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Realizing Operadic Plus-constructions as Nullifications*

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Abstract. In this paper we generalize the plus-construction given by M. Livernet for algebras over rational differential graded operads to the framework of cofibrant operads over an arbitrary ring (the category of algebras over such operads admits a closed model category structure). We follow the modern approach of J. Berrick and C. Casacuberta defining topological plus-construction as a nullification with respect to a universal acyclic space. We construct a universal H_*^Q-acyclic algebra $\mathcal{U}$ and we define $A \longrightarrow A^+$ as the $\mathcal{U}$-nullification of the algebra A. This map induces an isomorphism in Quillen homology and quotients out the maximal perfect ideal of $\pi_0(A)$. As an application, we consider for any associative algebra R the plus-constructions of $gl(R)$ in the categories of homotopy Lie and homotopy Leibniz algebras. This gives rise to two new homology theories for associative algebras, namely homotopy cyclic and homotopy Hochschild homologies. Over the rationals these theories coincide with the classical cyclic and Hochschild homologies.

Key words: cyclic homology, Hochschild homology, homotopical localization, operads, plus construction

1. Introduction

Quillen's plus construction for spaces was designed so as to yield a definition of higher algebraic K-theory groups of rings. Indeed, for any $i \geq 1$, $K_i R = \pi_i BGL(R)^+$, where $GL(R)$ is the infinite general linear group on the ring R. The study of the additive analogue, namely the Lie or Leibniz algebra $gl(R)$ has already produced a number of papers showing the strong link with cyclic and Hochschild homology (for classical background on these theories we refer to [22] and to the survey [23]). However there have always been restrictions, such as working over the rationals.

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For example M. Livernet has given a plus-construction for algebras over an operad in the rational context [19] by way of cellular techniques imitating the original topological construction given by D. Quillen in [27] (see also [26] for a plus-construction in the context of simplicial algebras). Specializing to the category of Lie, respectively Leibniz algebras, she proved then that the homotopy groups of $gl(R)^+$ are isomorphic to the cyclic, respectively Hochschild homology groups of R (this makes use of deep results of Kassel–Loday in [17] and Cuvier in [7]).

In the category of topological spaces plus-construction can be viewed as a localization functor, which has the main advantage to be functorial. This idea goes back to A.K. Bousfield and E. Dror Farjoun, but the work of J. Berrick and C. Casacuberta in [4] provides a very concrete model, i.e. a "small" universal acyclic space $B\mathcal{F}$ such that the nullification $P_{B\mathcal{F}}X$ is the plus-construction X^+.

Recently, thanks to the work of P. Hirschhorn [15] it appears possible to do homotopical localization in a very general framework. In fact, one can construct localizations in any closed model category satisfying some mild extra conditions (left proper and cofibrantly generated), such as categories of algebras over cofibrant operads. The category of Lie algebras over an arbitrary ring is not good enough for example. One needs to take first a cofibrant replacement $\mathcal{L}_\infty$ of the Lie operad and perform localization in the category of $\mathcal{L}_\infty$-algebras, which we call homotopy Lie algebras.

This allows to define a functorial plus-construction in the category of algebras over a cofibrant operad as a certain nullification functor with respect to an algebraic analogue $\mathcal{U}$ of Berrick and Casacuberta's acyclic space. This extends the results of M. Livernet to the non-rational case. In the following theorem $\mathcal{P}\pi_0(A)$ denotes the maximal $\pi_0\mathcal{O}$-perfect ideal of $\pi_0 A$.

THEOREM 1.1. *Let $\mathcal{O}$ be a rational or confibrant operad. Then the homotopical nullification with respect to $\mathcal{U}$ is a functorial plus-construction in the category of algebras over $\mathcal{O}$. It enjoys the following properties:*

(i) $P_{\mathcal{U}}A \simeq Cof(ev : \coprod_{[\mathcal{U},A]} \mathcal{U} \to A)$
(ii) $\pi_0(P_{\mathcal{U}}A) \cong \pi_0(A)/\mathcal{P}\pi_0(A)$
(iii) $H_*^{\mathcal{Q}} A \cong H_*^{\mathcal{Q}} P_{\mathcal{U}}A$

Of particular interest is the plus-construction in the category of homotopy Lie algebras. If we apply these constructions to the algebra $gl(R)$ of matrices of an associative algebra, and if we consider it as a homotopy Lie algebra, we obtain what we call the homotopy cyclic homology theory. Thus $HC_i^\infty(R)$ is defined as $\pi_i gl(R)^+$ for any $i \geq 0$. This theory corresponds to the classical cyclic homology over the rationals. We summarize in a proposition the computations of the lower homology groups (see

Proposition 5.1, 5.2, 5.3). They share a striking resemblance with the low dimensional algebraic K-groups ($st(R)$ stands for the Steinberg Lie algebra, see Section 6).

PROPOSITION 1.1. *Let k be a field and R be an associative k-algebra. Then*

(1) $HC_0^\infty(R)$ *is isomorphic to* $R/[R, R]$.
(2) $HC_1^\infty(R)$ *is isomorphic to* $Z(st(R)) \cong H_1^Q(sl(R))$.
(3) $HC_2^\infty(R)$ *is isomorphic to* $H_2^Q(st(R))$.
(4) $HC_3^\infty(R)$ *is isomorphic to* $H_3^Q(st(R))$.

The same kind of results hold for homotopy Hochschild homology, which is defined similarly using Leibniz algebras. As these new theories coincide over the rationals with the classical Hochschild and cyclic homology, the above proposition extends Livernet's computations, see Corollary 5.4.

The plan of the paper is as follows. First we introduce the notion of algebra over an operad and recall when and how one can do homotopy theory with these objects. We also explain what homotopical nullification functors are for algebras. The main theorem about the plus-construction appears then in Section 2 and Section 3 contains the properties of the plus-construction with respect to fibrations and extensions. The final section is devoted to the computations of the low dimensional additive K-theory groups.

2. Operads and algebras over an operad

We fix R a commutative and unitary ring. We work in the category **R-dgm** of differential N-graded R-modules and especially with chains (the differential decreases the degree by 1). For a classical background about operads and algebras over an operad we refer to [11, 12, 18, 21].

Σ_*-*modules.* A Σ_*-module is a sequence $\mathcal{M} = \{\mathcal{M}(n)\}_{n\geq 0}$ of objects $\mathcal{M}(n)$ in the category **R-dgm** together with an action of the symmetric group Σ_n. The category of Σ_*-modules is a monoidal category. We denote by $\mathcal{M} \circ \mathcal{N}$ the product of two Σ_*-modules and by **1** the unit of this product. The unit is defined by $\mathbf{1}(1) = R$ and $\mathbf{1}(i) = 0$ for $i \neq 1$. We denote by Σ_***-mod** the category of Σ_*-modules.

