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Theory of portfolios: New considerations on classic models and the Capital Market Line

Salvador Cruz Rambaud [a,*], José García Pérez [b], Miguel Angel Sánchez Granero [c], Juan Evangelista Trinidad Segovia [a]

[a] *Departamento de Dirección y Gestión de Empresas, Universidad de Almería, Carretera Sacramento s/n., La Cañada de San Urbano, Almería 04120, Spain*

[b] *Departamento de Economía Aplicada, Universidad de Almería, Almería, Spain*

[c] *Departamento de Geometría y Topología, Universidad de Almería, Almería, Spain*

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Abstract

The aim of this paper is to present an alternative method to obtain the efficient portfolio in Roy's model starting from the concepts of critical return and risk which are introduced here. This method will permit resolution of the main problem of Roy's model, that is to say, the impossibility of obtaining the portfolio in certain situations. The introduction of these new concepts will also allow the detection and solution of a problem associated with the calculation of the Capital Market Line. This work concludes by considering the possibility that investors allocate part of their budget for buying zero-risk assets.

Keywords: Markowitz; Tobin; Sharpe; Lintner; Capital Market Line; Efficient frontier; Portfolio; Return; Risk

1. Introduction

The publication in 1952 of the work *Portfolio Selection* in *The Journal of Finance* (Markowitz, 1952) marked, without doubt, the beginning of the classic theory of portfolios. This work, written by Harry Markowitz, was a milestone, because, for the first time, the relationship between return and risk was included in a financial model and the concept of the rational behaviour of the investor was introduced (see also Markowitz, 1959). Other authors, like Sharpe (1963) and Lintner (1964), starting from the ideas of Markowitz, followed the development of the theory of portfolios, giving rise to the *Diagonal Model* and the *Capital Market Line* (*CML*).

Parallel to the publication of *Portfolio Selection*, was the appearance, in the journal *Econometrica*, of the paper entitled *Safety First and the Holding of Assets*, written by Roy (1952). This

* Corresponding author.

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work is based on exactly the same premises as the work of Markowitz, the rational behaviour of the investor, although it remained unnoticed at the time.

Roy's model has been shown as highly flexible, since the variables of choice can be either the return or the risk, [1] and as very intuitive in its consideration of the risk, in the way that it is defined as the probability of not reaching the expected reservation return. From a practical point of view, in comparison with the classic models, better estimates of the return for the same level of risk are obtained and vice versa; moreover a closed and complete mathematical formulation of the model can be found, that is, a mathematical expression to obtain Roy's frontier, the optimum portfolio, the return and the risk. In the Thirties, Roy's criterion was introduced to deal with the optimal portfolio choice for an insurance company (De Finetti, 1940).

Nevertheless, the application of this model is not without problems. These are:

1. If the model is applied in its original form, the number of necessary estimates is very high, which complicates excessively the solution of the problem. However, in Cruz Rambaud et al. (1999a, op. cit.), the approaches of Sharpe (1963, op. cit.) were adapted to this model and so the number of estimates was reduced as in the Diagonal Model.
2. As a consequence of the mathematical approach, there exist some values of the return and of the risk for which this model is not operational. In the original model, the value of the minimum return is fixed and, from it, a line tangential to the Roy's frontier is traced, establishing the risk and obtaining the optimum composition of the portfolio at the point of tangency. This can mean that, for certain values of the minimum return, a point of tangency may not be obtained, as a consequence of the shape of the frontier. The case in which the risk is fixed is analogous (see Section 2.2).
3. In the original model, assets without risk cannot be introduced, because the matrix of variances–covariances would not be invertible, so the model would not have any solution.

The main objective of this work is to propose a solution to these last two limitations of the model. Thus, the organization of this paper is as follows: in the first part of Section 2 the main contributions of Roy's model are reviewed and then, in the second part, the concepts of critical return (d_c) and risk (R_c) are introduced. Solution (4) of Roy's model allows the definition of the debt function of a portfolio and, according to the debt capacity of an investor, a point of maximum debt of the Roy's frontier is determined. So, when the return d is greater than d_c or the risk R is greater than R_c, it is shown that the composition of the portfolio is determined by the point (σ_E, m_E) of the Roy's frontier. In Section 3, the possibility that the investor will place part of his budget in assets without risk is introduced. In this case, the discussion about the return and the risk will be around r^*, the return of assets without risk, and R^*, an upper bound of the probability of not reaching d^*, when all the investment is in assets with risk.

