Available online at www.sciencedirect.com
ScienceDirect
Topology and its Applications 154 (2007) 1481–1488
Topology and its Applications
www.elsevier.com/locate/topol

On cellularization for simplicial presheaves and motivic homotopy

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Received 8 March 2006; received in revised form 5 April 2006; accepted 5 April 2006

Abstract

We construct cellular homotopy theories for categories of simplicial presheaves on small Grothendieck sites and discuss applications to the motivic homotopy category of Morel and Voevodsky.

MSC: primary 55P60, 18F20; secondary 14F42

Keywords: Cellularization and localization; Simplicial presheaves; Motivic homotopy theory; Cellular objects

0. Introduction

Localization and cellularization techniques play a fundamental role in modern homotopy theory. For many applications it is often useful to approximate a given topological space by simpler ones. An important example is the cellularization functor, which associates to each pointed topological space a CW-complex in the sense of Whitehead built through the process of attaching cells. CW-complexes are the main building blocks of classical homotopy theory. More generally one would like to have a similar approximation of a space built out of copies of any given fixed space A. This general concept of cellularization was developed systematically for the category of topological spaces and simplicial sets by Dror Farjoun [3] building upon the general foundational work on homotopy localization of Bousfield. These localization and cellularization techniques were then extended under some technical conditions to arbitrary model categories by Nofech [15] and Hirschhorn [8].

In this note we will discuss cellularization techniques for model categories of simplicial presheaves on small Grothendieck sites and its applications to the motivic homotopy category of Morel and Voevodsky. More precisely, we show that the construction of A-cellular homotopy theories in the sense of Nofech [15] applies verbatim to the category of pointed simplicial presheaves on a small Grothendieck site, where A is a fixed given pointed simplicial presheaf. The machinery of Nofech especially allows to get functorial factorizations in the model categories involved. General localization techniques for simplicial presheaves on arbitrary small Grothendieck sites were developed systematically by Goerss and Jardine [5]. Localization techniques for presheaves play a crucial role in the construction of the motivic homotopy category by Morel and Voevodsky [14].

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0166-8641/$ – see front matter
doi:10.1016/j.topol.2006.04.027

Applying the framework of cellularization for simplicial presheaves on the Nisnevich site of smooth schemes of finite type over a noetherian base gives the construction of an unstable A-cellular motivic homotopy theory.

A systematic and concrete framework for stable cellularization in stable motivic homotopy theory was recently also developed by Dugger and Isaksen [4]. There the authors study explicitely the closed classes of stably cellular objects in the stable motivic homotopy category of Morel and Voevodsky.

The main aim of this paper is to introduce the formal homotopical framework for unstable cellularization of simplicial presheaves on an arbitrary small Grothendieck site along the lines of its corresponding setting in classical unstable homotopy theory and to apply it to obtain cellularization for unstable motivic homotopy theory. In future work we aim to study the fundamental properties of unstable cellularization for simplicial presheaves and give concrete applications of the techniques developed here in order to investigate cellularization functors in presheaf categories with respect to specific objects, like simplicial spheres and Eilenberg–Mac Lane objects.

1. Cellular homotopy theories for simplicial presheaves

In this section we construct cellular homotopy theories for categories of simplicial presheaves on a small Grothendieck site. A cellular homotopy theory in general is given as a closed model structure derived from a pointed simplicial model category with respect to a fixed cofibrant object. The general approach towards cellularization was developed by Dror Farjoun [3] in the context of pointed simplicial sets and pointed topological spaces. The framework in the context of general pointed closed model categories was developed systematically by Hirschhorn [8]. An independent account along similar lines was also given by Nofech [15]. We will follow this latter framework in order to construct cellular homotopy theories for simplicial presheaves.

Following Nofech [15], Definition 1.0 we will define for a given pointed simplicial closed model category $\mathcal{C}$ and a fixed cofibrant object A of $\mathcal{C}$ an A-cellular closed model category structure on $\mathcal{C}$.

For the basic notions of model categories we refer to the books of Quillen [16], Hovey [7] and Hirschhorn [8].

In what follows $\mathbf{Hom}(X, Y)$ always denotes the simplicial function complex in $\mathcal{C}$, $\mathbf{S}$ the category of simplicial sets and the subscript f will always denote fibrant approximation, e.g. if $\phi : X \to Y$ is a morphism in the category $\mathcal{C}$, then $\phi_f : X_f \to Y_f$ is the induced morphism between the fibrant approximations X_f and Y_f of the objects X and Y.