Operads. An operad $\mathcal{O}$ is a monad in the category of Σ_*-modules. Hence we have a product $\gamma : \mathcal{O} \circ \mathcal{O} \longrightarrow \mathcal{O}$ which is associative and unital. Equivalently the product γ defines a family of composition products

$$\gamma : \mathcal{O}(n) \otimes \mathcal{O}(i_1) \otimes \cdots \otimes \mathcal{O}(i_n) \longrightarrow \mathcal{O}(i_1 + \cdots + i_n)$$

which must satisfy equivariance, associativity and unitality relations (also called May's axioms). Moreover we suppose that $\mathcal{O}(1) = R$ and the chain complex $\mathcal{O}(0)$ is always understood to be zero, which means that our operads are *reduced* in the terminology from [3]. We write $\pi_k\mathcal{O}$ for the kth homology group of the underlying chain complexes of $\mathcal{O}$. In particular $\pi_0\mathcal{O}$ is an operad in the category of R-modules.

Let us denote by **Oper** the category of operads. There is a free operad functor:

$$F : \Sigma_*\text{-}\mathbf{mod} \longrightarrow \mathbf{Oper}$$

which is left adjoint to the forgetful functor. It can be defined using the formalism of trees.

Algebras over an operad. Let us fix an operad $\mathcal{O}$. An algebra over $\mathcal{O}$ (also called $\mathcal{O}$-algebra) is an object A of **R-dgm** together with a collection of morphisms

$$\theta : \mathcal{O}(n) \otimes_{R[\Sigma_n]} A^{\otimes n} \longrightarrow A$$

called evaluation products, which are equivariant, associative and unital. For an element $o \in \mathcal{O}(n)$ we will often use the shorter notation $o(a_1, \ldots, a_n)$ for the evaluation product $\theta(o \otimes a_1 \otimes \cdots \otimes a_n)$. Moreover we denote by $\pi_n(A)$ the nth homology group of the chain complex (A, d_A) and remark that $\pi_*(A)$ is a $\pi_*(\mathcal{O})$-algebra and that $\pi_0(A)$ is a $\pi_0(\mathcal{O})$-algebra in the category of graded R-modules and R-modules respectively.

There is a free $\mathcal{O}$-algebra functor to the category $\mathcal{O}$**-alg** of $\mathcal{O}$-algebras:

$$S(\mathcal{O}, -) : \mathbf{R\text{-}dgm} \longrightarrow \mathcal{O}\text{-}\mathbf{alg}$$

which is left adjoint to the forgetful functor. For any $M \in$ **R-dgm** it is given by $S(\mathcal{O}, M) = \bigoplus_{n \geq 0} \mathcal{O}(n) \otimes_{R[\Sigma_n]} M^{\otimes n}$.

Modules over operad algebras. Let A be an $\mathcal{O}$-algebra. An A-module M is a differential module equipped with evaluation products

$$\tau : \mathcal{O}(n) \otimes A^{\otimes n-p} \otimes M \otimes A^{p-1} \longrightarrow M$$

which are associative, unital and equivariant with respect to the action of Σ_{n-1} (acting on $\mathcal{O}(n)$ fixing the last variable and on $A^{\otimes n-p} \otimes M \otimes A^{\otimes p-1}$ by permuting the elements of $A^{\otimes n-p}$ and $A^{\otimes p-1}$). Here as well we use the notation $o(a_1, \ldots, m, \ldots, a_n)$ for $\tau(o \otimes a_1 \otimes \cdots \otimes m \otimes \cdots \otimes a_n)$. Equivalently A-modules are modules over the universal enveloping algebra $U(\mathcal{O}, A)$.

Classical operads.

(a) To any object M in **R-dgm** one associates the endomorphism operad given by:

$\mathrm{End}(M)(n) = Hom_{\mathbf{R\text{-}dgm}}(M^{\otimes n}, M).$

Any $\mathcal{O}$-algebra structure on M is given by a morphism of operads $\mathcal{O} \longrightarrow End(M)$.

(b) The operad $\mathcal{C}om$, defined by $\mathcal{C}om(n) = R$. The $\mathcal{C}om$-algebras are the differential graded commutative algebras.

(c) The operad $\mathcal{A}s$ defined by $\mathcal{A}s(n) = R[\Sigma_n]$. The $\mathcal{A}s$-algebras are precisely the differential graded associative algebras.

(d) The operad $\mathcal{L}ie$. A $\mathcal{L}ie$-algebra L is an object of **R-dgm** together with a bracket which is anticommutative and satisfies the Jacobi relation. If $2 \in R$ is invertible then a $\mathcal{L}ie$-algebra is a classical Lie algebra. Otherwise, the category of classical Lie algebras appears as full subcategory of the category of $\mathcal{L}ie$-algebras.

(e) The operad $\mathcal{L}eib$ which is the operad of Leibniz algebras. A Leibniz algebra L is equipped with a bracket $[-,-]$ of degree zero that satisfies the Leibniz identity

$$[x,[y,z]] = [[x,y],z] - [[x,z],y]$$

for any $x, y, z \in L$. If $[x,x] = 0$ for any $x \in L$, this identity is equivalent to the Jacobi identity hence Lie algebras are examples of Leibniz algebra. We have an epimorphism of operads

$\mathcal{L}eib \longrightarrow \mathcal{L}ie.$

Homotopy of operads. V. Hinich in [14] and C. Berger & I. Moerdijk in [3] proved that the category of operads is a closed model category. This structure is obtained via the free operad functor from the one on the category **R-dgm**, where the weak equivalences are the quasi-isomorphisms and the fibrations are epimorphisms in positive degrees. Thus a morphism $\mathcal{O} \to \mathcal{O}'$ is a weak equivalence if for each $n > 0$ the map $\mathcal{O}(n) \to \mathcal{O}'(n)$ is a quasi-isomorphism of chain complexes. The cofibrant operads are the retracts of the quasi-free operads.

Homotopy of algebras over an admissible operad. For any $d \geq 0$, let W_d be the following object of **R-dgm**:

$$\ldots 0 \longrightarrow R = R \longrightarrow 0 \to \cdots \to 0$$

concentrated in differential degrees d and $d+1$. Using the terminology of [3], we say that $\mathcal{O}$ is admissible if the canonical morphism of $\mathcal{O}$-algebras:

$$A \longrightarrow A \coprod S(\mathcal{O}, W_d)$$

is a quasi-isomorphism for any $\mathcal{O}$-algebra A and for all d. For any admissible operad $\mathcal{O}$ there exists a closed model structure on the category of $\mathcal{O}$-algebras see [14] and [3], which is transferred from **R-dgm** along the free-forgetful adjunction given by $S(\mathcal{O},-)$. As for operads the weak equivalences are the quasi-isomorphisms, the fibrations are the epimorphisms in positive degrees, and the cofibrant $\mathcal{O}$-algebras are the retracts of the quasi-free $\mathcal{O}$-algebras.

The category of $\mathcal{O}$-algebras is cofibrantly generated and cellular in the sense of P. Hirschhorn [15]. The set of generating cofibrations is

$$I = \{i_n : \mathcal{O}(x_n) \longrightarrow \mathcal{O}(x_n, y_{n+1})\}$$

where $\mathcal{O}(x_n)$ is the free $\mathcal{O}$-algebra on a generator of degree n and $\mathcal{O}(x_n, y_{n+1})$ is the free $\mathcal{O}$-algebra over the differential graded module $R < x_n, y_{n+1} >$ with two copies of R, one in degree n the other in degree $n+1$, the differential of y_{n+1} being x_n. The set of generating acyclic cofibrations is

$$J = \{j_n : 0 \longrightarrow \mathcal{O}(x_n, y_{n+1})\}.$$

Notice that the free algebra $\mathcal{O}(x_n)$ plays the role of the sphere S^n.