2. A new method to obtain the optimum portfolio in Roy's model

2.1. Preliminaries: Roy's model

The original approach of Roy's model was the following. Let us denote r the random variable that represents the behaviour of the return ex-post, $m_r = m$ its expected value and $\sigma_r = \sigma$ its standard deviation; we call d the amount below which it is not desirable that r will be, that is, the reservation return.

The probability of r adopting a position below any given quantity is not known by the investor. Nevertheless, an upper bound of probability can

[1] In its original form, Roy's model obtained the optimum portfolio for a minimum chosen return; García Pérez et al. (1998, pp. 423–429) showed that the model can be also expressed according to the risk.

be calculated using the inequality of Tchebycheff: [2]

$$P(|r-m| \geqslant m-d) \leqslant \frac{\sigma^2}{(m-d)^2},$$

which implies that

$$P(r \leqslant d) \leqslant \frac{\sigma^2}{(m-d)^2}.$$

Taking into account that the distribution function of r is unknown, in order to minimize the probability of r being smaller than d, Roy proposed to minimize $\frac{\sigma^2}{(m-d)^2}$, which is equivalent to maximizing $\frac{m-d}{\sigma}$. Graphically, this is equivalent to maximizing the slope of the line tangent to the Roy's frontier (Fig. 1).

At this point, Roy predetermined the amount d that represents the minimum (reservation) return that the investor wishes to obtain, and once the point of tangency is calculated and substituted in the inequality of Tchebycheff, the probability of the return is below that amount, this number will be the measurement of the risk according to which we will obtain the minimum return d. In this way, the risk represents the probability of the return r being below the value d.

Roy proved that the frontier is given by the hyperbola

$$\frac{(AW^{-1}A')(BW^{-1}B') - (AW^{-1}B')^2}{BW^{-1}B'}\left(\sigma^2 - \frac{1}{BW^{-1}B'}\right) = \left(m - \frac{AW^{-1}B'}{BW^{-1}B'}\right)^2, \tag{1}$$

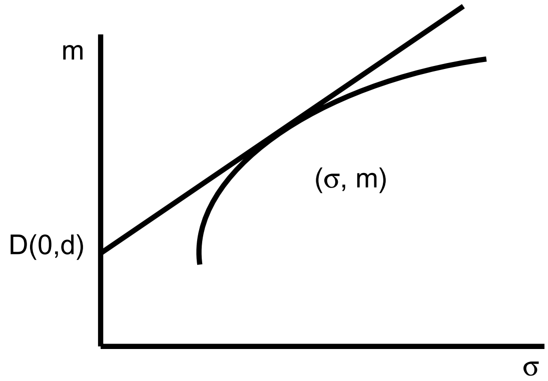

Fig. 1. Roy's model.

where $A = (r_1, \ldots, r_n)$ and r_i is the return on asset i; $B = (1, \ldots, 1)$; $X = (x_1, \ldots, x_n)$; W is the matrix of variances–covariances; $m = AX'$ and $\sigma^2 = XWX'$.

If we denote:

$$T = \frac{(AW^{-1}A')(BW^{-1}B') - (AW^{-1}B')^2}{BW^{-1}B'},$$

$$U = \frac{1}{BW^{-1}B'} \quad \text{and} \tag{2}$$

$$V = \frac{AW^{-1}B'}{BW^{-1}B'},$$

expression (1) could be written in the following form:

$$T(\sigma^2 - U) = (m - V)^2. \tag{3}$$

The optimum combination of assets is given by

$$X = \mu W^{-1}(A - dB), \tag{4}$$

where $A = (r_1, \ldots, r_n)$, being r_i the return on product i; $X = (x_1, \ldots, x_n)$ is the matrix of different combinations of products of the frontier of production possibilities; B is the matrix $(1, \ldots, 1)$; W is the matrix of variances–covariances; d the amount of the return chosen by the investor and μ is chosen so that $\sum_{i=1}^{n} x_i = 1$.

In the same way, the risk associated with the portfolio was determined by

$$R = \frac{1}{(A - dB)'W^{-1}(A - dB)}. \tag{5}$$

Another possibility, as is shown in García Pérez et al. (1998, op. cit.), consists of establishing the

[2] It is well known that Tchebycheff inequality is far from being accurate. Its relevance remains in its generality, not in its accuracy. Therefore the frontier taken into consideration is likely to be rather far from the correct but "unknown" one and so the reliability of Roy's model is, in principle, worse than that of Markowitz. More specifically, if the expectations, variances and covariances are known or, at least, satisfactorily estimated, the Markowitz's frontier is correct or satisfactorily estimated, while this cannot be said about Roy's frontier, owing to the established lack of precision of Tchebycheff inequality. Nevertheless, the inaccuracy of this inequality implies that the model is conservative, that is to say that the "true" risk is smaller than the one considered by the model.

risk R that one is prepared to support and thus to use the Roy's frontier to determine the reservation return associated with the given risk. This return was given by

$$d = V + \sqrt{\frac{UR}{1-TR}}\left(T - \frac{1}{R}\right) \tag{6}$$

and a combination of assets was determined by Eq. (4).

