2.1) we have,*

$$\lim_{t\to+\infty} sup\ u(t) \leq R\exp(M_B).$$

Proof. Let us fix $S > R$. We shall prove that

$$u(\tau) \leq Sexp(I_B) \text{ for some } \tau \in dom(u). \tag{2.4}$$

To show this, assume to the contrary that $u(t) > Sexp(I_B)$ for all $t \in dom(u)$, and define $v(t) = u(t)e^{B(t)}$. Since $u(t) \geq S$, and (2.2) holds, we obtain

$$\frac{v'(t)}{v(t)} = F(t, u(t)) - B'(t) \leq F(t, S) - F(t, R)$$

and hence, for a fixed $s \in dom(u)$, we have

$$\ln \frac{v(s)}{v(t)} \geq \int_s^t [F(\sigma, R) - F(\sigma, S)]d\sigma \to +\infty \text{ as } t \to +\infty;$$

since H2) holds. From this, $v(t) \to 0$ as $t \to +\infty$, and thus the same holds for u. But $u(t) \geq S > 0$, and this contradiction proves (2.4).

By (2.3)-(2.4) we have, $u(t) \leq Sexp(I_B)e^{B(t)-B(\tau)} \leq Sexp(M_B)$ for $t \geq \tau$, and the proof is complete. ■

Corollary 2.3 *Suppose that there exists a bounded and continuously differentiable function $B : \mathbb{R} \to \mathbb{R}$ such that $F(t, 0) \leq B'(t)$. Then,*

$$u(t) \to 0 \quad as \quad t \to +\infty$$

for any positive solution u of (2.1).

Proof. The assumptions in theorem 2.2 are satisfied for every $R > 0$, and the proof follows from that result. ■

Proposition 2.4 *In addition to H1)-H3), suppose that*
H4) There exists $\delta > 0$ such that

$$F(t, \delta) \geq A'(t); \quad t \in \mathbb{R}; \tag{2.5}$$

for some bounded and continuously differentiable function $A : \mathbb{R} \to \mathbb{R}$. Then, for each $\tau \in dom(u)$

$$u(t) \geq min\{u(\tau), \delta exp(-S_A)\}e^{A(t)-A(\tau)}; \quad t \geq \tau. \tag{2.6}$$

In particular, if $u(\tau) \geq \delta exp(-S_A)$, then

$$u(t) \geq \delta exp(-M_A); \quad for \quad t \geq \tau.$$

Proof. See proposition 2.1. ■

Remark 2.5 If δ satisfies (2.5), then $\delta \leq R$. To show this assume to the contrary that $\delta > R$. By (2.2) and (2.5), $A'(t) \leq F(t, \delta) \leq F(t, R) \leq B'(t)$, and hence

$$B(t) - A(t) - [B(0) - A(0)] = \int_0^t [B'(s) - A'(s)]ds$$
$$\geq \int_0^t [F(s, R) - F(s, \delta)]ds \to +\infty \text{ as } t \to +\infty.$$

Then $B - A$ is not bounded, and this contradiction ends the proof.

Theorem 2.6 (Persistence.) *If H4) is satisfied, then*

$$lim_{t\to+\infty} inf u(t) \geq \delta exp(-S_A)$$

for any positive solution u of (2.1).

Proof. See theorem 2.2. ■

3 A classification of the solutions of the logistic equation

In the following $\mathcal{C}_+$ denotes the set of all bounded continuous functions $\omega : \mathbb{R} \to \mathbb{R}$ such that $inf(\omega) > 0$.

Corollary 3.1 *If H1)-H4) are satisfied, then (2.1) has solution in* $\mathcal{C}_+$.

Proof. We shall use a well-know argument. Let us fix $x_0 \in [\delta exp(-M_A), R\ exp(M_B)]$ and define u_k, for each integer $k \geq 1$, as the solution of (2.1) determined by the initial condition $u_k(-k) = x_0$. By propositions 2.1 and 2.4,

$$\delta exp(M_A) \leq u_k(t) \leq R exp(M_B), \quad for \quad t \geq -k.$$

In particular, $\{u_k(0)\}$ is bounded, and so, we can assume that it converges to a point $\xi > 0$. Now it is easy to show that the solution u of (2.1), determined by the initial condition $u(0) = \xi$, satisfies the required property. ■

In this section, in addition to H1)-H4), we also assume that

H5) $\int_{-\infty}^{0} [F(t, x) - F(t, y)]dt = +\infty$ if $0 \leq x < y$.