Over the rational numbers all operads are admissible. This is not the case over an arbitrary ring, for example the operads $\mathcal{C}om$ and $\mathcal{L}ie$ over the integers are not admissible. However cofibrant operads and the operad $\mathcal{A}s$ are always admissible. In what follows we will consider only two types of admissible operads:

– Rational operads,
– Cofibrant operads.

The closed model category of algebras over such operads is left proper as we prove in [6].

Homology of algebras. Let $\mathcal{O}$ be an admissible operad, and let A be an $\mathcal{O}$-algebra. An element $a \in A$ is called decomposable if it lies in the ideal $A2$, the image of the evaluation products

$$\theta(n) : \mathcal{O}(n) \otimes_{R[\Sigma_n]} A^{\otimes n} \longrightarrow A$$

for any $n > 1$. We denote by $QA = A/A2$ the space of indecomposables of the algebra A. The Quillen homology of A, denoted by $H_*^Q(A)$, is the homology of $QS(\mathcal{O}, V)$ where $S(\mathcal{O}, V)$ is a cofibrant replacement of A. This does not depend on the choice of the cofibrant replacement and we always have $H_0^Q(A) = Q\pi_0(A)$.

Moreover, any cofibration sequence $A \longrightarrow B \longrightarrow C$ of $\mathcal{O}$-algebras yields a long exact sequence in Quillen homology.

As the operads $\mathcal{L}ie$ and $\mathcal{L}eib$ are Koszul operads, over $\boldsymbol{Q}$ one can compute their Quillen homology by way of a nice complex (we refer to Ginzburg and

Kapranov [12] and also to Fresse [9] for a more homotopical viewpoint on Koszul Duality).

$\mathcal{L}ie$-algebras. Let L be a $\mathcal{L}ie$-algebra over $\boldsymbol{Q}$. The homology of L, denoted here by $H_*^{Lie}(L)$, is computed using the Chevalley–Eilenberg complex $\mathrm{CE}_*(L)$. Now we can consider L as a $\mathcal{L}_\infty$-algebra and compute $H_*^Q(L)$. Koszul duality for rational Lie algebras gives the following isomorphism:

$$H_*^Q(L) \cong H_{*+1}^{Lie}(L).$$

$\mathcal{L}eib$-algebras. The same kind of results holds for $\mathcal{L}eib$-algebras. Consider a Leibniz algebra L. The homology of L, denoted by $H_*^{Leib}(L)$, is computed using the complex described in the foundational paper [24] (see also [20]). One has again a similar isomorphism:

$$H_*^Q(L) \cong H_{*+1}^{Leib}(L).$$

Quillen cohomology of discrete algebras. We refer the reader to [10] for more details about these constructions. A discrete algebra is an $\mathcal{O}$-algebra concentrated in differential degree 0. The structure of $\mathcal{O}$-algebra reduces then in fact to a structure of $\pi_0(\mathcal{O})$-algebra.

In the case of discrete algebras there is also a notion of Quillen cohomology with coefficents. Fix a discrete $\mathcal{O}$-algebra A and a discrete A-module M. A derivation $D : A \to M$ in $Der(A, M)$ is a linear map (which does not necessarily commute with the differential) such that for any $o \in \mathcal{O}(n)$ we have:

$$D(o(a_1, \dots, a_n)) = \sum_{i=1}^{n} o(a_1, \dots, D(a_i), \dots, a_n).$$

We can define Quillen cohomology by computing the derived functors of $Der(A, M)$, that is by taking A' a cofibrant replacement of A in the category of $\mathcal{O}$-algebras and computing the homology of the complex $Der(A', M)$.

The derived functor defined above has also a homotopical interpretation. The functor Q of indecomposable has a right adjoint $(-)_+$ defined as follows: If M is an object of **R-dgm** then $(M)_+$ is the trivial $\mathcal{O}$-algebra with M as underlying module. This adjoint pair of functors forms a Quillen pair. Now let us take a discrete A-module M and denote by $M[n]$ the nth suspension of M in the category **R-dgm** and define $K(M, n) = M[n]_+$. Then

$$H_Q^n(A, M) = [A, K(M, n)]_{\mathcal{O}-\mathbf{alg}} \cong [QA, M[n]]_{\mathbf{R-dgm}}.$$

Moreover $H_Q^1(A, M)$ classifies square zero extensions of A by M. A square zero extension is an exact sequence of A-modules

$$0 \to M \xrightarrow{i} B \xrightarrow{p} A \to 0,$$

such that p is a morphism of $\mathcal{O}$-algebras and

$$i(o(a_1, \dots, m, \dots, a_n)) = o(a'_1, \dots, i(m), \dots, a'_n)$$

for any $a_i \in A$, $m \in M$ and a'_i in $p^{-1}(a_i)$.

A square zero extension $0 \to M \xrightarrow{i} B \xrightarrow{p} A \to 0$ is universal if for any other square zero extension $0 \to M' \xrightarrow{i} B' \xrightarrow{p'} A \to 0$ there exists a unique morphism of $\mathcal{O}$-algebras $\phi : B \to B'$ such that $p'\phi = p$.

The set of isomorphism classes of square zero extensions is denoted by $Ex(A, M)$. By a classical result of Quillen [28] we have the following isomorphism:

$$H^1_Q(A, M) \cong Ex(A, M).$$

We also recall that for discrete $\mathcal{L}ie$-algebras (resp. for discrete $\mathcal{L}eib$-algebras, a result due to Gnedbaye [13, Theorem 3.3]) a square zero extension

$$0 \to M \xrightarrow{i} U \xrightarrow{p} L \to 0$$

is universal if and only if U is perfect and any square zero extension of U splits in the category of $\mathcal{L}ie$-algebras (resp. $\mathcal{L}eib$-algebras). Hence for any U-module M we have $Ex(U, M) = 0$. By representability of the Quillen homology we get that for any M, the set of homotopy classes $[QU, K(M, 1)]$ is trivial, thus over a field k we have $H^Q_0(U) = H^Q_1(U) = 0$ (take $M = k$).

Hurewicz Theorem. In her thesis [19, Theorem 2.13] M. Livernet proved a Hurewicz type theorem for algebras over an operad in the rational case. A result of Getzler and Jones about the construction of a cofibrant replacement for $\mathcal{O}$-algebras, which uses the Bar-Cobar construction, extends easily the proof of Livernet to admissible operads.

THEOREM 2.1. (Livernet). *Let A be an $\mathcal{O}$-algebra. Then there is a Hurewicz morphism:*

$$\mathrm{Hu} : \pi_*(A) \longrightarrow H^Q_*(A)$$

induced by the projection on indecomposable elements. It satisfies the following properties:

(i) If $\pi_k(A) = 0$ for $0 \leq k \leq n$ then Hu *is an isomorphism for $k \leq 2n + 1$ and an epimorphism for $k = 2n + 2$.*
(ii) If $\pi_0(A) = 0$ and $H^Q_k(A) = 0$ for $0 \leq k \leq n$ then Hu *is an isomorphism for $k \leq 2n + 1$ and an epimorphism for $k = 2n + 2$.*

Proof. In the case of 0-connected chain complexes we have a Quillen adjunction between $\mathcal{O}$-algebras and $B\mathcal{O}$-coalgebras, [11] and [2]. These two functors provide for any algebra A a cofibrant replacement of the form $S(\mathcal{O}, C(B\mathcal{O}, A))$ where $C(B\mathcal{O}, A)$ is the coalgebra over the cooperad $B\mathcal{O}$ obtained by applying the operadic bar construction. Now Livernet's arguments apply to $C(B\mathcal{O}, A)$.