Definition 1.1. Let $\mathcal{C}$ be a pointed simplicial closed model category and A be a cofibrant object of $\mathcal{C}$ and let $\mathbf{W_S}$ denote the class of weak equivalences of simplicial sets, and $\mathbf{F}_{\mathcal{C}}$ denote the class of fibrations in $\mathcal{C}$. An A-cellular closed model category structure $\mathcal{C}^A$ on the underlying category of $\mathcal{C}$ is a closed model category structure where the classes $(\mathbf{W}_{\mathcal{C}^A}, \mathbf{F}_{\mathcal{C}^A}, \mathbf{C}_{\mathcal{C}^A})$ of weak equivalences, fibrations and cofibrations of $\mathcal{C}^A$ are defined as follows:

$$\mathbf{W}_{\mathcal{C}^A} = \{\phi\colon \mathbf{Hom}(A, \phi_f) \in \mathbf{W_S}\},$$
$$\mathbf{F}_{\mathcal{C}^A} = \mathbf{F}_{\mathcal{C}},$$
$$\mathbf{C}_{\mathcal{C}^A} = \{j\colon\ j \textit{ has Left Lifting Property (LLP) with respect to } (\mathbf{W}_{\mathcal{C}^A} \cap \mathbf{F}_{\mathcal{C}^A})\}.$$

The cofibrations and weak equivalences of $\mathcal{C}^A$ are also called respectively A-cofibrations and A-equivalences. Any weak equivalence in $\mathcal{C}$ is an A- equivalence for any cofibrant object A of $\mathcal{C}$.

In the presence of a set of generators of trivial cofibrations in $\mathcal{C}$, i.e. a set of trivial cofibrations $\{t_j\}$ such that a morphism ϕ is a fibration if and only if any of the t_j has the Left Lifting Property (LLP) with respect to ϕ with s-definite domains and codomains (see [1, 4.2]), Nofech proved a rather general theorem for the existence of A-cellular closed model category structures [15, Theorem 2.1].

Theorem 1.2 *(Nofech). Let $\mathcal{C}$ be a pointed proper simplicial closed model category with arbitrary colimits having a set $\{t_j\}$ of generators of trivial cofibrations and let A be a cofibrant, s-definite object of $\mathcal{C}$. Then there exists an A-cellular closed model category structure $\mathcal{C}^A$ admitting functorial factorizations.*

Having an A-cellular closed model category structure allows to factor any morphism $f : X \to Y$ in $\mathcal{C}$ into an A-cofibration followed by a fibration which is simultaneously an A-equivalence

$$X \xrightarrow{\mathbf{C}_{\mathcal{C}^A}} Z \xrightarrow{\mathbf{W}_{\mathcal{C}^A} \cap \mathbf{F}_{\mathcal{C}}} Y.$$

Factoring morphisms functorially in this way allows to construct A-cellular approximations or A-colocalizations of an object X. There exists cellularization functors CW_A and nullification functors P_A together with morphisms $CW_A X \to X$ being terminal up to homotopy from A-cellular objects into X and $X \to P_A X$ being initial up to homotopy from X into A-acyclic objects.

If the map to be factorized is $* \to X$ this gives $CW_A X$, the A-cellular approximation or A-colocalization of X

$$* \to CW_A X \to X.$$

If the map to be factorized is $X \to *$ this gives $P_A X$, the A-localization of X

$$X \to P_A X \to *.$$

We will now apply the construction of A-cellular closed model categories to the category of simplicial presheaves on a Grothendieck site. For the general notations and definitions of Grothendieck topologies and simplicial presheaves we refer the reader to [6,9].

Let **C** be a small Grothendieck site and **S**Pre(**C**) be the category of simplicial presheaves on **C**. The objects of **S**Pre(**C**) are the contravariant functors $X : \mathbf{C}^{op} \to \mathbf{S}$ from **C** to the category of simplicial sets **S** and the morphisms are given as the natural transformations between these functors. It is sometimes also useful to view a simplicial presheaf **C** simply as a simplicial object in the category of presheaves of sets on **C**.

The homotopy theory of simplicial presheaves on a Grothendieck site **C** is determined by the given topology on the site **C** (see [9]).

