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The two-sided power distribution for the treatment of the uncertainty in PERT *

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Abstract. The aim of this paper is to include the Two-Sided Power (TSP) distribution in the PERT methodology making use of the advantages that this four-parameter distribution offers. In order to be completely determined, a distribution of this type needs, the same as the beta distribution, a new datum apart from the three usual values a (pessimistic), m (most likely) and b (optimistic). To solve this question, when using the beta distribution in the PERT context, we are looking for the maximum similarity with the normal and so it is required that the distribution has the same variance as the normal or its same kurtosis, giving rise to the constant variance and mesokurtic families, respectively. Nevertheless, while this approach can be only applied to the beta distribution for some values in the range of the standardized mode, in the case of the TSP distribution this methodology leads always to a solution. A detailed analysis comparing the beta and TSP distribution based on their PERT means and variances is presented indicating better results for the second.

Key words: Beta, TSP distribution, PERT, mesokurtic, constant variance

1. Introduction

Recently, van Dorp and Kotz (2002a, 2002b) have introduced the so-called *Two-Sided Power* (TSP) distribution which is a generalization of the triangular one. This distribution, the same as the beta, belongs to the Pearson system and is defined in the following way. Let X be a random variable that follows a TSP distribution.

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Then the density function of X is given by:

$$f(x/a,m,b,n) = \begin{cases} \dfrac{n}{b-a}\left(\dfrac{x-a}{m-a}\right)^{n-1}, \text{ if } a < x \leq m \\ \dfrac{n}{b-a}\left(\dfrac{b-x}{b-m}\right)^{n-1}, \text{ if } m \leq x < b \end{cases}. \tag{1}$$

This distribution, denoted by TSP(a,m,b,n), with $a \leq m \leq b$ and $n > 0$, verifies the following properties:

1. If $n > 1$, then the mode of this distribution is m and the value assigned by the density function is $\frac{n}{b-a}$.
2. If $0 \leq n \leq 1$ and $a < m < b$, then the mode is a or b and $f(\cdot/a,m,b,n)$ converges to infinity when x tends to a or b.
3. If $n = 1$, then the TSP degenerates into the uniform distribution.
4. If $n = 2$, the TSP becomes into a triangular distribution with parameters a, m and b.
5. Finally, if $a = 0$ and $m = b = 1$, then $f(\cdot/a,m,b,n)$ is a power distribution and, if $a = m = 0$ and $b = 1$, its reflection would be obtained.

In this paper, the problem of finding the value of n from the three expert's subjective estimations is solved. In the context of PERT we are interested in the bell-shaped and unimodal distributions, so we will work with the TSP distributions verifying the condition $n > 1$. Nadarajah (2002) corresponds with van Dorp and Kotz where he proposes them a much more flexible distribution, but van Dorp and Kotz highlight the intuitive meaning of n, because the expected value of X has the following expression:

$$E(X) = \frac{a + (n-1)m + b}{n+1}. \tag{2}$$

Thus, in order to obtain the expected value of the random variable, the endpoints a and b are weighted by $\frac{1}{n+1}$, while the mode m is weighted by $\frac{n-1}{n+1}$ (observe that the sum of the weights is then 1). In our opinion this property places the TSP distribution in the context of PERT. In this way, there exist some previous papers from McCrimmon and Ryavec (1964) and Johnson (1997) trying to analyze the possibilities of replacing the beta distribution with the triangular one.

On the other hand, it is well-known that in the classical expression of PERT :

$$E(X) = \frac{a + km + b}{k+2}, \text{var}(X) = \frac{(b-a)^2}{(k+2)^2}, \tag{3}$$

the value 4 is usually assigned to the parameter k. In this way, the question, presented by Sasieni (1986), on the weighting of the modal value in (3), the initial answers from Gallagher (1987) and Littlefield and Randolph (1987) and the later works of Kamburowski (1997), Herrerías, García and Cruz (1999), García, Cruz and Herrerías (2003), and Herrerías, García and Cruz (2003) have stated the conditions for several subfamilies of beta distributions.

The question is the following: with the three classical estimates of PERT, a, m and b, it is impossible to determine a unique beta distribution, because this is a four-parameter distribution. So, in order to solve this problem, we can opt either to ask for more information to the expert (García, Cruz and Andújar, 1998; García, 1999; García and Cruz, 2001) or to introduce some restrictions on the family of beta distributions.