Perfect algebras over an operad. Consider an algebra A over an operad in the category of R-modules. The algebra A is called $\mathcal{O}$-perfect if any element in A is decomposable i.e. $A = A2$ or $QA = 0$. We define $\mathcal{P}A$, the maximal $\mathcal{O}$-perfect ideal of A, by transfinite induction.

Let A_0 be the ideal $A2$. We define the ideals A_α inductively by setting $A_\alpha = (A_{\alpha-1})2$ if α is a successor ordinal and $A_\alpha = \cap_{\beta<\alpha} A_\beta$ if α is a limit ordinal. Then we set $\mathcal{P}A = \lim_\alpha A_\alpha$.

This inverse system actually stabilizes for some ordinal β, hence $A_\beta 2 = A_\beta$ and $\mathcal{P}A = A_\beta$. Of course, if $QA = 0$ then we have $\mathcal{P}A = A$. We also notice that for any $\mathcal{O}$-algebra A we have $\mathcal{P}(A/\mathcal{P}A) = 0$.

Consider an epimorphism $f: \mathcal{O} \longrightarrow \mathcal{O}'$ of operads and let A be an $\mathcal{O}'$-algebra. Then if A is $\mathcal{O}'$-perfect it is also $\mathcal{O}$-perfect. Thus we have an inclusion $\mathcal{P}A \subseteq \mathcal{P}'A$ of the $\mathcal{O}$-perfect ideal into the $\mathcal{O}'$-perfect ideal of A.

3. A Functorial Additive Plus-construction

The theory of homological and homotopical localization of topological spaces developed by P. Bousfield and E. Dror Farjoun (see e.g. [5, 8]) has an analogue in the category of algebras over a cofibrant operad $\mathcal{O}$. This takes place in the more general framework established by P. Hirschhorn in [15]. Our category $\mathcal{O}$-**alg** of algebras over a cofibrant operad $\mathcal{O}$ is indeed cellular and left proper. We recall first how to build mapping spaces in a model category which is not supposed to be simplicial, and what is meant by homotopical nullification with respect to an object in this context. We apply then this theory to construct a plus-construction for algebras over a cofibrant operad.

Mapping spaces. One way to construct mapping spaces up to homotopy in a model category is to find a cosimplicial resolution $X^\bullet$ of the source X (as in [15, Definition 18.1.1], see also [16, Chapter 5]). When X is cofibrant and Y fibrant, define the mapping space $map(X, Y)$ to be the simplicial set $mor_{\mathcal{O}}$-**alg**(X^*, Y).

In a pointed model category there is always at least one morphism $X \to Y$, namely the trivial one, which serves as base point for the mapping space. The homotopy groups of a pointed mapping space can then be computed, see [16, Lemma 6.1.2].

PROPOSITION 3.1. *Let X be a cofibrant and Y a fibrant $\mathcal{O}$-algebra. Then* $\pi_n map(X, Y) \cong [\Sigma^n X, Y]$. □

X-nullification. Let $\mathcal{O}$ be an admissible operad and fix an $\mathcal{O}$-algebra X. One says that an algebra Z is X-local or X-null if the space $map(X, Z)$ is weakly homotopy equivalent to a point. By Proposition 3.1 this is equivalent to requiring that $[\Sigma^k X, Z]$ be trivial for all $k \geq 0$. A morphism of $\mathcal{O}$-algebras

$h: A \to B$ is called an *X-equivalence* if it induces a weak homotopy equivalence $map(h, Z): map(B, Z) \to map(A, Z)$ for every X-local algebra Z. Theorem 4.1.1 from [15] ensures then the existence of an X-nullification functor, i.e. a continuous functor $P_X : \mathcal{O}\text{-}\mathbf{alg} \to \mathcal{O}\text{-}\mathbf{alg}$ together with a natural transformation $\eta : Id \to P_X$ from the identity functor to P_X, such that $\eta_A : A \to P_X A$ is an X-equivalence and $P_X A$ is X-local for any $\mathcal{O}$-algebra A.

EXAMPLE: An interesting example is when X is the free $\mathcal{O}$-algebra $\mathcal{O}(x)$ with one generator x in dimension n. This plays the role of the n-dimensional sphere, hence $\mathcal{O}(x)$-nullification gives rise to a functorial n-Postnikov section in this category.

Plus-construction. A *Quillen plus-construction* of an algebra A over an operad $\mathcal{O}$ is a Quillen homology equivalence $\eta : A \to A^+$ which quotients out the perfect radical on π_0, that is

$$\pi_0(A^+) \cong \pi_0(A)/\mathcal{P}\pi_0(A).$$

This definition parallels the classical one introduced by D. Quillen for spaces in [27]. Recall that by definition of the radical we have $\mathcal{P}\big(\pi_0(A)/\mathcal{P}\pi_0(A)\big) = 0$ so the image of the plus-construction consists of algebras B with $\mathcal{P}\pi_0(B) = 0$. Therefore if the plus-construction can be constructed as a nullification the local objects will be those algebras B with $\mathcal{P}\pi_0(B) = 0$ (compare with Proposition 3.3) and hence the following universal property will hold: for any morphism $g : A \to B$ to an $\mathcal{O}$-algebra B with $\mathcal{P}\pi_0(B) = 0$, there exists up to homotopy a unique map $\tilde{g} : A^+ \to B$ such that $\tilde{g}\eta = g$. In particular this will imply that the plus-construction is unique up to quasi-isomorphism.

We now construct an H_*^Q-acyclic algebra $\mathcal{U}$ such that the associated nullification $A \to P_{\mathcal{U}} A$ is the plus-construction.

$\mathcal{O}$-trees. A *rooted tree* T is a directed graph in which any vertex v has one ingoing arrow a_v, except one distinguished vertex, the root, that has no ingoing arrow. We require moreover that the following additional conditions are satisfied: Each vertex v has a finite number of outgoing arrows, denoted by $val(v)$; the set $suc(v)$ of successor vertices of v, i.e. those which are connected to v by an ingoing arrow is finite and totally ordered; and finally, the vertices v of odd level have at least 2 successors. The root has level 0, and inductively we say that a vertex v has level k if $v \in suc(u)$ for some u of level $k-1$.

Let $\mathcal{O}$ be any operad. An *$\mathcal{O}$-tree* is a pair (T, ϕ) where T is a rooted tree and ϕ is a function which associates to each vertex v of odd level a multilinear operation $o_n \in \mathcal{O}(n)_0$ where n is equal to the number $val(v)$ of outgoing arrows. It is best to think about the even level vertices as elements which are composed together following a recipe given by the operations corresponding to the odd level vertices. In general there are uncountably many $\mathcal{O}$-trees, even in the case of the operad $\mathcal{L}ie$, as explained in [4]. Maybe the following $\mathcal{L}ie$-tree explaining how an element (corresponding to the root) decomposes as a certain infinite sequence of brackets is the simplest example: The rooted tree has 2^n vertices of even level $2n$, each of which has precisely one successor, the odd level vertices have each two successors; the function ϕ associates to each odd level vertex the operation $[-,-]$. It corresponds to the decomposition of an element as a commutator $[a_1, a_2]$, where a_1, a_2 are themselves commutators $[a_3, a_4]$, $[a_5, a_6]$ respectively, and so on.