Let X be a simplicial presheaf on **C**. The *nth homotopy sheaf* $\tilde{\pi}_n(X)$ is the sheaf associated to the nth homotopy presheaf $\pi_n(X)$ defined as:

$$(x_0 : U \to X_0) \mapsto \pi_n\big(X(U), x_0\big).$$

Definition 1.3. A morphism $f : X \to Y$ of simplicial presheaves on **C** is a local weak equivalence if the following holds:

(1) the induced map of sheaves $f_* : \tilde{\pi}_0(X) \to \tilde{\pi}_0(Y)$ is an isomorphism of sheaves,
(2) the diagram of morphisms of sheaves

$$\begin{tikzcd} \tilde{\pi}_n(X) \arrow[r, "f_*"] \arrow[d] & \tilde{\pi}_n(Y) \arrow[d] \\ \tilde{X}_0 \arrow[r, "f_*"] & \tilde{Y}_0 \end{tikzcd}$$

is Cartesian, where the bottom row is given by the induced morphism on the sheaves associated to the presheaves of vertices of X and Y.

Local weak equivalences $f : X \to Y$ between simplicial presheaves can also be characterized locally, if the topos Shv(**C**) has enough points, by the property that all induced maps $f_* : X_x \to Y_y$ of stalks are weak equivalences in the category **S** of simplicial sets, i.e. are isomorphisms of homotopy groups of ordinary simplicial sets in all stalks. Jardine [10] gave another elegant definition using the framework of Boolean localization.

Definition 1.4. A morphism $f : X \to Y$ of simplicial presheaves is a *cofibration* if it is a monomorphism in **S**Pre(**C**) or equivalently if all induced functions between sets $f : X_n(U) \to Y_n(U)$ for all $n \geqslant 0$ and all objects U of **C** are one-to-one.

Definition 1.5. A morphism $f : X \to Y$ of simplicial presheaves is a *global fibration* if it has the Right Lifting Property (RLP) with respect to all morphisms of simplicial presheaves which are simultaneously cofibrations and local weak equivalences.

The homotopy theory of simplicial presheaves is now described by the following general theorem (see [9] and [10, Theorem 24]).

Theorem 1.6 *(Jardine). Let* $\mathbf{C}$ *be a small Grothendieck site. Then with the class of local weak equivalences, cofibrations and global fibrations the category* $\mathbf{S}\mathrm{Pre}(\mathbf{C})$ *of simplicial presheaves on* $\mathbf{C}$ *has a proper simplicial closed model category structure.*

An analogous theorem holds for the category $\mathrm{Shv}(\mathbf{C})$ of simplicial sheaves on the cite $\mathbf{C}$, essentially due to Joyal [12]. It follows from the theorem for presheaves in observing that the sheafification morphism $\eta : X \to \tilde{X}$ is a local weak equivalence (see again [10] for an elegant proof using Boolean localization). It turns out that the associated sheaf functor induces an equivalence

$$\mathbf{Ho}\big(\mathbf{S}\mathrm{Pre}(\mathbf{C})\big) \simeq \mathbf{Ho}\big(\mathbf{S}\mathrm{Shv}(\mathbf{C})\big)$$

on the homotopy categories associated to the closed model structures (see [10, Theorem 27]).

For the discussion of cellularization it is essential that we work in a pointed closed model category. For a small Grothendieck site $\mathbf{C}$ we denote by $\mathbf{S}\mathrm{Pre}(\mathbf{C})_*$ the category of pointed simplicial presheaves on $\mathbf{C}$ whose objects are simply pairs (X, x) consisting of a usual simplicial presheaf on $\mathbf{C}$ together with a morphism of presheaves $x : * \to X$, where $*$ is the constant presheaf and whose morphisms are usual morphisms of simplicial presheaves compatible with the basepoints.

The forgetful functor

$$F : \mathbf{S}\mathrm{Pre}(\mathbf{C})_* \to \mathbf{S}\mathrm{Pre}(\mathbf{C})$$

has a left adjoint, the functor of adding basepoints

$$+ : \mathbf{S}\mathrm{Pre}(\mathbf{C}) \to \mathbf{S}\mathrm{Pre}(\mathbf{C})_*$$

defined on objects by $+ : X \mapsto X_+$, where X_+ is the pointed simplicial presheaf $X \amalg *$ pointed by the canonical embedding $* \to X \amalg *$.