In this way, by imposing the condition that the beta distribution has the same kurtosis as the normal one (i.e., $\beta_2 = 3$), we would obtain the so-called *mesokurtic family of beta distributions*. In this case, the equation relating the value of k with the standardized mode M is:

$$k^3(5M^2 - 5M + 1) + k^2(16M^2 - 16M + 2) - 5k - 4 = 0. \tag{4}$$

Given the values a, m and b, starting from $M = \frac{m-a}{b-a}$, it is well-known (see García, Cruz and Herrerías, 2003) that we will be able to solve the cubic equation (4) and obtain a unique value of k, provided that $0 \leq M \leq 0.2763933\ldots$ or $0.7236067 \cdots \leq M \leq 1$.

On the other hand, the normal distribution has a 99.7% of mass of probability between $\mu - 3\sigma$ and $\mu + 3\sigma$, that is to say, its range is about six times the standard deviation (see Yu Chuen-Tao, 1974). So, by imposing the condition that the standardized beta distribution has a standard deviation $\frac{1}{6}$ (i.e., $\sigma^2 = \frac{1}{36}$), we would obtain the so-called *constant variance family of beta distributions* (Herrerías, García y Cruz, 2003). In this case, the equation relating the value of k with the standardized mode M is:

$$k^3 + k^2[7 - 36(M - M^2)] - 20k - 24 = 0. \tag{5}$$

This cubic equation (5) has always a unique solution $k > 0$, for every value $0 < M < 1$. It is well-known that the unique intersection of both families holds for $k = 4$ which is the value assigned to the beta distribution of the classical PERT. Finally, the so-called *Caballer family* is composed by the beta distributions $\beta(p, q, a, b)$ with $p = h \pm \sqrt{2}$ and $q = h \mp \sqrt{2}$, $h > 0$. The study of these three families can be seen in García, Cruz and Herrerías (2003).

The organization of this paper is as follows: In Sect. 2 the different families of the Standard Two-Sided Power (STSP) distributions are presented showing the advantages over their corresponding beta distributions. In particular, the mesokurtic and the constant variance families of distributions STSP are introduced here. In Sect. 3 the means and variances estimated from the values a, m and b are compared, using different subfamilies of betas and STSP distributions. Finally, Sect. 4 summarizes and concludes.

2. The different families of STSP distributions

In the field of PERT and starting from the three usual values a, m and b whose meaning is well-known, it would be impossible to determine a unique TSP distribution, because this is a four-parameter (a, b, m and n) distribution. Hence, it is necessary to restrict the choice of a unique TSP distribution to some of its subfamilies.

Standardizing the random variable X, that is to say making the transformation

$$T = \frac{X-a}{b-a},$$

we obtain a new random variable T whose density function is given by:

$$f(t/M,n) = \begin{cases} n\left(\dfrac{t}{M}\right)^{n-1}, & \text{if } 0 < t \leq M \\ n\left(\dfrac{1-t}{1-M}\right)^{n-1}, & \text{if } M \leq t < 1 \end{cases} \tag{6}$$

and whose distribution function is:

$$F(t/M,n) = \begin{cases} M\left(\dfrac{t}{M}\right)^{n}, & \text{if } 0 \leq t \leq M \\ 1-(1-M)\left(\dfrac{1-t}{1-M}\right)^{n}, & \text{if } M \leq t \leq 1 \end{cases}. \tag{7}$$

This standardized distribution will be called the *Standard Two-Sided Power* (STSP) distribution. On the other hand, the (ordinary) moments of order p are:

$$\mu'_p = E(T^p) = \frac{nM^{p+1}}{n+p} + \sum_{i=0}^{p}(-1)^i \binom{p}{p-i}\frac{n}{n+i}(1-M)^{i+1}, \tag{8}$$

being

$$E(T) = \frac{(n-1)M+1}{n+1} \tag{9}$$

and

$$\mathrm{var}(T) = \frac{n-2(n-1)M(1-M)}{(n+2)(n+1)^2} \tag{10}$$

the expected value and the variance, respectively, of the random variable T.