A universal H_*^Q-acyclic algebra. We first define a direct system $\{U_r, \phi_r\}$ of free $\mathcal{O}$-algebras associated to a given $\mathcal{O}$-tree (T, ϕ), by induction on r: Let U_0 be the free $\mathcal{O}$-algebra on one generator x in dimension 0 (corresponding to the root). Let $n = val(root)$ and $suc(root) = \{v_1, \ldots, v_n\}$. For each $j = 1, \ldots, n$, let $k_j = val(v_j)$ and $o_{k_j} = \phi(v_j)$ be the multilinear operation in $\mathcal{O}(k_j)_0$ associated to the vertex v_j. Choose k_j free generators $x_{1j1}, x_{1j2}, \ldots, x_{1jk_j}$ in dimension 0 corresponding to the vertices in $suc(v_j)$ of level 2. The first index indicates half of the level, the second is the index of the vertex of odd degree. Let then U_1 be the free $\mathcal{O}$-algebra on those $k_1 + \ldots + k_n$ generators and define $\phi_1 : U_0 \to U_1$ on the generator x by

$$\phi_1(x) = \sum_{j=1}^{n} o_{k_j}(x_{1j1}, x_{1j2}, \ldots, x_{1jk_j}).$$

Inductively, we define then U_r as the free $\mathcal{O}$-algebra on as many generators as there are vertices of level $2r$, and $\phi_r : U_{r-1} \to U_r$ is given on each generator of U_{r-1} by a similar formula as the above one for $\phi_1(x)$. Define $\mathcal{U}_{(T,\phi)}$ as the homotopy colimit of the direct system $\{U_r, \phi_r\}$ associated to the $\mathcal{O}$-tree (T, ϕ).

LEMMA 3.2. *Let $\mathcal{O}$ be a rational or cofibrant operad. Then for any $\mathcal{O}$-tree (T, ϕ), the $\mathcal{O}$-algebra $\mathcal{U}_{(T,\phi)}$ is H_*^Q-acyclic, i.e. $H_*^Q(\mathcal{U}_{(T,\phi)}) = 0$.*

Proof. To compute the homotopy colimit of the direct system described above, one has to replace each map $\phi_r : U_{r-1} \to U_r$ by a cofibration. Thus $\mathcal{U}_{(T,\phi)}$ is free on generators x_I of degree 0 and y_I of degree 1 where I is a multi-index of the form rjs, r indicating half of the level where these generators are created, $1 \leq s \leq k_j$, and the differential is given by $d(y_I) = x_I - \phi_r(x_I)$. In the space $Q\mathcal{U}_{(T,\phi)}$ of indecomposables the differential identifies y_I with x_I, so that the Quillen homology is trivial. □

Notice that a morphism from $\mathcal{U}_{(T,\phi)}$ to some $\mathcal{O}$-algebra A is the choice of an element in degree zero (the image of the root) together with one way to write it as a succession of operations in the operad modulo some boundaries, the succession of operations being imposed by the chosen $\mathcal{O}$-tree. The algebra $\mathcal{U}$ is now defined as a coproduct taken over all $\mathcal{O}$-trees (T, ϕ):

$$\mathcal{U} = \coprod_{(T,\phi)} \mathcal{U}_{(T,\phi)}$$

As mentioned before this is in general an uncountable coproduct of algebras, which are all concentrated in degree 0 and 1. Given an $\mathcal{O}$-algebra X, we choose one representative for any homotopy class of maps $\mathcal{U} \to X$ and call the coproduct of all them $\coprod_{[\mathcal{U},X]} \mathcal{U} \to X$ the evaluation map. As in the case of the plus-construction for spaces, the homotopy cofiber of the evaluation map will be equivalent to X^+, as we show in the theorem below. Let us first compute what happens at the level of π_0.

PROPOSITION 3.3. *Let $\mathcal{O}$ be a rational or cofibrant operad and X be an $\mathcal{O}$-algebra. Then $\mathcal{P}\pi_0 X$ is the image of the evaluation map $ev : \coprod_{[\mathcal{U},X]} \mathcal{U} \to X$ on π_0. In particular X is $\mathcal{U}$-null if and only if $\mathcal{P}\pi_0 X = 0$.*

Proof. An element in $\pi_0 X$ is in the image of the evaluation map if and only if there is some representative $x \in X_0$ which lies in the image of a morphism from $\mathcal{U}_{(T,\phi)}$ for some $\mathcal{O}$-tree (T, ϕ). This means precisely that $[x] \in \mathcal{P}\pi_0 X$. The second assertion is clear by Proposition 3.1 since the suspension of an H^Q_*-acyclic object such as $\mathcal{U}$ (see Lemma 3.2) is 0-connected and H^Q_*-acyclic (Quillen homology commutes with suspension), therefore trivial by the Hurewicz Theorem 2.1. □

Notice in the proof above that the morphism hitting x need not be unique. We do not claim that $[\mathcal{U}, X]$ is isomorphic to the perfect $\pi_0\mathcal{O}$-ideal of $\pi_0 X$.

The cone of $\mathcal{U}$. In order to do some computations with this H^Q_*-acyclic algebra, we need to describe how to construct the cone of it. Let us simply describe the cone on $\mathcal{U}_{(T,\phi)}$ for a fixed tree T. For each generator x_I in degree 0 we add a generator $\bar{x}_I$ in degree 1, and for each generator y_I in degree 1 we add a generator $\bar{y}_I$ in degree 2. The differential is as follows: $dy_I = x_I - \phi_r(x_I)$, as in $\mathcal{U}_{(T,\phi)}$, $d\bar{x}_I = x_I$, so we kill π_0, and $d\bar{y}_I = y_I - \bar{x}_I - u_I$ where u_I is a decomposable element of degree 1 such that $du_I = \phi_r(x_I)$. Such an element exists indeed since $\phi(x_I)$ is a sum of decomposable elements in degree 0 of type $o(x_{J_1}, \ldots, x_{J_k})$, which are hit for example by the differential of $o(\bar{x}_{J_1}, x_{J_2}, \ldots, x_{J_k})$.