2.2. Calculation of critical values

The frontier (3) has an oblique asymptote defined by the equation

$$m = V + \sqrt{T}\sigma$$

which determines a risk $R_c = \frac{1}{T}$ and a return $d_c = V$, that we call *critical risk and return*, respectively (Fig. 2). This means that if we fix a minimum return $d > d_c$, there will not be any line tangent to the Roy's frontier crossing the point $(0, d)$ and so the optimum combination of assets will not be given by Eq. (4) and the risk associated with the said minimum return will not be determined by (5).

Analogously, if we fix a risk $R > R_c$, there will not be any straight line tangential to the Roy's frontier with slope $\frac{1}{\sqrt{R}}$, from which, in this case, Eqs. (4) and (6) will not be valid either, not having any problem in those cases in which $d < d_c$ and $R < R_c$.

On the other hand, it is evident that if $R < R_c$, the more favourable case will be given by Eqs. (4) and (6) and it should not be logical to consider any return less than the one given by (6). Similarly, if $d < d_c$, the more favourable case will be determined by Eqs. (4) and (5) and it should not be reasonable to have a risk greater than the one obtained in (5).

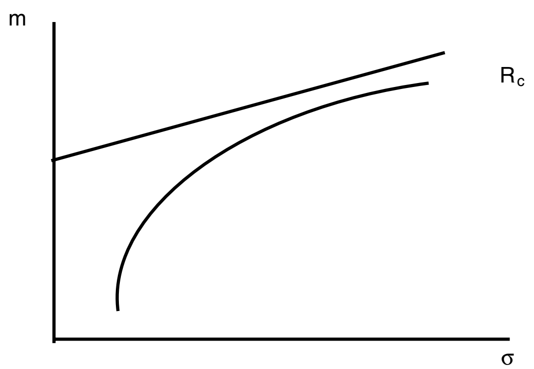

Fig. 2. Obtaining the critical return.

Usually, in practice, the critical risk is very low, but sometimes we are willing to support an upper risk to obtain a higher return. The solution (4) of the new model to determine the optimum portfolio in Roy's model allows that some x_i was negative in the vector X. Stated in another way, the optimum combination of the portfolio permits the possibility of a debt situation in some assets which compound it.

So, it is natural to find an expression of the total debt which is necessary to invest in the optimum portfolio. In effect, if $I = \{1, 2, \ldots, n\}$, we are going to denote:

$$J = \{j \in I / x_j > 0\}$$

and

$$K = \{k \in I / x_k < 0\}.$$

Obviously, the total debt is $-\sum_{k \in K} x_k$. However,

$$-2\sum_{k \in K} x_k = -\sum_{k \in K} x_k + \sum_{j \in J} x_j - \sum_{k \in K} x_k - \sum_{j \in J} x_j = \sum_{i \in I} |x_i| - 1.$$

Thus,

$$-\sum_{k \in K} x_k = \frac{1}{2}\left(\sum_{i \in I} |x_i| - 1\right). \tag{7}$$

For example, if the optimum portfolio requires that $x_1 = 2$, $x_2 = 3$, $x_3 = 1.5$, $x_4 = -3$ and $x_5 = -2.5$, the necessary debt is 550% of the amount at investor's disposal which is the half the complement to the unit of the sum of the absolute values of x_is:

$$5.5 = \frac{1}{2}(12 - 1).$$

In this way, the investor would have to carry out two simultaneous operations:

1. A stock credit sale for the amount of the debt function.

2. And, at the same time, a stock credit purchase for the same amount in the assets with a positive coefficient.

On the other hand, and for a concrete investor, we define the *debt capacity E* as the maximum percentage of debts acceptable by the investor, this rate being referred to the available capital. Any debt less than or equal to E will be allowed by the investor. Such a percentage depends on the investor being willing to get into debt in each asset, that, for his part, is according to the standard deviation of the portfolio.

The last paragraph justifies the following definition for a specific investor. We will call *debt function* to the following function of σ:

$$D(\sigma) = \sum_{i=1}^{n} |x_i(\sigma)| - 1.$$

That is, for a dispersion σ, the investor is willing to invest $x_1(\sigma), \ldots, x_n(\sigma)$ in the assets $1, \ldots, n$, respectively, from where we obtain its debt by means of expression (7).