Using the well-known relationships between the central moments:

$$\mu_p = E[T-E(T)]^p \tag{11}$$

and the ordinary moments (see Stuart and Ord, 1994), the coefficient of kurtosis, β_2, of the standardized random variable can be calculated according to M and n, remaining:

$$\beta_2 = \frac{n+2}{(n+3)(n+4)} \cdot \\ \cdot \frac{AM^4+BM^3+CM^2+DM+E}{4(n-1)^2M^4-8(n-1)^2M^3+[4(n-1)^2+4n(n-1)]M^2-4n(n-1)M+n^2}, \tag{12}$$

being A, B, C, D and E polynomials in n of the following form:

$$\begin{cases} A = 24n^3 - 72n^2 + 120n - 72 \\ B = -48n^3 + 144n^2 - 240n + 144 \\ C = 60n^3 - 120n^2 + 156n - 96 \\ D = -36n^3 + 48n^2 - 36n + 24 \\ E = 9n^3 - 3n^2 + 6n \end{cases}. \tag{13}$$

It can be easily shown that, in the case in which $n = 1$ (uniform distribution), starting from the expression (12), the coefficient of kurtosis is $\frac{9}{5}$ and, in the case in which $n = 2$ (classical triangular distribution), the coefficient of kurtosis is $\frac{12}{5}$.

Defining the *family of constant variance* as the set composed by the STSP distributions with the same variance as the normal distribution (i.e., $\frac{1}{36}$), in the case of working with standardized random variables, the following equation:

$$n^3 + 4n^2 + (-72M^2 + 72M - 31)n + (72M^2 - 72M + 2) = 0 \tag{14}$$

holds.

This equation will allow us to obtain, for every value $0 < M < 1$, a unique value of $n > 1$. So, we can state that, given the three usual values a, m and b, then a unique unimodal STSP distribution of constant variance is determined. This result allows the use of this family in the context of PERT.

On the other hand, we can define the *mesokurtic family* as the set of STSP distributions whose coefficient of kurtosis (β_2) is equal to 3. Then, making the expression (12) equal to 3 and re-arranging some terms, we would obtain the following equation:

$$an^4 + bn^3 + cn^2 + dn + e = 0, \tag{15}$$

being a, b, c, d and e polynomials in M of the following form:

$$\begin{cases} a(M) = 2M^4 - 4M^3 + 6M^2 - 4M + 1 \\ b(M) = -14M^4 + 28M^3 - 22M^2 + 8M - 1 \\ c(M) = -2M^4 + 4M^3 - 22M^2 + 20M - 6 \\ d(M) = 62M^4 - 124M^3 + 94M^2 - 32M + 2 \\ e(M) = -48M^4 + 96M^3 - 56M^2 + 8M \end{cases}. \tag{16}$$

It can be shown that, for every $0 < M < 1$, the equation (15) has a unique solution verifying the condition $n > 1$, which allows us to state that there always exists a STSP unimodal distribution belonging to the mesokurtic family. This result improves the obtained one by the mesokurtic family of beta distributions in which it is impossible to obtain a solution for the values of

$$0.2763933 \cdots < M < 0.7236067 \ldots$$

In Fig. 1, we can see the different values of n for the corresponding values $0 < M < 1$, hundredth by hundredth. As we can observe, the STSP distribution of constant variance increases the weighting n of the mode as the value of M tends to the endpoints 0 and 1, while the mesokurtic STSP distribution behaves in the opposite way, increasing the weighting n of the mode, as the value of M is approaching the center of the interval $0 < M < 1$. Thus, the riskier the expert when supplying the modal value, the greater the value of n, that is to say, the STSP distribution of constant variance weights more the modal value, while with the STSP mesokurtic distribution occurs just the opposite.

In other words, for every $0 < M < 1$, it will be possible to choose a mesokurtic STSP distribution. Kotz and van Dorp (2004) present a symmetric and mesokurtic