Proposition 3.2 *Let* $u : \mathbb{R} \to (0, +\infty)$ *be a solution of (2.1).*
a) If there exist $T \in \mathbb{R}$ *and* $\eta \in (0, \delta)$ *such that* $u(t) < \eta e^{-S_A}$ *for* $t < T$, *then* $u(t) \to 0$ *as* $t \to -\infty$.
b) If there exist $T \in \mathbb{R}$ *and* $S > R$ *such that* $u(t) > Se^{I_B}$ *for* $t < T$, *then* $u(t) \to +\infty$ *as* $t \to -\infty$.

Proof. We only prove a). To do this, we define $v(t) = u(t)e^{-A(t)}$ and we remark that $u(t) \leq \eta$ and

$$\frac{v'(t)}{v(t)} = F(t, u(t)) - A'(t) \geq F(t, \eta) - F(t, \delta)$$

for $t < T$. From this, for a fixed $s \in dom(u)$,

$$\ln \frac{v(s)}{v(t)} \to +\infty \text{ as } t \to -\infty,$$

(since H5) holds) and the proof follows easily. ■

Theorem 3.3 *Assume H1)-H5) hold. If $u \in \mathcal{C}_+$ is a solution of (2.1), then $\delta exp(-S_A) \leq u(t) \leq Rexp(I_B)$. In particular, $\delta exp(-M_A) \leq u(t) \leq Rexp(M_B)$.*

Proof. Let us fix $\eta \in (0, \delta)$. By proposition 3.2 a), there exists a sequence $\tau_n \to -\infty$ such that $u(\tau_n) \geq \eta e^{-S_A}$. From this and (2.6), $u(t) \geq \eta e^{-S_A}$ if $t \geq \tau_n$. Hence, $u(t) \geq \eta e^{-S_A}$, for all $t \in \mathbb{R}$ and so, $u(t) \leq \delta e^{-S_A}$ since $\eta \in (0, \delta)$ is arbitrary. The rest of the proof is analogous. ■

Let us define $\mathcal{X} = \{u(0) : u \in \mathcal{C}_+ \ is\ a\ solution\ of\ (2.1)\}$. By corollary 3.1, $\chi = \emptyset$ and by theorem 3.3 $\mathcal{X}$ is bounded and closed. In particular, (2.1) has a maximal (and a minimal) solution in $\mathcal{C}_+$.

Theorem 3.4 *Assume that H1) - H5) hold and let $u : \mathbb{R} \to (0, \infty)$ be a solution of (2.1)*

a) If $u(0) < \inf(\mathcal{X})$, then $u(t) \to 0$ as $t \to \infty$.

b) If $u(0) > \sup(\mathcal{X})$, then $u(t) \to +\infty$ as $t \to -\infty$.

Proof. We only prove a). To this end, let us fix $\eta \in (0, \delta)$ and assume that there exists a sequence $\tau_n \to -\infty$ such that $u(\tau_n) \geq \eta e^{-S_A}$. By the argument in theorem 3.3, $u(t) \geq \eta e^{-S_A}$, for all $t \in \mathbb{R}$.

Let us fix $x \in \mathcal{X}$ and let $v \in \mathcal{C}_+$ be the solution of (2.1) determined by the initial condition $v(0) = x$. Since $u(0) < v(0)$ then, $u < v$, and by theorem 3.3, $u(t) \leq Rexp(M_B)$. Thus, $u \in \mathcal{C}_+$, and this contradiction ($u(0) \in \mathcal{X}$) shows the existence of $T \in \mathbb{R}$ such that $u(t) < \delta exp(-S_A)$ for $t < T$. The proof now follows from proposition 3.2 a). ■

4 Asymptotic properties. Uniqueness

Let I be an interval. We denote by $\mathcal{C}_+(I)$ the set of all bounded continuous functions $u : I \to \mathbb{R}$ such that $\inf(u) > 0$.

Proposition 4.1 *Assume $H1$) holds and that*

$$\int_{-\infty}^{0} [F(t, u_0(t)) - F(t, u_1(t))]dt = +\infty \tag{4.1}$$

if $u_0, u_1 \in \mathcal{C}_+(-\infty, 0]$ are uniformly continuous and $u_1 - u_0 \in \mathcal{C}_+(-\infty, 0]$. Suppose further that the restriction of F to $\mathbb{R} \times K$ is bounded for any compact set $K \subset (0, +\infty)$. Then, (2.1) has at most one solution in $\mathcal{C}_+$.