We call a morphism of pointed simplicial presheaves a local weak equivalence, cofibration or global fibration if it belongs to the corresponding class as a morphism of simplicial presheaves without basepoints. From the theorem of Jardine it follows therefore, that the category of pointed simplicial presheaves $\mathbf{S}\mathrm{Pre}(\mathbf{C})_*$ has the structure of a pointed proper simplicial closed model category. The forgetful functor F and its left adjoint $+$ both preserve weak equivalences and therefore induce a pair of adjoint functors on the homotopy categories $\mathbf{Ho}(\mathbf{S}\mathrm{Pre}(\mathbf{C}))$ and $\mathbf{Ho}(\mathbf{S}\mathrm{Pre}(\mathbf{C}))_*$. Analogous statements also hold for the category of pointed simplicial sheaves $\mathbf{S}\mathrm{Shv}(\mathbf{C})_*$ on $\mathbf{C}$ (see also [14, 2.2.5]).

Let K be any simplicial set and U an object of the site $\mathbf{C}$, then we define the simplicial presheaf $L_U K$ on objects V of $\mathbf{C}$ by

$$L_U K(V) = \coprod_{\phi : V \to U} K.$$

Morphisms of simplicial presheaves $L_U K \to X$ are therefore in one-to-one correspondence with maps of simplicial sets $K \to X(U)$.

Let us also recall the half-smash product. If K is a pointed simplicial set and X a pointed simplicial presheaf, the half-smash product $X \rtimes K$ is the simplicial presheaf defined on objects U of the site $\mathbf{C}$ by

$$(X \rtimes K)(U) = X(U) \rtimes K = \big(X(U) \times K\big)/(* \times K).$$

Definition 1.7. Let $f : A \to B$ be a cofibration in $\mathbf{S}\mathrm{Pre}(\mathbf{C})$. A simplicial presheaf Z on $\mathbf{C}$ is *f-local* if Z is globally fibrant and if the map $Z \to *$ has the Right Lifting Property (RLP) with respect to all presheaf cofibrations

$$(B \times Y) \cup_{(A \times Y)} \big(A \times L_U \Delta[n]\big) \xrightarrow{(f,j)_*} B \times L_U \Delta[n]$$

arising jointly from $f : A \to B$ and the inclusions $j : Y \hookrightarrow L_U \Delta[n]$ for any object U of $\mathcal{C}$.

A morphism $g : X \to Y$ is an f-weak equivalence if the induced map of simplicial sets

$$g^* : \mathbf{Hom}(Y, Z) \to \mathbf{Hom}(X, Z)$$

is a weak equivalence for all f-local objects Z in $\mathbf{S}\mathrm{Pre}(\mathbf{C})$. A morphism $g : X \to Y$ is an *f-fibration* if it has the Right Lifting Property (RLP) with respect to all cofibrations in $\mathbf{S}\mathrm{Pre}(\mathbf{C})$ which are simultaneously also f-weak equivalences.

One of the main theorems of Goerss and Jardine [5, Theorem 4.8] now says that localizing with respect to a cofibration f of simplicial presheaves gives again a simplicial closed model structure on $\mathbf{S}\mathrm{Pre}(\mathbf{C})$.

Theorem 1.8 *(Goerss–Jardine). Let $\mathbf{C}$ be a small Grothendieck site. Then with the class of f-weak equivalences, cofibrations and f-fibrations the category $\mathbf{S}\mathrm{Pre}(\mathbf{C})$ of simplicial presheaves on $\mathbf{C}$ has a simplicial closed model category structure.*

Analogous results hold again also for the category $\mathbf{S}\mathrm{Shv}(\mathbf{C})$ of simplicial sheaves on the site $\mathbf{C}$.

In the special case of a rational point $f : * \to I$ of a simplicial presheaf on a small Grothendieck site $\mathbf{C}$, i.e. when formally collapsing a simplicial presheaf I to a point, it turns out that the f-local simplicial closed model category structure on $\mathbf{S}\mathrm{Pre}(\mathbf{C})$ is actually proper (see [11, Appendix A]). This does not depend on the object I being an interval for the site $\mathbf{C}$ as defined by Morel and Voevodsky, where a similar result is proven for the category of simplicial sheaves on a Grothendieck site with interval (see [14, Theorem 2.3.2]). This general result in turn can be used to construct the motivic homotopy category.