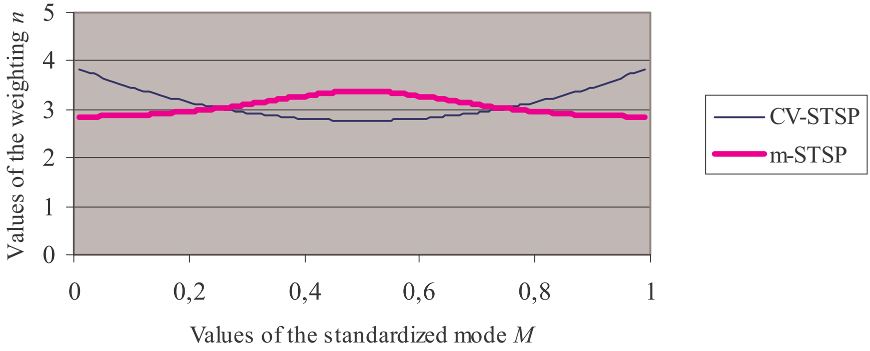

Fig. 1

STSP distribution, taking $a = -\frac{1}{2}$, $b = \frac{1}{2}$, $m = 0$ and $n = 3.37228$, showing that the joint characteristics of symmetry and kurtosis equal to 3 are not exclusive of the normal distribution. In this paper we state that there always exists a mesokurtic STSP distribution, for every value of M, and so the family of mesokurtic STSP distributions is more flexible than the family of mesokurtic beta distributions, whereby this is significantly improved. In this way, recall that the reason for the classical weighting $k = 4$ in the PERT methodology is in the intention of find a bounded and non necessarily symmetric distribution as similar as possible to the normal distribution (variance and kurtosis).

After the last results, we can state that the STSP distribution improves the beta one in order to reach the previous objective. In effect, if, starting from the usual values a, m and b, we aim to obtain a beta distribution with kurtosis equal to 3, this will not be possible for values $0.2763933\cdots < M < 0.7236067\ldots$. Nevertheless, for every $0 < M < 1$, we can obtain both a STSP distribution with constant variance and a mesokurtic one.

On the other hand, solving the system composed by equations (14) and (15), the unique solutions for M and $n > 1$ are:

$$M = 0.747133\ldots, n = 3.02344\ldots$$

and

$$M = 0.252867\ldots, n = 3.02344\ldots,$$

that correspond to STSP distributions whose distribution functions can be seen in Fig. 2 (these are the unique distributions verifying simultaneously $\sigma^2 = \frac{1}{36}$ and $\beta_2 = 3$).

In conclusion, we can state that the classical distribution of PERT with an underlying beta distribution (Herrerías, García and Cruz, 2002) is obtained with the value $k = 4$, while the "classical" distribution of PERT with an underlying STSP distribution will be obtained with the value $n = 3.02344\ldots$ Thus, the expected value of this distribution, given some initial values a, m and b, would be:

$$E(X) = \frac{a + 2.02344\ldots m + b}{4.02344\ldots}. \tag{17}$$

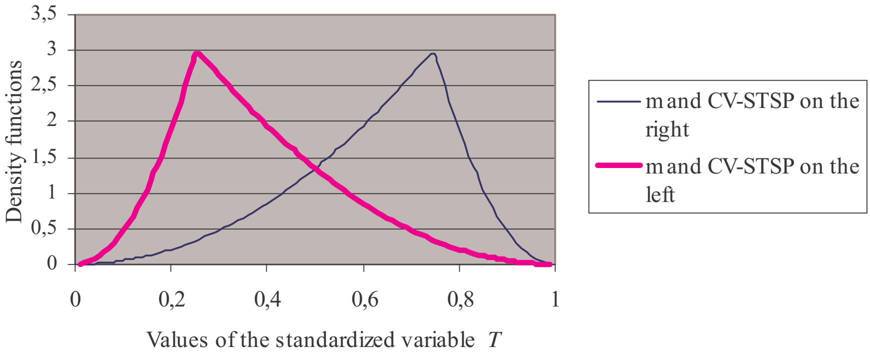

Fig. 2

3. Study of means and variances estimated from the values a, m and b, using subfamilies of betas and STSP distributions

By comparing the expressions (2) and (3), the relationship between n and k is immediate:

$$n - 1 = k$$

or

$$n = k + 1.$$

Hence, we can write the equations of the expected value and the variance of the standardized beta, according to n:

$$\begin{cases} E(X) = \dfrac{(n-1)M + 1}{n+1} \\ \operatorname{var}(X) = \dfrac{(n-1)^2 M(1-M) + n}{(n+2)(n+1)^2} \end{cases}, \tag{18}$$

while the expressions of the mean and the variance for the STSP distribution are those of expressions (9) and (10), respectively. This would allow us to carry out a compared study of each one of the subfamilies, obtaining their intersection and relative position. We will start with the weighting.