Proof. Let $u_0 < u_1$ be solutions of (2.1) in $\mathcal{C}_+$ and define $\Delta = \dfrac{u_0}{u_1}$. Then $\inf(\Delta) > 0$ and

$$\Delta' = \Delta[F(t, u_0(t)) - F(t, u_1(t))] \geq 0.$$

From this, $\Delta(t)$ has a finite limit $\lambda \in (0, 1)$ as $t \to -\infty$. In particular,

$$\int_{-\infty}^{0} [F(t, u_0(t)) - F(t, u_1(t)]dt = \lim(\frac{\Delta(0)}{\lambda}) < +\infty. \tag{4.2}$$

On the other hand; $\Delta(t) \leq \Delta(0)$ for $t \leq 0$, and so,

$$u_1(t) - u_0(t) \geq [1 - \Delta(0)]u_1(t) \geq [1 - \Delta(0)] \inf(u_1) > 0$$

if $t \leq 0$, that is, $u_1 - u_0 \in \mathcal{C}_+(-\infty, 0]$. Obviously, u_0, u_1 are uniformly continuous since, u_0', u_1' are bounded. From this, (4.1) holds. This contradicts (4.2) and the proof is complete. ■

Let $K \subset [0, \infty)$ be compact and let $G : K \to \mathbb{R}$ be a function. We write $G \in \alpha_K(F)$ (resp. $G \in w_K(F)$) if there exists a sequence $t_n \to -\infty$ (resp. $t_n \to +\infty$) such that

$$F(t_n, x) \to G(x) \quad \text{as} \quad n \to \infty \quad \text{uniformly for} \quad x \in K. \tag{4.3}$$

Remark 4.2 Suppose that the restriction of F to $\mathbb{R} \times K$ is bounded and uniformly continuous for each compact subset K of $(0, \infty)$. Assume further that $\alpha_K(F)$ contains a strictly decreasing function for any compact subset $K \subset (0, \infty)$. If H1) holds, then F satisfies the assumptions in proposition (4.1).

Proof. Assume, to the contrary, that there exist $u_0, u_1 \in \mathcal{C}_+(-\infty, 0]$ uniformly continuous such that $u_1 - u_0 \in \mathcal{C}_+(-\infty, 0]$ and

$$\int_{-\infty}^{0} F(t, u_0(t)) - F(t, u_1(t))]dt < +\infty.$$

Now fix a compact set $K \subset (0, \infty)$ such that $u_0(t), u(t) \in K$ for all $t \in \mathbb{R}$. Since the restriction of F to $\mathbb{R} \times K$ is uniformly continuous, then $F(t, u_0(t)) - F(t, u_0(t))$ is uniformly continuous and hence.

$$F(t, u_0(t)) - F(t, u_1(t)) \to 0 \quad \text{as} \quad t \to -\infty.$$

Now, let $G \in \alpha_K(F)$ be strictly decreasing and fix a sequence $t_n \to -\infty$ satisfying (4.3). Since u_0, u_1 are bounded, we can assume, without loss of generality, that $u_i(t_n) \to x_i$; $i = 0, 1$. From this, $G(x_0) = G(x_1)$ and, consequently, $x_0 = x_1$. Thus, $u_1(t_n) - u_0(t_n) \to 0$ as $n \to \infty$, and this contradiction ($u_1 - u_0 \in \mathcal{C}_+(-\infty, 0]$) ends the proof. ■

Proposition 4.3 *In addition H1)-H4), assume that F is bounded on $\mathbb{R} \times K$ for any compact set K of $[0, \infty)$. Suppose further that,*

$$\int_{\tau}^{\infty} [F(t, v_0(t)) - F(t, v_1(t))]dt = +\infty \tag{4.4}$$

if $v_0, v_1 \in \mathcal{C}_+[\tau, \infty)$ are uniformly continuous and $v_1 - v_0 \in \mathcal{C}_+[\tau, \infty)$. If u_0, u_1 are positive solutions of (2.1) then

$$u_1(t) - u_0(t) \to 0 \quad \textit{as} \quad t \to +\infty.$$

Proof. Let us fix $\tau \in dom(u_0) \bigcap dom(u_1)$. Without loss of generality, we can suppose that $u_0 < u_1$. Define $0 < \Delta(t) = \dfrac{u_0(t)}{u_1(t)}$ for $t \geq \tau$. Then $\Delta(t) < 1$ and $\Delta'(t) = \Delta(t)[F(t, u_0(t)) - F(t, u_1(t))] \geq 0$. In particular, $\Delta(t)$ has a finite limit $\lambda \in (0, 1]$ as $t \to +\infty$.