THEOREM 3.4. *Let $\mathcal{O}$ be a rational or cofibrant operad. Then the homotopical nullification with respect to $\mathcal{U}$ is a functorial plus-construction in the category of algebras over $\mathcal{O}$. It enjoys the following properties:*

(i) $P_{\mathcal{U}}A \simeq Cof(ev : \coprod_{[\mathcal{U},A]} \mathcal{U} \longrightarrow A)$
(ii) $\pi_0(P_{\mathcal{U}}A) \cong \pi_0(A)/\mathcal{P}\pi_0(A)$
(iii) $H_*^Q(A) \cong H_*^Q(P_{\mathcal{U}}A)$

Proof. Consider the cofibration sequence

$$\coprod_{[\mathcal{U},A]} \mathcal{U} \xrightarrow{ev} A \longrightarrow B$$

Clearly $A \to B$ is a $P_{\mathcal{U}}$-equivalence. So it remains to show that B is $\mathcal{U}$-local, or equivalently by the preceding proposition that $\mathcal{P}\pi_0(B) = 0$. Let us thus compute $\pi_0 B$. Consider actually the more elementary cofiber C_α of a single map $\alpha : \mathcal{U}_{(T,\phi)} \to A$. Such a map corresponds to an element $a \in \mathcal{P}\pi_0 A$ together with a decomposition following the pattern indicated by the tree (T, ϕ). Let us replace A by a free algebra $\mathcal{O}(V)$ and construct now C_α as the push-out of $\mathcal{O}(V) \leftarrow \mathcal{U}_{(T,\phi)} \hookrightarrow C(\mathcal{U}_{(T,\phi)})$. The models of these algebras we exhibited earlier show that $C_\alpha = \mathcal{O}(V) \coprod \mathcal{O}(\bar{x}_I, \bar{y}_I)$ with $d\bar{x}_I = a_I = \alpha(x_I)$ and $d\bar{y}_I = b_I - \bar{x}_I - \alpha(u_I)$. Clearly $\pi_0 C_\alpha \cong \pi_0 A / < a >$. Likewise $\pi_0 B \cong \pi_0 A / \mathcal{P}\pi_0(A)$ (which incidentally proves (ii)).

Hence $\mathcal{P}\pi_0(B) = 0$, which shows that $B \simeq P_{\mathcal{U}}A$ and the third property is now a direct consequence of the first one and the long exact sequence in Quillen homology for the above cofibration. □

From now on we will denote the $\mathcal{U}$-nullification of an $\mathcal{O}$-algebra A simply by A^+.

Naturality. We conclude this section with a discussion of the naturality of the plus-construction with respect to the operad. We denote by $\mathcal{U}'$ the universal H_*^Q-acyclic $\mathcal{O}'$-algebra as constructed above and $A^{+'} = P_{\mathcal{U}'}A$ the associated plus-construction.

PROPOSITION 3.5. *Let* $\mathbf{f} : \mathcal{O} \longrightarrow \mathcal{O}'$ *be a map of operads, then there is a map of* $\mathcal{O}$*-algebras* $f : \mathcal{U} \longrightarrow \mathcal{U}'$.

Proof. The map $\mathbf{f}$ induces a map between the directed systems $\{U_r, \phi_r\}$ and $\{U'_r, \mathbf{f}(\phi_r)\}$. Where $\{U_r, \phi_r\}$ is the directed system associated to a $\mathcal{O}$-tree (T, ϕ) and $\{U'_r, \mathbf{f}(\phi_r)\}$ is the directed system associated to the $\mathcal{O}'$-tree $(T, \mathbf{f}(\phi))$ where each vertex is of the form $\mathbf{f}(o)$. There is a natural transformation between the directed systems of $\mathcal{O}$-algebras, thus also a map between their homotopy colimits.

PROPOSITION 3.6. *Let* $\mathbf{f} : \mathcal{O} \longrightarrow \mathcal{O}'$ *be a quasi-isomorphism of operads, and suppose that either we work over* $\boldsymbol{Q}$*, or the operads* $\mathcal{O}$ *and* $\mathcal{O}'$ *are cofibrant. Then* $f : \mathcal{U} \to \mathcal{U}'$ *is a quasi-isomorphism of* $\mathcal{O}$*-algebras.*

Proof. The result follows from the fact that free algebras over the operads $\mathcal{O}$ and $\mathcal{O}'$ and over the same generators are quasi-isomorphic as $\mathcal{O}$-algebras. □

As a consequence, when replacing an operad by a cofibrant one to do homotopy, the choice of this cofibrant operad does not matter.

COROLLARY 3.7. *Let A be an $\mathcal{O}'$-algebra, and let* $\mathbf{f}: \mathcal{O} \longrightarrow \mathcal{O}'$ *be a morphism of operads. Under the same assumptions as in the preceding proposition, the map* $A^+ \longrightarrow A^{+'}$ *is a quasi-isomorphism of $\mathcal{O}$-algebras.*

4. Fibrations and the Plus-construction

Let $\mathcal{O}$ be an operad which is either cofibrant or taken over the rationals. This section is devoted to the analysis of the behavior of the plus-construction with fibrations. In particular we will be interested in the homotopy fiber AX of the map $X \to X^+$. As one should expect it, AX is the universal H_*^Q-acyclic algebra over X in the sense that any map $A \to X$ from an H_*^Q-acyclic algebra A factors through AX. The most efficient tool to deal with such questions is the technique of fiberwise localization in our model category of $\mathcal{O}$-algebras. To our knowledge, such a tool had not been developed up to now in any other context than spaces, and we refer therefore to the separate paper [6] for the following claim:

THEOREM 4.1. *Let $\mathcal{O}$ be a rational or cofibrant operad. Let $p: E \twoheadrightarrow B$ be a fibration of $\mathcal{O}$-algebras inducing a surjection on π_0 and call F the fiber of p. There exists then a commutative diagram*

$$\begin{array}{ccccc} F & \to & E & \to & B \\ \downarrow & & \downarrow & & \downarrow \\ F^+ & \to & E^+ & \to & B \end{array}$$

where both lines are fibrations and the map $E \to \bar{E}$ is a $P_{\mathcal{U}}$-equivalence.

The main ingredient in the proof of this theorem is the fact that the category of $\mathcal{O}$-algebras satisfies (a weak version of) the cube axiom. From the above theorem we infer that the plus-construction sometimes preserves fibrations.

THEOREM 4.2. *Let $\mathcal{O}$ be a rational or cofibrant operad. Let $F \longrightarrow E \longrightarrow B$ be a fibration of $\mathcal{O}$-algebras inducing a surjection on π_0. If the basis B is local with respect to the $\mathcal{U}$-nullification then we have a fibration*

$$F^+ \longrightarrow E^+ \longrightarrow B.$$

Proof. By Theorem 4.1 this is a direct consequence of the fact that the total space sits in a fibration where both the fiber and the base space are $\mathcal{U}$-local and hence is also $\mathcal{U}$-local. □

The fiber of the plus-construction. Another consequence of the fiberwise plus-construction is that the homotopy fiber AX is H_*^Q-acyclic.

PROPOSITION 4.3. *Let $\mathcal{O}$ be a rational or cofibrant operad. The fiber AX of the plus-construction $X \longrightarrow X^+$ is H_*^Q-acyclic for any $\mathcal{O}$-algebra X.*

Proof. Consider the fibration $AX \to X \to X^+$. The plus-construction preserves this fibration by the above theorem, since $\pi_0 X^+ \cong \pi_0 X / \mathcal{P}\pi_0 X$. Hence $(AX)^+$ is contractible, as it is the fiber of the identity on X^+. This means that $H_*^Q(AX) = 0$ and we are done. □

Cellularization. We can go a little further in the analysis of the fiber AX. Our next result says precisely that the map $AX \to X$ is a $CW_{\mathcal{U}}$-equivalence, where $CW_{\mathcal{U}}$ is Farjoun's cellularization functor ([8, Chapter 2]). We do not know whether AX is actually the $\mathcal{U}$-cellularization of X, but we know it is H_*^Q-acyclic by Proposition 4.3.