First, observe that the weighting values of the mesokurtic beta have an exponential increase with two vertical asymptotes at $M = 0.2763933\ldots$ and $M = 0.7236067\ldots$ and that, moreover, they are greater than those of the mesokurtic STSP.

The beta of constant variance behaves the same as the mesokurtic beta and the mesokurtic STSP, increasing the weighting as the value of M approaches the center of the interval, $M = 0.5$, that is to say, it attributes the value of maximum weighting to the values of smaller predicament for the expert; however, the STSP distribution of constant variance acts on the contrary, increasing the weighting as M approaches the endpoints and, therefore, the expert takes a bigger predicament.

On the other hand, the weighting values $(k = n - 1)$ in the beta of constant variance are always greater than the weighting values in the STSP of constant

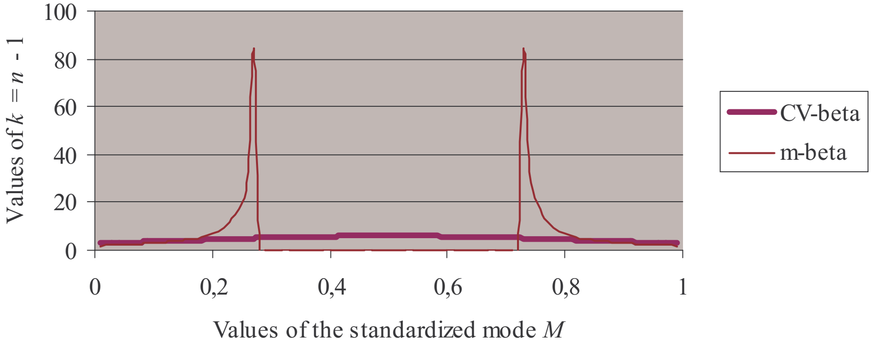

Fig. 3

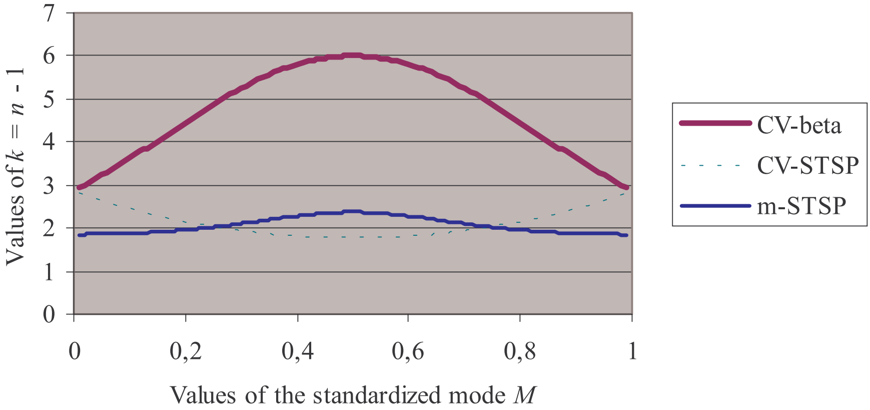

Fig. 4

variance and in the mesokurtic STSP. In definitive, we can conclude that, starting from a, m and b, for the beta distribution it is not always possible to approach the kurtosis of the normal one (3), while the STSP can get it for every triple a, m and b, even with smaller weighting values. This allows us to conclude that the distribution STSP is more flexible in the context of PERT than the beta, since it is always possible to obtain a generalized triangular distribution of van Dorp and Kotz with the same kurtosis as the normal one, for every triple a, m and b, assigning to m a weighting smaller than with the beta distribution in those cases in which it is possible to make it with this last distribution.

3.1. Analysis of the different estimates

If we work with the standardized variable $T = \frac{X-a}{b-a}$ and we give the values $0.01, 0.02, \ldots, 0.99$ to the mode, we would have, for each one of the underlying distributions (beta of constant variance – CV-beta –, STSP of constant variance -CV-STSP- and mesokurtic STSP – m-STSP –), the observations (x_i, y_i^j), where $i = 1, \ldots, 99$ and $j = 1, 2, 3$, and:

- x_i are the modal values,
- y_i^j are the estimates of the mean,
- $j = 1$ indicates that the underlying distribution is the CV-beta,
- $j = 2$ indicates that the underlying distribution is the CV-STSP,
- $j = 3$ indicates that the underlying distribution is the m-STSP.