Claim. $\lambda = 1$. To show this assume $\lambda < 1$. By proposition 2.1 and theorem 2.6, the restriction of u_i to $[\tau, \infty)$ belongs to $\mathcal{C}_+[\tau, \infty)$. Note also that,

$$u_1(t) - u_0(t) \geq u_1(t) - \lambda u_0(t) = (1 - \lambda) u_1(t),$$

and so, $u_1 - u_0 \in \mathcal{C}_+[\tau, \infty)$. Therefore, (4.4) holds. Note that the restriction of u_i to $[\tau, \infty)$ is uniformly continuous since the restriction of u_i' is bounded in this interval. On the other hand,

$$\int_\tau^0 [F(t, u_0(t) - F(t, u_1(t))]dt = \ln \frac{\lambda}{\Delta(\tau)} < +\infty$$

and this contradiction proves the claim.

Hence $\dfrac{u_0(t)}{u_1(t)} \to 1$ as $t \to +\infty$ and the proof follows from the fact that the restriction of u_i to $[\tau, \infty)$ is bounded. ■

Remark 4.4 The assumptions in proposition 4.3 are satisfied if:
a) The restriction of F to $\mathbb{R} \times K$ is bounded and uniformly continuous on $\mathbb{R} \times K$, for any compact set $K \subset [0, \infty)$.
b) $\omega_K(F)$ contains a strictly decreasing function for any compact set $K \subset [0, \infty)$.

Proof. See remark 4.2. ■

Using the arguments in propositions 4.1 and 4.3 we obtain

Proposition 4.5 *Assume (4.1) holds if* $u_0, u_1, u_1 - u_0 \in \mathcal{C}_+(-\infty, 0]$. *If* $H1$) *holds then (2.1) has at most one solution in* $\mathcal{C}_+$.

Proposition 4.6 *In additions to* $H1$)-$H4$)*, assume that (4.4) holds if* $v_0, v_1, v_1 - v_0 \in \mathcal{C}_+[\tau, \infty)$. *If* u_0, u_1 *are positive solution of (2.1) then*

$$u_0(t) - u_1(t) \to 0 \quad as \quad t \to +\infty.$$

Remark 4.7 a) Suppose that for each compact subset K of $(0, \infty)$ the exists a continuous function $b_K : \mathbb{R} \to [0, \infty)$ such that

$$F(t, x) - F(t, x) \geq b_k(t)(y - x) \quad \text{if} \quad x \leq y \quad \text{and} \quad x, y \in K.$$

If $\int_{-\infty}^0 b_K(t)dt = +\infty$, then the assumptions in proposition 4.5 are satisfied.
b) We have a parallel remark concerning proposition 4.6 (In this case, it is assumed that $K \subset [0, \infty)$ and $\int_0^\infty b_K(t)dt = +\infty$).

5 Comparison with earlier results

In [9] the following result was proved.

Theorem 5.1 *Suppose that $F(t,x)$ has continuous partial derivative $F_x(t,x)$ defined on $\mathbb{R} \times \mathbb{R}_+$ such that*

$$F_x(t,x) \leq -\lambda(t)\gamma(x) \tag{5.1}$$

for some continuous functions $\gamma : [0,\infty) \to (0,\infty)$, $\lambda : \mathbb{R} \to [0,\infty)$ such that

$$\int_0^\infty \lambda(t)dt = +\infty. \tag{5.2}$$

If there exist positive numbers β, R such that

$$F(t,R) \leq \beta, \quad \int_t^{t+1} F(s,R)ds \leq 0; \quad t \in \mathbb{R}; \tag{5.3}$$

then (2.1) is dissipative.

Further, if there exist positive numbers α, δ such that

$$F(t,\delta) \geq -\alpha, \quad \int_t^{t+1} F(s,\delta)ds \geq 0; \quad t \in \mathbb{R}, \tag{5.4}$$

then (2.1) is persistent.