PROPOSITION 4.4. *Let $\mathcal{O}$ be a rational or cofibrant operad. We have $map(\mathcal{U}, AX) \simeq map(\mathcal{U}, X)$ for any $\mathcal{O}$-algebra X.*

Proof. Apply $map(\mathcal{U}, -)$ to the fibration $AX \to X \to X^+$ so as to get a fibration of simplicial sets

$$map(\mathcal{U}, AX) \to map(\mathcal{U}, X) \to map(\mathcal{U}, X^+)$$

By construction X^+ is $\mathcal{U}$-local, so that the base space $map(\mathcal{U}, X^+)$ is contractible. Therefore $map(\mathcal{U}, AX) \simeq map(\mathcal{U}, X)$. □

On the level of components, this implies we have an isomorphism $[\mathcal{U}, AX] \cong [\mathcal{U}, X]$, which means that any element in the $\mathcal{O}$-perfect ideal $\mathcal{P}\pi_0 X$ together with a given decomposition can be lifted in a unique way to such an element in $\pi_0 AX$.

PROPOSITION 4.5. *Let $\mathcal{O}$ be a rational or cofibrant operad. The fibration $AX \to X \to X^+$ is also a cofibration (up to homotopy).*

Proof. By definition X^+ is the homotopy cofiber of the evaluation map $\coprod\mathcal{U} \to X$. By the above proposition this map admits a lift to AX. By considering the composite $\coprod\mathcal{U} \to AX \to X$, we get a cofibration

$$Cof\Big(\coprod\mathcal{U} \to AX\Big) \to Cof\Big(\coprod\mathcal{U} \to X\Big) \to Cof(AX \to X)$$

The first homotopy cofiber is $(AX)^+$, which is contractible, and the second is X^+. The third is thus X^+ as well. □

Preservation of square zero extensions. Let us finally study the effect of the plus-construction on a square zero extension $M \to B \to A$, as introduced at the end of Section 1. In the case of Lie or Leibniz algebras this notion coincides of course with the classical one of central extension, as exposed e.g. in [17]. Following [28], [10, chapter 5], such a square zero extension is classified by an element in the first Quillen cohomology group $H^1_Q(A;M) \cong [A, K(M,1)]$. Recall that $K(M,1)$ denotes the suspension of the $\mathcal{O}$-trivial module M, given as $\mathcal{O}$-algebra by the chain complex M concentrated in degree 1. As for group extensions, the homotopy fiber of the classifying map $A \to K(M,1)$ (the k-invariant of the extension) is precisely B.

PROPOSITION 4.6. *Let $\mathcal{O}$ be a rational or cofibrant operad. Let $M \hookrightarrow B \to A$ be a square zero extension of discrete $\mathcal{O}$-algebras. Then the plus-construction yields a fibration $M \to B^+ \to A^+$.*

Proof. Let us consider the k-invariant and the associated fibration $B \to A \to K(M,1)$. The base is 0-connected, thus $P_{\mathcal{U}}$-local. Theorem 4.2 yields next another fibration $B^+ \to A^+ \to K(M,1)$, so that the homotopy fiber of $B^+ \to A^+$ is M. □

5. Applications to Algebras of Matrices

Recollections on algebras of matrices. Let k be a field and R be an associative k-algebra. Consider $gl(R)$ the union of the $gl_n(R)$'s. This is a $\mathcal{L}ie$-algebra and also a $\mathcal{L}eib$-algebra for the classical bracket of matrices. The trace $tr : gl(R) \longrightarrow R/[R,R]$ is a morphism of $\mathcal{L}ie$ and $\mathcal{L}eib$-algebras, whose kernel is by definition the algebra $sl(R)$.

We define the Steinberg algebra $st(R)$ for the operad $\mathcal{L}ie$ and the Leibniz Steinberg algebra $stl(R)$ for the operad $\mathcal{L}eib$ (following the notation in [24]) by taking the free algebra in the adequate category over the generators $u_{i,j}(r)$, $r \in R$ and $1 \leq i \neq j$ with the relations

(a) $u_{i,j}(m \cdot r + n.s) = m \cdot u_{i,j}(r) + n \cdot u_{i,j}(s)$ for $r, s \in R$ and $m, n \in \cdot$
(b) $[u_{i,j}(r), u_{k,l}(s)] = 0$ if $i \neq l$ and $j \neq k$.
(c) $[u_{i,j}(r), u_{k,l}(s)] = u_{i,l}(rs)$ if $i \neq l$ and $j = k$.

We have the following extension of algebras in the category of $\mathcal{L}ie$-algebras:

$$Z(st(R))\longrightarrow st(R)\longrightarrow sl(R)$$

where the center of the Steinberg algebra $Z(st(R))$ is the kernel of the canonical map between $st(R)$ and $sl(R)$. Following the work of C. Kassel and J.L. Loday this is a universal square zero extension [17, Proposition 1.8].

Likewise the center of the Leibniz Steinberg algebra is the kernel of the canonical map $stl(R) \to sl(R)$ and the extension of $\mathcal{L}eib$-algebras

$$Z(stl(R))\longrightarrow stl(R)\longrightarrow sl(R)$$

is a universal square zero extension [24, Theorem 4.4].

Now we can consider all these algebras as algebras over cofibrant replacements $\mathcal{L}_\infty$ and $\mathcal{L}eib_\infty$ of the operads $\mathcal{L}ie$ and $\mathcal{L}eib$.

Let us a recall that central extensions of discrete $\mathcal{L}_\infty$ and $\mathcal{L}eib_\infty$ algebras are exactly the same as central extensions of $\mathcal{L}ie$ and $\mathcal{L}eib$-algebras. This comes from the fact that the category of discrete $\mathcal{O}$-algebras is equivalent to the category of $\pi_0(\mathcal{O})$-algebras.

Homology theories. In the category of $\mathcal{L}_\infty$-algebras we define homotopy cyclic homology HC^∞:

$$HC^\infty_*(R) = \pi_*(gl(R)^+).$$

Likewise in the category of $\mathcal{L}eib_\infty$-algebras we define homotopy Hochschild homology:

$$HH^\infty_*(R) = \pi_*(gl(R)^+).$$

By Corollary 3.7, we notice that these definitions do not depend on the choice of the cofibrant replacement of the operads $\mathcal{L}ie$ or $\mathcal{L}eib$. These theories define two functors from the category of associative algebras to the categories of $\mathcal{L}ie$ and $\mathcal{L}eib$ graded algebras. We recall that the homotopy of an $\mathcal{L}_\infty$-algebra (resp. a $\mathcal{L}eib_\infty$-algebra) is a graded $\mathcal{L}ie$-algebra (resp. a $\mathcal{L}eib$-algebra).