From the general discussion above we now get immediately the following existence theorem for A-cellular homotopy theories on the categories of pointed simplicial presheaves and sheaves on a small Grothendieck site.

Theorem 1.9. *Let $\mathbf{S}\mathrm{Pre}(\mathbf{C})_*$ be the category of pointed simplicial presheaves on a small Grothendieck site $\mathbf{C}$ and let A be a fixed object of $\mathbf{S}\mathrm{Pre}(\mathbf{C})_*$. Then there exists an A-cellular closed model category structure $(\mathbf{S}\mathrm{Pre}(\mathbf{C})_*)^A$ admitting functorial factorizations.*

Proof. Every object A of $\mathbf{C}$ is cofibrant and also s-definite. Actually every object A of $\mathbf{C}$ is in fact accessible (see [6, I, Définition 9.3]), which is a stronger notion than s-definiteness (see [14, 2.2.3]). Therefore the existence of an A-cellular closed model category structure $(\mathbf{S}\mathrm{Pre}(\mathbf{C})_*)^A$ follows directly from Nofech's existence theorem (Theorem 1.2 above) and Jardine's theorem (Theorem 1.6 above). For this we observe that generators of trivial cofibrations are given as the morphisms

$$L_U i[n,k] : L_U V[n,k] \to L_U \Delta[n]$$

and the factorization necessary for proving the closed model category axiom CM5I is constructed using these generators and the morphisms of the form

$$A \rtimes i[n] : A \rtimes \partial\Delta[n] \to A \rtimes \Delta[n]$$

and to obtain the factorization for proving the axiom CM5II we use again the generators of trivial cofibrations as usual [16]. □

An analogous statement also holds again for the category $\mathrm{Shv}(\mathbf{C})_*$ of pointed simplicial sheaves on an arbitrary small Grothendieck site $\mathbf{C}$.

Example 1.10. Cellularization with respect to constant simplicial presheaves.

Let A be a constant simplicial presheaf on a small Grothendieck site $\mathbf{C}$, i.e. the value of the functor A is just a constant simplicial set A. This gives basically a presheaf version of classical cellularization with respect to a given simplicial set A as studied by Dror Farjoun [3].

Example 1.11. Cellularization with respect to abelian Eilenberg–Mac Lane objects.

Let A be a presheaf of abelian groups on a small Grothendieck site $\mathbf{C}$. Following Morel and Voevodsky [14, p. 55] we can define the abelian Eilenberg–Mac Lane object $K(A,n)$ in $\mathbf{S}\mathrm{Pre}(\mathbf{C})_*$ associated to A as

$$K(A,n) = \Gamma\big(A[n]\big)$$

where $A[n]$ is the chain complex of presheaves with the only one nontrivial term A in degree n and $\Gamma(-)$ is the right adjoint functor of the normalized chain functor $N(-)$, which associates to every presheaf of simplicial abelian groups on $\mathbf{C}$ the normalized chain complex (see [14, Proposition 1.25], [13, p. 93]).

In future work we aim to study cellularizations and localizations with respect to Eilenberg–Mac Lane objects and more generally with respect to GEMs, i.e. infinite weak products of these Eilenberg–Mac Lane objects in analogy to the classical theory developed by Dror Farjoun [3]. We also like to extend results on localizations of abelian Eilenberg–Mac Lane objects obtained in the framework of classical homotopy theory by the second author in joint work with Casacuberta and Tai [2] to the case of simplicial presheaf. Localization functors with respect to Eilenberg–Mac Lane objects for presheaf categories were also studied in relation with homology localizations by Goerss and Jardine [5].

Following Dror Farjoun [3], 2D we can also discuss closed classes of pointed simplicial presheaves or sheaves on an arbitrary small Grothendieck site. These are certain full subcategories of $\mathbf{S}\mathrm{Pre}(\mathbf{C})_*$ or $\mathrm{Shv}(\mathbf{C})_*$ respectively which are closed under weak equivalence and arbitrary pointed homotopy colimits. Especially important in the context of cellularization here is the closed class of A-cellular objects for a given pointed simplicial presheaf or sheaf A. A-cellular objects are precisely those which can be built out of A up to homotopy in analogous fashion as CW-complexes are built out of spheres.

