A Note for Cyclic 3-Dimensional Competitive Systems.

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Abstract

In this paper we consider the class $\mathcal{C}$ of all $T$-periodic competitive and dissipative 3-dimensional systems which have a cyclic in the boundary and such that the origin is a source. We use the ideas in [1] in order to give a correct proof of theorem 1.2 of [1] where we study the coexistence state for these system.

Key Words: Competitive systems, coexistence states, dissipative systems.

1 Introduction

Let $\mathcal{C}$ be the class of all $T$-periodic competitive and dissipative systems which have a cyclic in the boundary and such that the origin is a source.

In a recent paper by Tineo [1], it was proved that the property of having a coexistence state is a generic property in $\mathcal{C}$ with respect to a suitable topology.
Unfortunately the proof in [1] contains a mistake, but fortunately the ideas in that paper are correct and we use these in order to give a correct proof of theorem 1.2 of [1]. Our proof is based in on improvement of theorem 1.1 in that paper.

To be more precise, let us consider the system

\[ x'_i = x_i F_i(t, x), \quad x = (x_1, x_2, x_3), \quad 1 \leq i \leq 3, \]  

\[ \text{(1.1)} \]

where \( F_1, F_2, F_3 : \mathbb{R} \times \mathbb{R}^3_+ \to \mathbb{R} \) are continuous functions which are T-periodic in \( t \) and locally Lipschitz continuous in \( x \). We shall assume that the following hypotheses hold:

\( H_1 \) System (1.1) is competitive. That is, \( F_i(t, x) \) is decreasing with respect to \( x_j \) for all \( i \neq j \).

\( H_2 \) System (1.1) is dissipative.

\( H_3 \) \( \int_0^T F_i(t, 0) dt > 0 \) for all \( i \). This condition implies that the trivial solution is a source.

We say that (1.1) is \( \tau \)-cyclic (resp. \( \sigma \)-cyclic) if the species \( x_i \) (resp. \( x_{i+1} \)) is carried to extinction by \( x_{i+1} \) (resp. \( x_i \)) in the subsystem obtained from (1.1) by letting \( x_{i-1} = 0 \), \( i \in \mathbb{Z} \). (Here and henceforth we shall use the mod 3 notation).

**Remark.** If \((H_1)\) holds then \((H_2)\) is equivalent to saying the system

\[ z' = z F_i(t, ze_i) \]

\[ \text{(1.2)} \]

is dissipative for \( i \leq i \leq 3 \). Here and henceforth, \((e_1, e_2, e_3)\) denotes the canonical vector basis of \( \mathbb{R}^3 \). Thus, if \((H_1)-(H_3)\) holds then (1.2) has a minimal positive T-periodic solution that shall denote by \( v_i \).
In [1] is was proved that if (1.1) is $\tau-$cyclic then,

$$I_i := \int_0^T F_{i+1}(t, v_i(t)e_i)dt \geq 0 \geq J_i := \int_0^T F_i(t, v_{i+1}(t)e_i)dt$$

Also in theorem 1.1 it was proved that if $(H_1)-(H_3)$ hold and (1.1) is $\tau-$cyclic, then this system has a coexistence state if $I_i > 0$ $\forall i$. We shall complete this result as follows.

**Theorem 1.1** If $J_i < 0$ $\forall i$, then (1.1) has a coexistence state.

Using this result we prove our main result.

**Theorem 1.2** Let $F$ satisfying $(H_1)-(H_3)$. If $F$ is $\tau-$cyclic then the system

$$x_i = F^\epsilon_i(t, x),$$

(1.3)

is $\tau-$cyclic and has a coexistence state for any $\epsilon > 0$, where

$$F^\epsilon_i(t, x) := F_i(t, x) + \epsilon[F_i(t, x_{i+1}e_{i+1}) - F_i(t, 0)] \quad \epsilon \in (0,1).$$

(1.4)

In [1], the author define,

$$F^\epsilon_i(t, x) := F_i(t, x) + \epsilon[F_i(t, 0) - F_i(t, x_{i-1}e_{i-1})] \quad \epsilon \in (0,1).$$

but, in this case, system (1.3) is not, in general, competitive.

2 The Proofs

*Proof of Theorem 1.1* Let $\Delta$ be 2-cell given by theorem 2.1 in [1] and let $\pi : \Delta \rightarrow \Delta$ the Poincare map of (1.1). We know that $\pi$ is an orientation-preserving homeomorphism onto $\Delta$. The proof of our result follows as in [1], applying the main result in [2] to $\pi^{-1}$. 

\[\blacksquare\]
Proof of Theorem 1.2 Obviously, (1.3) is competitive for all $\epsilon > 0$. On the other hand, if $z_{i+1} = 0$, we have

$$F^{\epsilon}_{i+1}(t, x) = F_{i+1}(t, x) + \epsilon[F_{i+1}(t, x_{i+2}e_{i+2}) - F_{i+1}(t, 0)] \leq F_{i+1}(t, x),$$

$$F^{\epsilon}_{i+2}(t, x) = F_{i+2}(t, x) + \epsilon[F_{i+2}(t, x_{i+3}e_{i+3}) - F_{i+2}(t, 0)] = F_{i+2}(t, x) + \epsilon[F_{i+2}(t, x_{i}e_{i}) - F_{i+2}(t, 0)] = F_{i+2}(t, x) \geq F_{i+2}(t, x),$$

and by proposition 4.1 of [1], (1.3) is $\tau$-cyclic.

Note now that, the minimal positive $T$-periodic solution of the logistic equation $z' = zF^\epsilon(t, ze_i)$ is also $v_i$, since $F^\epsilon(t, ze_i) \equiv F_i(t, ze_i)$. Finally

$$\int_0^T F^\epsilon_i(t, v_{i+1}(t)e_{i+1})dt = (1 + \epsilon)J_i - \epsilon \int_0^T F_i(t, 0)dt \leq -\epsilon \int_0^T F_i(t, 0)dt < 0,$$

and the proof follows from Theorem 1.1.

Remark. Let $F^\epsilon(t, x)$ be defined by (1.4). It is easy to show that $F^\epsilon(t, x) \to F(t, x)$ as $\epsilon \to 0^+$ uniformly on $\mathbb{R} \times K$ for any compact subset $K$ of $\mathbb{R}$. We shall construct a family $\{F^\epsilon, \epsilon > 0\}$, satisfying the conclusions of Theorem 1.2, such that $F^\epsilon(t, x) \to F(t, x)$ as $\epsilon \to 0^+$ uniformly on $\mathbb{R} \times \mathbb{R}_3^+$. To this end let us define

$$\varphi_i(t, w) = \begin{cases} v_i(t) - \frac{1}{v_i(t)}[w - v_i(t)]^2 & \text{if } 0 \leq w \leq v_i(t) \\ v_i(t) & \text{if } w \geq v_i(t) \end{cases}$$

and

$$F^\epsilon(t, x) = F_i(t, x) + \epsilon[F_i(t, \varphi_{i+1}(t, x_{i+1})e_{i+1}) - F_i(t, 0)].$$

Using the arguments in Theorem 1.2, if is easy to show that this family $\{F^\epsilon\}$ has the required properties.
References