When we consider these two theories over $\boldsymbol{Q}$, we have quasi-isomorphisms $\mathcal{L}eib_\infty \to \mathcal{L}eib$ and $\mathcal{L}ie_\infty \to \mathcal{L}ie$. Hence our functorial plus-construction is homotopy equivalent to M. Livernet's one and in particular they have the same homotopy groups. Using deep theorems of C. Cuvier in [7], J.-L. Loday and D. Quillen in [25], M. Livernet proved in [19, Proposition 5.2 and 5.3] that $\pi_n(sl(R)^+)$ is isomorphic to $HC_n(R)$ (respectively $HH_n(R)$) for any integer $n \geq 1$. Therefore our theories coincide as well with the classical cyclic and Hochschild homologies (we check the case $n = 0$ in Proposition 5.1 below):

$$HC^\infty_n(R) \cong HC_n(R),$$

$$HH^\infty_n(R) \cong HH_n(R).$$

The names homotopy Hochschild and homotopy cyclic homology we use in the present paper come of course from these results. We do not know if the above isomorphisms remain valid over $\boldsymbol{Z}$. However, using the properties of our construction, we are able to compute the first four groups of HC^{∞} and HH^{∞}. These results form perfect analogues of the classical computations in algebraic K-theory, see for example [29, Theorem 4.2.10], and [1, Theorem 3.14] for a topological approach.

Abelianization. In order to compute $HH_0^{\infty}(R)$ and $HC_0^{\infty}(R)$ we use the following fibration given by the trace:

$$sl(R) \longrightarrow gl(R) \longrightarrow R/[R, R].$$

PROPOSITION 5.1. *Let R be an associative k-algebra. Then $HH_0^{\infty}(R)$ and $HC_0^{\infty}(R)$ are both isomorphic to $R/[R, R]$. Moreover $sl(R)^+$ is the 0-connected cover of $gl(R)^+$.*

Proof. The commutator subgroup of $gl(R)$ as well as $sl(R)$ (i.e. the perfect radical in either the category of Lie or Leibniz algebras) is $sl(R)$. Therefore so is the perfect radical in $\mathcal{L}_{\infty}$ and $\mathcal{L}eib_{\infty}$ (this is the case for any discrete algebra). Hence $\pi_0 gl(R)^+ \cong R/[R, R]$ and $\pi_0 sl(R)^+ = 0$. Now Theorem 4.2 yields a fibration

$$sl(R)^+ \longrightarrow gl(R)^+ \longrightarrow R/[R, R].$$

which shows that $sl(R)^+$ is the 0-connected cover of $gl(R)^+$. □

The center of the Steinberg algebra. In order to compute $HH_1^{\infty}(R)$ and $HC_1^{\infty}(R)$, we use the Steinberg Lie, respectively the Steinberg Leibniz, algebra $st(R)$ and the following square zero extension:

$$Z(st(R)) \longrightarrow st(R) \longrightarrow sl(R).$$

This is the universal central extension of the perfect algebra $sl(R)$. In particular $st(R)$ is superperfect, meaning that $H_1^Q(st(R)) = 0$. □

PROPOSITION 5.2. *Let R be an associative k-algebra. Then $HH_1^{\infty}(R)$ is isomorphic to the center of the Steinberg Leibniz algebra $Z(stl(R)) \cong H_1^Q(sl(R))$. For $2 \leq i \leq 3$, $HH_i^{\infty}(R)$ is isomorphic to $H_i^Q(stl(R))$, the Quillen homology of the Steinberg Leibniz algebra in the category of $\mathcal{L}eib_{\infty}$-algebras.*

Proof. As $sl(R)^+$ is the 0-connected cover of $gl(R)^+$ by the preceding proposition, we have an isomorphism $\pi_1 gl(R)^+ \cong \pi_1 sl(R)^+$. By the Hurewicz Theorem 2.1, this is isomorphic to $H_1^Q(sl(R))$.

Moreover Proposition 4.6 shows that $Z(stl(R)) \to stl(R)^+ \to sl(R)^+$ is a fibration. Both $sl(R)$ and $stl(R)$ are perfect algebras, so their plus-constructions are 0-connected. Actually $stl(R)^+$ is even 1-connected since $H_1^Q(stl(R)) = 0$. The homotopy long exact sequence allows now to conclude that $\pi_1 sl(R)^+ \cong Z(stl(R))$.

As $stl(R)^+$ is the 1-connected cover of $gl(R)^+$, the Hurewicz Theorem 2.1 tells us that the next two Quillen homology groups coincide with the corresponding homotopy groups. □

The same arguments apply in the category of homotopy Lie algebras as well.

PROPOSITION 5.3. *Let R be an associative k-algebra. Then* $HC_1^\infty(R)$ *is isomorphic to* $Z(st(R)) \cong H_1^Q(sl(R))$. *For* $2 \leq i \leq 3$, $HC_i^\infty(R)$ *is isomorphic to* $H_i^Q(st(R))$, *the Quillen homology of the Steinberg Lie algebra in the category of* $\mathcal{L}_\infty$*-algebras.*

As explained in the first section there is an isomorphism over Q between the Quillen homology H_*^Q and H_{*+1}^{Lie}, respectively H_{*+1}^{Leib}. Together with the fact that the theories up to homotopy coincide with their classical analogues over Q, the three computations we made above yield the following isomorphisms.

COROLLARY 5.4. *Let R be an associative algebra over Q. Then*

(1) $HC_0(R) \cong HH_0(R) \cong R/[R, R]$,
(2) $HC_1(R) \cong H_2^{Lie}(sl(R))$, $HH_1(R) \cong H_2^{Leib}(sl(R))$,
(3) $HC_2(R) \cong H_3^{Lie}(st(R))$, $HH_2(R) \cong H_3^{Leib}(st(R))$,
(4) $HC_3(R) \cong H_4^{Lie}(st(R))$, $HH_3(R) \cong H_4^{Leib}(st(R))$.

Computation (1) is well known and trivial. The only reason why it does not appear in [19] is that $sl(R)$ is used there instead of $gl(R)$. Computations (2), (3) and (4) are non trivial results (for (2) and (3) we refer to [17] for Lie algebras and to [24, Corollary 4.5] and [13, Theorem 2.5] for Leibniz algebras). Notice that the results of Kassel–Loday, as well as those of Loday–Pirashvili and Gnedbaye, actually hold over any ring, which proves that $HC_n^\infty(R) \cong HC_n(R)$ and $HH_n^\infty(R) \cong HH_n(R)$ in full generality for $n \leq 2$.

Hochschild and cyclic homology both enjoy Morita invariance, and these homology theories are well behaved with respect to products. These facts however are not obvious (see [22, Theorems 1.2.4 and 2.2.9] for Morita invariance). In the case of our homotopy versions, they are straightforward to check.

Morita invariance. These theories are obviously Morita invariant since $gl(gl(R))$ is isomorphic to $gl(R)$. Hence we have $HC_*^\infty(gl(R)) \cong HC_*^\infty(R)$ and $HH_*^\infty(gl(R)) \cong HH_*^\infty(R)$.

Products. Let R and S be two associative k-algebras, and form the product in the category of associative algebras $R \times S$. We want to compute $HC_*^\infty(R \times S)$ and $HH_*^\infty(R \times S)$. Observe that $gl(R \times S)$ is isomorphic as a Lie algebra to the product $gl(R) \times gl(S)$. As nullifications preserve products (this is a consequence of the fiberwise localization [6]) one has:

PROPOSITION 5.5. *Let R and S be two associative k-algebras. Then:*

(i) $HC_*^\infty(R \times S) \cong HC_*^\infty(R) \oplus HC_*^\infty(S)$
(ii) $HH_*^\infty(R \times S) \cong HH_*^\infty(R) \oplus HH_*^\infty(S)$.

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