In all cases, it is graphically observed that the relationship between the standardized mode and the estimate of the mean starting from it is practically linear, whereby it is logical to propose a linear fitting between both parameters of the following form:

$$y_i^j = \theta_1^j + \theta_2^j x_i,$$

where $i = 1, 2, \ldots, 99$ and $j = 1, 2, 3$. These fittings lead to the following solutions:

$$\mu(\text{CV-beta}) = 0.17\hat{3} + 0.65\hat{3}M$$

$$\mu(\text{CV-STSP}) = 0.228 + 0.544M$$

$$\mu(\text{m-STSP}) = 0.255 + 0.490M$$

with a 99 percent significance level. A contrast of structural change of Chow (1960) allows to be show that the three solutions are statistically different, i.e. that they come from three different populations. Said otherwise, the hypothesis that:

$$\theta_1^j = \theta_2^j \text{ and } \theta_2^j = \theta_2^k,$$

if $j \neq k$, with $j, k \in \{1, 2, 3\}$, is rejected. The statistical meaning of this hypothesis is that the estimates of the mean using the three former fittings are essentially different.

Moreover, it can be shown that, in all cases, $\theta_1^j + \theta_2^j = 1$ which implies that, for $M = 0.5$, the estimated value for the mean is 0.5 and that, if we consider the original values (a, m and b), the weightings of the optimistic and pessimistic values are the same.

On the other hand, the estimated variance starting from the mode, in the case of the m-STSP distribution, presents a parabolic relationship, which can be shown carrying out a quadratic regression:

$$\text{var(m-STSP)} = 0.4121 - 0.7536M + 0.7536M^2,$$

with a 99 percent significance level. As it can be observed, this function has a minimum at $M = 0.5$ and increases in a parabolic way as M approaches the endpoints of the interval. That is to say, when the expert makes that the modal value approaches the endpoints of the interval (pessimistic and optimistic values) previously supplied by himself, we will consider that said expert is not very reliable. This distribution exhibits this characteristic when reaching the biggest variance as the mode approaches the endpoints, situation that the models of constant variance do not capture. So the use of this model (m-STSP) is more logical in the PERT methodology. As it is well-known, the m-beta could not be used because it had not any solution for all the values of the standardized mode.

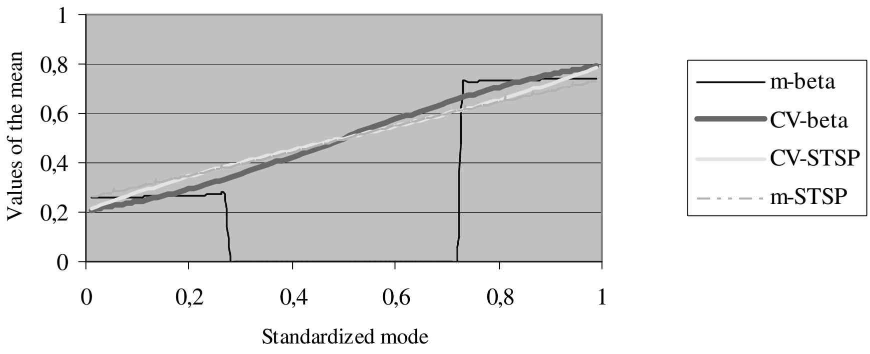

Fig. 5

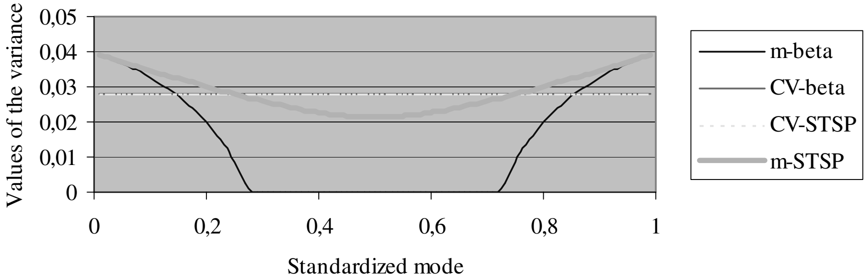

Fig. 6

3.2. Comparison of means

Figure 5 shows the estimated means for different values $0 < M < 1$. The points of intersection have the following coordinates:

Type of distribution		mesokurtic	
		Beta	STSP
of constant variance	Beta	(0.853553,5) (0.146437,5)	–
	STSP	(0.9254,3.55562) (0.0746,3.55562)	(0.747133,3.02344) (0.252867,3.02344)

In order to study the position of the estimated values, we can observe Table 1.