Moreover, if (5.1)-(5.4) hold and u_0, u_1 are positive solution of (2.1) then

$$u_1(t) - u_0(t) \to 0 \quad as \quad t \to +\infty.$$

Remark 5.2 Note that (5.1)-(5.2) imply (4.4), since the assumptions in remark 4.7, part b), are satisfied with $b_K(t) = \min\{\gamma(x) : x \in K\}\lambda(t)$.

Using lemma 5.6 below it is easy to show that (5.3) (resp. (5.4)) implies $H3$) (resp. $H4$)) and so, theorem 5.1 is a consequence of theorems 2.2, 2.6 and proposition 4.6.

Lemma 5.3 *Given a continuous function $\alpha : [a,d] \to (-\infty, 0]$ and a number $\rho \geq 0$, $\rho > \alpha(d)$, there exists a continuous function $\beta : [a,d] \to \mathbb{R}$ such that $\alpha(t) \leq \beta(t) \leq \rho$, for $a \leq t \leq a$; $\beta(a) = \alpha(a)$ and $\beta(d) = \rho$.*

Proof. Let $L : [a, d] \to \mathbb{R}$ be the affine map determined by the conditions $L(a) = \alpha(a)$, $L(d) = \rho$. Since $L(d) > \alpha(d)$, there exists $c \in [a, d)$ such that $L(c) = \alpha(c)$ and $L(t) > \alpha(t)$ for all $t \in (c, d]$. Let us define $\beta : [a, d] \to \mathbb{R}$ by $\beta(t) = \alpha(t)$ for $a \leq t \leq c$ and $\beta(t) = L(t)$ for $a \leq t \leq d$. Obviously $\alpha \leq \beta$. On the other hand, the slop of L is negative and hence, $L(t) \leq \rho$ for all $t \in [c, d]$. From this $\beta \leq \rho$. The rest of the proof is trivial. ∎

Lemma 5.4 *Let $\alpha : [a, c] \to \mathbb{R}$ be a continuous function such that $\alpha < 0$ in $[a, c)$ and $\alpha(c) = 0$. Given a positive number $\epsilon > 0$, there exists a continuous function $\beta : [a, c] \to (-\infty, 0]$ such that $\beta(a) = \alpha(a)$ $\beta(c) = 0$, $\alpha \leq \beta$ and*

$$\int_a^c \beta(s)ds \geq -\epsilon.$$

Proof. Let us fix a number $\delta \in (0, c - a)$ such that

$$\int_a^{a+\delta} \alpha(s)ds \geq -\epsilon.$$

By lemma 5.3, there exists a continuous function $\gamma : [a, a + \delta] \to (-\infty, 0]$ such that, $\alpha \leq \gamma$, $\gamma(a) = \alpha(a)$ and $\gamma(a + \delta) = 0$. We define $\beta : [a, c] \to \mathbb{R}$ by $\beta(t) = \gamma(t)$ on $[a, a + \delta]$ and $\beta(t) = 0$ for $t \in [a + \delta, c]$. Thus,

$$\int_a^c \beta(s)ds = \int_a^{a+\delta} \gamma(s)ds \geq \int_a^{a+\delta} \alpha(s)ds$$

and the proof is complete. ∎

Lemma 5.5 *Let $\alpha : [a, b] \to \mathbb{R}$ be a continuous function such that $\int_a^b \alpha(t)dt \leq 0$. Given a positive number $M \geq \max(\alpha)$, there exists a continuous function $\beta : [a, b] \to \mathbb{R}$ such that $\alpha(t) \leq \beta(t) \leq M$ for all $t \in [a, b]$, $\beta(a) = \alpha(a)$, $\beta(b) = \alpha(b)$ and*

$$\int_a^b \beta(t)dt = 0.$$

Proof. Note that if $\int_a^b \alpha(t)dt = 0$, it suffices to take $\beta \equiv \alpha$. Thus, we shall assume that $\int_a^b \alpha(t)dt < 0$. Let $\mathcal{C}$ be the space of all continuous functions

$\beta : [a,b] \to \mathbb{R}$ provided with the used sup norm and let $\mathcal{F} \subset \mathcal{C}$ be the subset consisting of all points $\beta \in \mathcal{C}$ such that

$$\begin{aligned} \beta &\equiv \alpha \quad \text{in} \quad \{a,b\} \cup \alpha^{-1}[0,\infty) \\ \alpha(t) &\leq \beta(t) \leq M; \quad t \in [a,b] \end{aligned}$$

Obviously, $\mathcal{F}$ is convex and $\alpha \in \mathcal{F}$. On the other hand, the map $I : \mathcal{F} \to \mathbb{R}$; $I(\beta) = \int_a^b \beta(s)ds$; is continuous and $I(\alpha) \leq 0$. Thus, it suffices to show the existence of a $\beta_* \in \mathcal{F}$ such that $I(\beta_*) \geq 0$.