Suppose that, for each σ, the combination X is uniquely determined. In any case, we are interested in the maximum debt acceptable by the investor, that is, that value of σ, σ_E, such that

$$D(\sigma) > E \text{ for every } \sigma > \sigma_E.$$

The existence of σ_E is guaranteed because $D(0) = 0$ and $\lim_{\sigma\to\infty} D(\sigma) = \infty$. This value, σ_E, determines a point (σ_E, m_E) in the Roy's frontier which will be called *point of maximum debt*, in such a way that any point in the frontier with a standard deviation greater than that of this point is not possible for the investor, because its possibilities of debt would be surpassed.

Fixing $R > R_c$ or $d > d_c$, the composition of the portfolio is determined by the point (σ_E, m_E) of the efficient frontier. To find this composition, we will use formula (4), for which we need to calculate the minimum return d_E and the risk R_E associated with the line $m = d_E + \frac{\sigma}{\sqrt{R_E}}$ tangential to the Roy's frontier at point (σ_E, m_E) (Fig. 3).

This tangent is given by the derivative of the frontier at this point, that is,

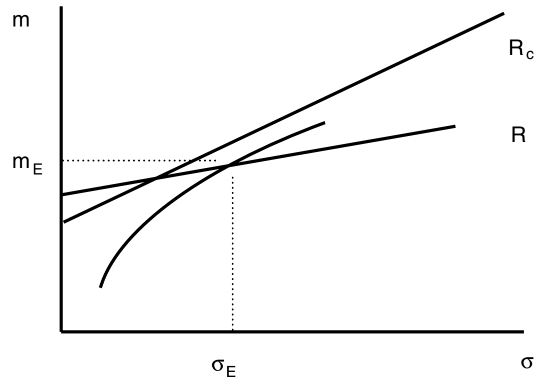

Fig. 3. Optimum portfolio when $R > R_c$ or $d > d_c$.

$$\frac{\partial m}{\partial \sigma} = \frac{1}{\sqrt{R_E}} = \frac{T\sigma_E}{m_E - V},$$

from where

$$d_E = m_E - \frac{T\sigma_E^2}{m_E - V} \tag{8}$$

and so the optimum combination for R is determined by

$$X = \mu W^{-1}(A - d_E B), \tag{9}$$

where μ is chosen such that $\sum_{i=1}^{n} x_i = 1$.

3. The problem of the incorporation of zero-risk assets

3.1. Preliminaries: Capital Market Line

A logical extension of the model of Markowitz is the introduction of the possibility for the investor to place part of his budget in assets without risk, or to borrow to achieve a certain leverage. This extension of the model of Markowitz was initiated by Tobin (1958), and later on by Sharpe (1963, op. cit.), and Lintner (1964, op. cit.).

The main contribution of Tobin was the Theorem of Separation, which stated that *the optimum portfolio formed by individual assets with risk does not depend on the attitude towards risk of the investor.*

Supposing homogeneous behaviour on behalf of the investors, all of them will buy the same securities and in the same proportion of the portfolio given by the market, although depending on the size of their budget. In an efficient market, the mere play of supply and demand will imply that the composition of the portfolios of every investor was the same as that of the market portfolio, becoming the optimum portfolio the *market portfolio*. The representative line of this equilibrium situation is called the *Capital Market Line* or *CML*.

The achievement of the *CML* is done starting from the equation of the Roy's frontier in the context of the existence of assets without risk and supposing an efficient market. The rest of this subsection summarizes Lintner's work (1964, op. cit.) knowledge of which is necessary in order to follow the rest of our paper.

In order to reach the new Roy's frontier, Lintner supposed that an asset without risk exists with a return r^*, in which you can invest or buy (borrow). We will call r the random variable that represents the behaviour of the return ex-post of assets with risk of a portfolio, m_r its expected value, and σ_r its standard deviation. Let w be the proportion invested in assets with risk. Then the random variable of the return on the total portfolio will be

$$y = (1 - w)r^* + wr = r^* + w(r - r^*),$$

where $w < 1$ indicates that the investor invests part of his money in assets without risk, while $w > 1$ indicates that the investor borrows at an interest rate of r^*; the mean and variance of the total return are given by

$$m_y = r^* + w(m_r - r^*),$$

and

$$\sigma_y^2 = w^2\sigma_r^2$$

and, eliminating w from the system of equations, we get

$$m_y = r^* + \theta\sigma_y,$$

where

$$\theta = \frac{m_r - r^*}{\sigma_r}$$

which is usually denominated as the *line of opportunity of the market for a portfolio with a given risk*.