This way, it is said that a distribution is *more moderate in mean* when its estimated mean, coming from this distribution, is nearer the value 0.5, as in the case of the standardized normal N(0,1). Therefore, we can conclude that the STSP distribution is more moderate in mean than the beta in all cases and that, up to in the interval $0.252867 < M < 0.747133$, the mesokurtic STSP is more moderate than the STSP of constant variance.

Table 1 Distributions in order of moderation

Intervals of M	Distributions in order of moderation			
$0 < M < 0.0746$	m-STSP	m-beta	CV-STSP	CV-beta
$0.0746 < M < 0.146437$	m-STSP	CV-STSP	m-beta	CV-beta
$0.146437 < M < 0.252867$	m-STSP	CV-STSP	CV-beta	–
$0.252867 < M < 0.747133$	CV-STSP	m-STSP	CV-beta	–
$0.747133 < M < 0.853553$	m-STSP	CV-STSP	CV-beta	–
$0.853553 < M < 0.9254$	m-STSP	CV-STSP	m-beta	CV-beta
$0.9254 < M < 1$	m-STSP	m-beta	CV-STSP	CV-beta

Table 2 Distributions in order of conservatism

Intervals of M	Distributions in order of conservatism			
$0 < M < 0.146447$	m-STSP	m-beta	CV-STSP	CV-beta
$0.146437 < M < 0.252867$	m-STSP	CV-beta	CV-STSP	m-beta
$0.252867 < M < 0.747133$	CV-STSP = CV-beta		m-STSP	-
$0.747133 < M < 0.853553$	m-STSP	CV-beta = CV-STSP		m-beta
$0.853553 < M < 1$	m-STSP	m-beta	CV-STSP = CV-beta	

3.3. Comparison of variances

In the context of PERT (Taha, 1981; Herrerías, 1989), distributions with maximum variance are preferable, since, in situations of uncertainty, it is better to approximately guess right than to make a mistake when reducing the variance. This way, it is said that a distribution is *more conservative* when its estimated variance, coming from this distribution, is greater. As we can observe in Fig. 6, it is possible to build Table 2.

This allows us to conclude that, given the three habitual estimates a, m and b of PERT, if we need to choose a distribution to estimate the mean and the variance, we can opt for that of maximum moderation and bigger conservatism, being always preferable the STSP distribution to the beta. More specifically, the distribution STSP of constant variance is the best option in the interval $0.252867 < M < 0.747133$ for the standardized mode, and the mesokurtic STSP in the rest of the interval $0 < M < 1$.

This statement can be supported by the behavior of the kurtosis. In effect, by comparing the family of STSP distributions according to n and M with the family of beta distributions according to $k = n - 1$ and M, the first of them allows to choose a distribution with more kurtosis than the second one. More concretely, in the case of symmetric distributions (Fig. 7), the STSP distribution surpasses the beta distribution in kurtosis, for all values of n. In effect, if we calculate the kurtosis of the STSP distribution according to n in the case in which $M = 0.5$ (symmetry):

$$\beta_2 = 6\frac{(n+1)(n+2)}{(n+3)(n+4)},$$

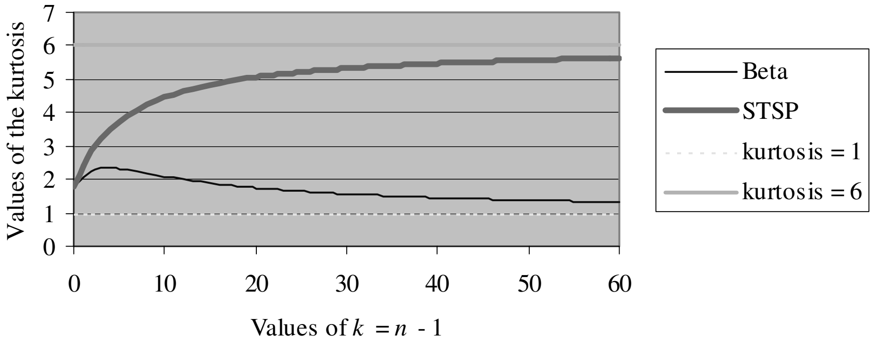

Fig. 7

while the kurtosis of the beta distribution according to k, also in the case in which $M = 0.5$:

$$\beta_2 = \frac{(k+3)(k^3+32k^2+60k+48)}{(k+4)(k+5)(k+2)^2}.$$

Observe that, in symmetric distributions, the value of the kurtosis of the beta distribution is less than that of the normal one (3), while, for the STSP distribution, we can find weighting values leading to coefficients of kurtosis greater or less than that of the normal one (see Fig. 7). Therefore, this distribution can be an alternative

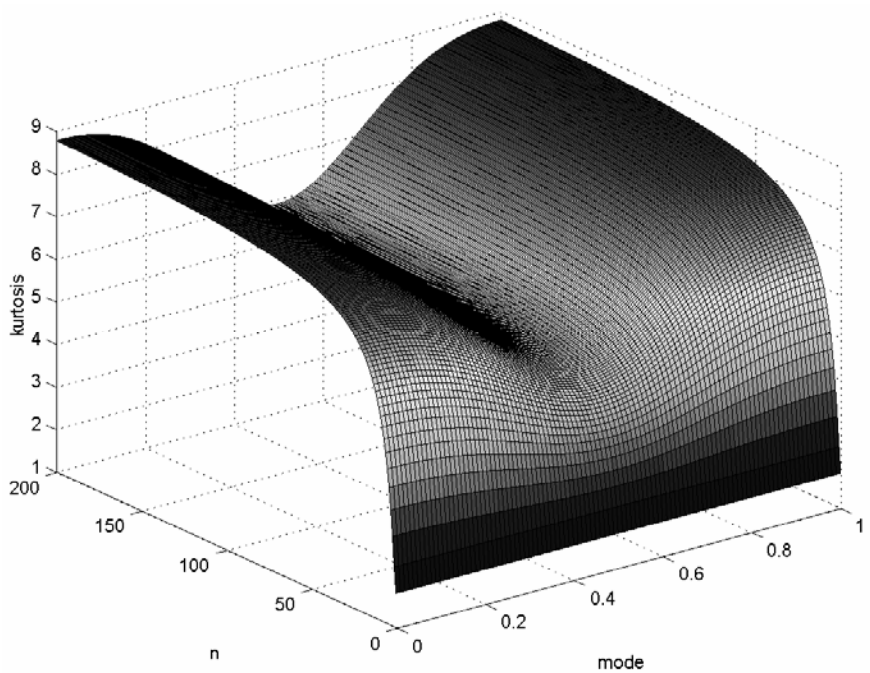

Fig. 8

to the normal and other distributions when trying to fit sample distributions that, being symmetrical, show a leptokurtic sample behavior.

Figure 8 shows a three-dimensional image of the coefficient of kurtosis, according to M and n, of a symmetric STSP distribution. It can be observed that, when the distribution is asymmetric, the kurtosis get the maximum value of 9, while, when it is symmetric, its maximum value is only 6.

4. Conclusion

In the context of PERT, the family of STSP distributions always improves the family of beta distributions due to the following reasons:

1. In the answer to Nadarajah, van Dorp and Kotz supply an intuitive interpretation for the value of n, because $n-1$ is the weighting of the modal value. Nevertheless, this parameter is not intuitive in the PERT context, because it is not possible to ask the expert about this value. In effect, there is not a question with an easy interpretation for the expert which allows us to obtain the value of n. In this paper, by restricting the underlying distribution to the mesokurtic or constant variance TSP families, the value of n can be directly obtained from the three expert's subjective estimations.
2. It is always possible to choose a mesokurtic STSP distribution, while it does not happen the same with the beta.
3. Besides the classical beta distribution, there would also exist a "classical" STSP, for $n = 3,02344\ldots$
4. As it is shown in the paper, the distribution STSP is more moderate in mean and more conservative in variance for all values of the standardized mode.
5. TSP distributions offer more flexibility in terms of their kurtosis values, which could be of use in other applications as well (for the comparison of moment ratio diagrams of the beta and STSP distribution, see van Dorp and Kotz (2002b)).

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