As usual, we define $\alpha^+(t) = \max\{0, \alpha(t)\}$. Note that, if $\alpha(a) \geq 0$ and $\alpha(b) \geq 0$, then $\alpha^+ \in \mathcal{F}$ and $I(\alpha^+) \geq 0$. So, we can assume that either $\alpha(a) < 0$ or $\alpha(b) < 0$.

Now let us consider the following cases and subcases:

Case 1. $\alpha^+ \equiv 0$.

Subcase 1.1. $\alpha(a) < 0$ and $\alpha(b) \geq 0$.

In this case there exists $c \in (a,b]$ such that $\alpha < 0$ on $[a,c)$ and $\alpha(c) = 0$. By lemma 5.4, there exists a continuous function $\gamma : [a,c] \to (-\infty, 0]$ such that $\gamma(a) = \alpha(a)$, $\gamma(c) = 0$ and $\int_a^c \gamma(s)ds \geq \int_a^b \alpha^+$. Let us define $\beta_* : [a,b] \to \mathbb{R}$ by

$$\begin{aligned} \beta_*(t) &= \gamma(t) \quad \text{for} \quad t \in [a,c] \\ \beta_*(t) &= \alpha^+(t) \quad \text{for} \quad t \in [c,d] \end{aligned}$$

Obviously, $\beta_* \in \mathcal{F}$ and

$$\int_a^b \beta_* = \int_a^b \gamma + \int_c^b \alpha^+ = \int_a^c \gamma + \int_a^b \alpha^+ \geq 0.$$

Subcase 1.2. $\alpha(a) \geq 0$ and $\alpha(b) < 0$.

The proof of this case is similar to subcase 1.2.

Subcase 1.3. $\alpha(a) < 0$ and $\alpha(b) < 0$.

Let us fix $c < d$ on (a,b) such that $\alpha < 0$ on $[a,c) \cup (d,b]$ and $\alpha(c) = \alpha(d) = 0$. By lemma 5.4 there are continuous functions $\gamma_0 : [a,c] \to (-\infty, 0]$, $\gamma_1 : [d,b] \to (-\infty, 0)$ such that $\gamma_0(a) = \alpha(a)$, $\gamma_0(c) = \gamma_1(b) = 0$, $\gamma_1(d) = \alpha(d)$,

$$\int_a^c \gamma_0 \geq -\frac{1}{2}\int_a^b \alpha^+, \quad \int_d^b \gamma_1 \geq -\frac{1}{2}\int_a^b \alpha^+.$$

Define $\beta_* : [a,b] \to \mathbb{R}$ by $\beta_* = \gamma_0$ on $[a,c]$, $\beta_* = \alpha^+$ on $[c,d]$ and $\beta_* = \gamma_1$ on $[d,b]$. It is easy to show that $\beta_* \in \mathcal{F}$ and $I(\beta_*) \geq 0$.

Case 2. $\alpha \leq 0$. Let us fix $\delta > 0$, $\delta < \left(b - \dfrac{a}{2}\right)$, such that

$$M(b-a) \geq 2\delta[M - \min(\alpha)].$$

By lemma 5.3 there exist continuous functions $\gamma_0 : [a, a+\delta] \to \mathbb{R}$, $\gamma_1 : [b-\delta, b] \to \mathbb{R}$ such that $\gamma_0(a) = \alpha(a)$, $\gamma_0(a+\delta) = M = \gamma_1(b-\delta)$, $\gamma_1(b) = \alpha(b)$, $M \geq \gamma_0 \geq \alpha$ on $[a, a+\delta]$ and $M \geq \gamma_1 \geq \alpha$ on $[b-\delta, b]$. Define $\beta_* : [a,b] \to \mathbb{R}$ by $\beta_* = \gamma_0$ on $[a, a+\delta]$, $\beta_* \equiv M$ on $[a+\delta, b-\delta]$ and $\beta_* = \gamma_1$ on $[b-\delta, b]$. Obviously $\beta_* \in \mathcal{F}$. On the other hand:

$$\begin{aligned} \int_a^b \beta_* &= M(b-a-2\delta) + \int_a^{a+\delta} \gamma_0 + \int_{b-\delta}^b \gamma_1 \\ &\geq M(b-a-2\delta) + 2\min(\alpha)\delta \\ &\geq 0 \end{aligned}$$

and the proof is complete. ■

In the following, $\mathcal{J}$ denotes the set of all strictly increasing sequences $\tau : \mathbb{Z} \to \mathbb{R}$ such that

$$\tau_n := \tau(n) \to \pm\infty \quad \text{as} \quad n \to \pm\infty$$
$$0 < \inf\{\tau_{n+1} - \tau_n : n \in \mathbb{Z}\} \leq \sup\{\tau_{n+1} - \tau_n : n + \mathbb{Z}\} < +\infty.$$

Given a continuous function $\varphi : \mathbb{R} \to \mathbb{R}$, we write $\varphi \in \mathcal{A}_0$ (resp. $\varphi \in \mathcal{A}^0$) if φ is bounded above (resp. below) and if there exists $\tau \in \mathcal{J}$ such that

$$\int_{t_n}^{\tau_{n+1}} \varphi(s)ds \leq 0 \quad (\text{resp.} \int_{\tau_n}^{\tau_{n+1}} \varphi(s)ds \geq 0); \quad n \in \mathbb{Z}. \tag{5.5}$$

Lemma 5.6 *If $\varphi \in \mathcal{A}_0$, then there exists a bounded and continuously differentiable function $B : \mathbb{R} \to \mathbb{R}$ such that $\varphi(t) \leq B'(t)$ for all $t \in \mathbb{R}$.*

Proof. Let us write $M = \sup(\varphi)$. If $M = 0$, if suffices to take $B \equiv 0$. Thus, we shall assume that $M > 0$.

Let us fix $\tau \in \mathcal{J}$ satisfying (5.5) and let $n \in \mathbb{Z}$. By lemma 5.5, there exists a continuous function $\beta_n : [\tau_n, \tau_{n+1}] \to \mathbb{R}$ such that $\beta_n(\tau_n) = \varphi(\tau_n)$, $\beta_n(\tau_{n+1}) = \varphi(\tau_{n+1})$, $M \geq \beta_n \geq \varphi$ and

$$\int_{\tau_n}^{\tau_{n+1}} \beta_n(s)ds = 0. \tag{5.6}$$

In particular, we can define a continuous function $\beta : \mathbb{R} \to \mathbb{R}$ by

$$\beta(t) := \beta_n(t) \quad \text{if} \quad \tau_n \le t \le \tau_{n+1}.$$

Note that $\varphi \le \beta \le M$.

To end the proof, it suffices to show that the function $B(t) = \int_{\tau_0}^{t} \beta(s)ds$ is bounded. To this end, let us first remark that for each $t \in \mathbb{R}$, there exists and unique $n \in \mathbb{Z}$ such that $\tau_n \le t \le \tau_{n+1}$. In this case, we write $[t] = \tau_n$. Note that, by (5.4);

$$B(t) = \int_{[t]}^{t} \beta(s)ds. \tag{5.7}$$

Assume now that there exists a sequence $\{t_k\}$ in $\mathbb{R}$ such that $|B(t_k)| \to +\infty$. Then, $|t_k| \to +\infty$ since B is continuous. On the other hand, by (5.7), there exists $\sigma_k \in [[t_k], t_k]$ such that

$$B(t_k) = (t_k - [t_k])\beta(\sigma_k) \tag{5.8}$$

and hence $|\beta(\sigma_k)| \to +\infty$ as $k \to +\infty$, since $0 \le t_k - [t_k] \le \sup\{\tau_{n+1} - \tau_n : n \in \mathbb{Z}\} < +\infty$. But $\beta(t) \le M$ for all $t \in \mathbb{R}$, and so, $\beta(\sigma_k) \to -\infty$ as $k \to +\infty$. From this and (5.8),

$$B(t_k) \to -\infty \quad \text{as} \quad k \to +\infty \tag{5.9}$$

Given $t \in \mathbb{R}$ we write $(t) = \tau_{n+1}$, if $[t] = \tau_n$. Since $\int_{[t]}^{(t)} \beta(s)ds = 0$ and (5.7) holds, we have

$$B(t) = -\int_{t}^{(t)} \beta(s)ds$$

and hence $B(t_k) = -((t_k) - t_k)\beta(s_k)$ for some $s_k \in [t_k, (t_k)]$. By (5.9), $\beta(s_k) \to +\infty$ since $\{(t_k) - t_k\}$ is bounded. This contradicts the fact that $\beta \le M$ and the proof is complete. ■

6 Average concepts

In this section we show assumptions (2.2) and (2.5) can be seen as a sort of average restrictions.