Definition 1.12. A full subcategory $\mathcal{C}$ of the category $\mathbf{S}\mathrm{Pre}(\mathbf{C})_*$ of pointed simplicial presheaves on an arbitrary small Grothendieck site $\mathbf{C}$ is called a closed class if it is closed under weak equivalences and arbitrary pointed homotopy colimits, i.e. if for any diagram $D : I \to \mathcal{C}$ the object $\mathrm{hocolim}_I$ is also in $\mathcal{C}$.

Example 1.13. Closed classes of cellular objects.

Let $\mathcal{A}$ be a set of objects in $\mathbf{S}\mathrm{Pre}(\mathbf{C})_*$. The class of *$\mathcal{A}$-cellular objects* is the smallest closed class that contains the given set $\mathcal{A}$. Especially important is the case of cellularization with respect to a single object A in $\mathcal{A}$. This class is called the class of A-cellular objects. It is built through transfinite induction by starting with the full subcategory containing the single simplicial presheaf A and closing it inductively under arbitrary pointed homotopy colimits and weak equivalences. The closed class of cellular objects and their basic properties in general model categories are discussed in detail by Dugger and Isaksen [4, 2].

2. Applications to motivic homotopy theory

In this section we apply the preceding results to construct a motivic cellular homotopy theory associated to the motivic homotopy category of Morel and Voevodsky.

Let $(Sm|_S)_{Nis}$ be the Nisnevich site of smooth schemes of finite type over a noetherian scheme S (see [14] for all the relevant definitions and notations) and $\mathbf{S}\mathrm{Pre}((Sm|_S)_{Nis})$ be its category of simplicial presheaves. The Morel–Voevodsky motivic homotopy theory arises by formally inverting a rational point $f : * \to \mathbb{A}^1$ of the affine line $\mathbb{A}^1$, which plays the role of an interval for the Nisnevich site $(Sm|_S)_{Nis}$.

Definition 2.1. A simplicial presheaf Z on $(Sm|_S)_{Nis}$ is motivic fibrant if Z is globally fibrant and if the map $Z \to *$ has the Right Lifting Property (RLP) with respect to all presheaf cofibrations

$$\left(\mathbb{A}^1 \times A\right) \cup_A B \xrightarrow{(f,j)} \mathbb{A}^1 \times B$$

arising jointly from $f : * \to \mathbb{A}^1$ and the cofibrations $j : A \to B$.

A morphism $g : X \to Y$ is a *motivic weak equivalence* if the induced map of simplicial sets

$$g^* : \mathbf{Hom}(Y, Z) \to \mathbf{Hom}(X, Z)$$

is a weak equivalence for all motivic fibrant objects Z in $\mathbf{S}\mathrm{Pre}((Sm|_S)_{Nis})$.

A morphism $g : X \to Y$ is a motivic fibration if it has the Right Lifting Property (RLP) with respect to all cofibrations in $\mathbf{S}\mathrm{Pre}((Sm|_S)_{Nis})$ which are simultaneously also motivic weak equivalences.

As in the case of formally collapsing a simplicial presheaf to a point on an arbitrary Grothendieck site we have the following main theorem, which gives the motivic homotopy theory of Morel and Voevodsky (see [14], [11], Theorem 1.1 and also [5, Remark 4.11]).

Theorem 2.2 *(Morel–Voevodsky, Jardine). Let* $\mathbf{S}\mathrm{Pre}((Sm|_S)_{Nis})$ *be the category of simplicial presheaves on the Nisnevich site* $(Sm|_S)_{Nis}$*. Then with the class of motivic weak equivalences, cofibrations and motivic fibrations the category* $\mathbf{S}\mathrm{Pre}((Sm|_S)_{Nis})$ *has a proper simplicial closed model category structure.*

Again a similar theorem holds for the category $\mathbf{S}\mathrm{Shv}((Sm|_S)_{Nis})$ of simplicial sheaves on the Nisnevich site $(Sm|_S)_{Nis}$ and the forgetful functor and sheafification functor give a pair of adjoint functors inducing an equivalence of the motivic homotopy categories

$$\mathbf{Ho}\big(\mathbf{S}\mathrm{Pre}\big((Sm|_S)_{Nis}\big)\big) \simeq \mathbf{Ho}\big(\mathbf{S}\mathrm{Shv}\big((Sm|_S)_{Nis}\big)\big)$$

(see [11, Theorem 1.2]).

As in the general case we get now the following existence theorem for a motivic cellular homotopy theory.