The maximization of θ leads us to the existence of a unique composition of the portfolio with risk, that, for any value of σ_y, will lead to an optimum composition of the total portfolio which maximizes the return.

Let x_i be the proportion of the asset (with risk) i in the portfolio with risk, with mean of the return r_i and variance σ_i, and let σ_{ij} be the covariance between the returns of assets i and j.

Under these conditions, Lintner proved that the optimum composition of the portfolio with risk (when θ is maximum) was given by

$$x_i = \frac{\sum_j r^{ij}(r_j - r^*)}{\sum_i \sum_j r^{ij}(r_j - r^*)}, \tag{10}$$

where r^{ij} is the element ij of the inverse of the matrix of variances–covariances.

3.2. Zero-risk assets for Roy's model

It is clear that, in this situation, the model of Roy cannot be directly applied, because we cannot introduce the asset without risk as one more, since in this case the matrix of variances–covariances would not be invertible; nevertheless, as it was shown in Cruz Rambaud et al. (1999b, op. cit.), the optimum combination can be obtained, as well as in the case of Lintner, starting from the model of Roy without any problem.

Nevertheless, starting from Roy's model, the problem can be solved, because the Roy's frontier would result determined by the segment between the return of the asset without risk and the point of maximum debt. The equation of this straight line is given by

$$m = r^* + \frac{1}{\sqrt{R^*}}\sigma, \tag{11}$$

where

$$R^* = \frac{\sigma_E^2}{(m_E - r^*)^2}.$$

Associated with this line, there exists a risk R^* which represents an upper bound of the probability of not reaching the return of the asset without risk, being the total investment in assets with risk.

Roy's model also provides an intuitive approach to the idea of risk under these assumptions, since once d, the minimum desirable return, and R, the risk we are willing to bear, have been fixed, we can find the proportion w to invest in the optimum combination of a portfolio with risk. Such combination of the portfolio with risk would be the result of the intersection of line (11) with the following line:

$$m - d = \frac{\sigma}{\sqrt{R}},$$

of which

$$\sigma = \frac{r^* - d}{\sqrt{R^*} - \sqrt{R}} \sqrt{RR^*}$$

and from there

$$w = \frac{\sigma}{\sigma_E} = \frac{r^* - d}{\sigma_E\left(\sqrt{R^*} - \sqrt{R}\right)} \sqrt{RR^*},$$

that represents the proportion to invest in the optimum combination of the portfolio in assets with risk given by (9) in order to obtain a profit greater than d with a probability at least equal to $1 - R$. The expected return and the variance of such a portfolio are given by

$$m_y = r^* + w(m_E - r^*)$$

and

$$\sigma_y^2 = w^2 \sigma_E^2.$$

Finally, it is necessary to point out that every combination of minimum return and risk is not acceptable, because the following considerations must be taken into account: It is clear that if $d < r^*$ then one must choose

$$R \leqslant \frac{\sigma_E^2}{(m_E - d)^2},$$

since, otherwise, the intersection between the two lines would give rise to a point of the Roy's frontier with standard deviation greater than σ_E, that would exceed the debt capacity of the investor or, if $R < R^*$, such lines would not intersect for positive values of w. Analogously, if $d > r^*$ then we must choose $d < m_E$ and $R \geqslant \frac{\sigma_E^2}{(m_E - d)^2}$. Likewise, if $R > R^*$, we must choose $d > r^*$ and $d \leqslant m_E - \frac{\sigma_E}{\sqrt{R}}$.

Therefore, the Roy's frontier is reduced to a straight line (namely, to a segment) which depends on the debt capacity of the investor and, taking into account that this percentage is not equal for all investors, we can affirm that there is not a capital market line common to all investors.

4. Conclusions

The advances presented in this paper contribute, without doubt and to a greater extent, to the solution of the problems arising from the practical application of the classical models for the obtaining of investment portfolios. The concept of risk as *the probability of not reaching the reservation return*, provides a more intuitive capacity in the use of these models. Moreover, the determination by the investor of his critical level, allows him to choose a riskier composition of his portfolio, in exchange for a greater expected return.

On the other hand, the definition of the debt function allows the investor to obtain a greater expectation of returns, subject only to his maximum capacity of debt.

Finally, the introduction of these new concepts provides some ideas which are valid for the resolution of the Capital Market Line in a new context in which the return of zero-risk assets is greater than the critical return, this situation being otherwise impossible with the use of classical models.

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