Given a bounded continuous function $h : \mathbb{R} \to \mathbb{R}$, we define the upper (resp. lower) average of h, denoted $\overline{D}(h)$ (resp. $\underline{D}(h)$) as the infimum (resp. supremum) of all real numbers λ such that (see [2]),

$$h(t) \leq \lambda + B'(t) \ (resp.\ h(t) \geq \lambda + B'(t) \tag{6.1}$$

for some bounded continuously differentiable function $B : \mathbb{R} \to \mathbb{R}$.

Obviously, $\overline{D}(h) \leq \sup(h)$ (resp. $\inf(h) \leq D(h)$), since (6.1) is valid for $\lambda = \sup(h)$ (resp. $\lambda = \inf(h)$) and $B \equiv 0$.

Proposition 6.1 $\underline{D}(h) \leq \overline{D}(h)$.

Proof. Suppose that the result is false and fix $\epsilon > 0$ such that

$$\mu := \underline{D}(h) - \overline{D}(h) - 2\epsilon > 0.$$

Now, fix bounded continuously differentiable functions $A, B : \mathbb{R} \to \mathbb{R}$ such that

$$\underline{D}(h) - \epsilon + A'(t) \leq h(t) \leq B'(t) + \overline{D}(h) + \epsilon$$

then, $B'(t) \geq \mu + A'(t)$ and hence, B is not bounded. This contradiction ends the proof. ∎

We remark that condition (2.2) implies $\overline{D}(F(\cdot, R)) \leq 0$, and condition (2.5) implies $\underline{D}(F(\cdot, \delta)) \geq 0$. Using the inequalities (5.3), (5.4), (5.5), we define

$$\begin{aligned}
\overline{C}(h) &= \inf_{\tau \in \mathcal{J}} \sup_{n \in \mathbb{Z}} \frac{1}{\tau(n+1) - \tau(n)} \int_{\tau_n}^{\tau(n+1)} h(s)ds \\
\underline{C}(h) &= \sup_{\tau \in \mathcal{J}} \inf_{n \in \mathbb{Z}} \frac{1}{\tau(n+1) - \tau(n)} \int_{\tau_n}^{\tau(n+1)} h(s)ds
\end{aligned}$$

$$\begin{aligned}
\overline{B}(h) &= \inf_{T>0} \sup_{t \in \mathbb{R}} \frac{1}{T} \int_t^{t+T} h(s)ds \\
\underline{B}(h) &= \sup_{T>0} \inf_{t \in \mathbb{R}} \frac{1}{T} \int_t^{t+T} h(s)ds.
\end{aligned}$$

It is easy to show that

$$\overline{C}(h) \leq \overline{B}(h) \quad \text{and} \quad \underline{C}(h) \geq B(h).$$

Note also that by lemma 5.5,

$$\overline{D}(h) \leq \overline{C}(h) \quad \text{and} \quad \underline{D}(h) \geq \underline{C}(h).$$

From this,

$$\overline{A}(h) \leq B(h) \leq \underline{C}(h) \leq \underline{D}(h) \leq \overline{D}(h) \leq \overline{C}(h) \leq \overline{B}(h) \leq \overline{A}(h)$$

where $\underline{A}(h)$ and $\overline{A}(h)$ were defined by,

$$\begin{aligned} \underline{A}(h) &= \lim_{T \to +\infty} \inf_{t-s \geq T} \frac{1}{t-s} \int_s^t h(\sigma) d\sigma \\ \overline{A}(h) &= \lim_{T \to +\infty} \sup_{t-s \geq T} \frac{1}{t-s} \int_s^t h(\sigma) d\sigma. \end{aligned}$$

These definitions were used in [7] to establish the existence of coexistence states for non-autonomous competitions systems of Lotka-Volterra type and for to generalize in [8] to the periodic case a well case known work of Mottoni and Schiaffino [3] concerning periodic systems for species in competition.

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