Theorem 2.3. *Let* $\mathbf{S}\mathrm{Pre}((Sm|_S)_{Nis})_*$ *denote the category of pointed simplicial presheaves on the Nisnevich site* $(Sm|_S)_{Nis}$ *and let* A *be an object of* $\mathbf{S}\mathrm{Pre}((Sm|_S)_{Nis})_*$*. Then there exists an* A*-cellular closed model category structure* $(\mathbf{S}\mathrm{Pre}((Sm|_S)_{Nis})_*)^A$ *admitting functorial factorizations.*

Proof. Again every object A of $\mathbf{S}\mathrm{Pre}((Sm|_S)_{Nis})_*$ is cofibrant, and s-definite as above. Therefore the existence of an A-cellular closed model category structure $(\mathbf{S}\mathrm{Pre}((Sm|_S)_{Nis})_*)^A$ follows directly from Nofech's existence theorem (Theorem 1.2 above) and the theorems of Morel–Voevodsky and Jardine (Theorems 2.2 and 1.6 above) on the existence of a proper simplicial model category structure on $\mathbf{S}\mathrm{Pre}((Sm|_S)_{Nis})$ using again similar morphisms as in the proof of Theorem 1.9 to establish the necessary factorizations in the model category axioms CM5I and CM5II. □

An analogous statement also holds again for the category $\mathbf{S}\mathrm{Shv}((Sm|_S)_{Nis})_*$ of pointed simplicial sheaves on the Nisnevich site $(Sm|_S)_{Nis}$.

Example 2.4. Cellularization using motivic spheres.

Following Morel and Voevodsky [14], let $S^{1,0}$ be the constant simplicial presheaf of $\mathbf{S}\mathrm{Pre}((Sm|_S)_{Nis})_*$, whose sections are given by the simplicial circle $\Delta^1/\partial\Delta^1$ and $S^{1,1}$ be $\mathbb{A}^1 - \{0\}$ with basepoint given by 1. For $p \geqslant q \geqslant 0$ the motivic spheres are defined as in [14]

$$S^{p,q} = \big(S^{1,0} \wedge \cdots \wedge S^{1,0}\big) \wedge \big(S^{1,1} \wedge \cdots \wedge S^{1,1}\big)$$

with $p - q$ copies of $S^{1,0}$ and q copies of $S^{1,1}$.

In [4] Dugger–Isaksen call an object of $\mathbf{S}\mathrm{Pre}((Sm|_S)_{Nis})_*$ unstably cellular, if it is $\mathcal{A}$-cellular, where $\mathcal{A}$ is the set of all motivic spheres $S^{p,q}$ for $p \geqslant q \geqslant 0$. Using the fundamental properties of the closed class of $\mathcal{A}$-cellular objects, they are able to show that $\mathbb{A}^n - \{0\}$ and the projective spaces $\mathbb{P}^n$ are unstably cellular. In general, for a scheme to be unstably cellular is very restrictive and means that it is close of being a linear variety, which is basically built inductively out of affine spaces $\mathbb{A}^n$ [17].

Example 2.5. Cellularization using motivic Eilenberg–Mac Lane spaces.

In [18, 6] Voevodsky introduced motivic Eilenberg–Mac Lane spaces, representing motivic cohomology. Following the definition of [18, 6.1] we set

$$K\big(\mathbb{Z}(n), 2n\big) = L\big(\big(\mathbb{P}^1, \infty\big)^n\big)$$

where $L(-)$ is the algebro-geometric analogue of the infinite symmetric product functor of algebraic topology and the definition is motivated by the classical Dold–Thom correspondence. These spaces assemble the motivic Eilenberg–Mac Lane spectrum, which represents motivic cohomology.

It is an interesting problem to investigate cellularization and the behaviour of the cellularization functor with respect to these specific motivic Eilenberg–Mac Lane objects and we like to study the associated cellular homotopy theory further.

Acknowledgements

The first author likes to thank the Universidad de Almería for financial support. He also thanks Daniel Juan-Pineda and the organizing committee of the JAMEX III conference in Oaxaca, Mexico for the possibility of contributing to this wonderful event. The second author was supported by the Spanish Ministry of Education and Science under MEC-FEDER Grant MTM2004-03629. Both authors like to thank C. Casacuberta for his interest and support. We are also grateful for many useful suggestions of improvements by the referee.

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