

# THE PARAMETERS OF THE CLASSICAL PERT. AN ASSESSMENT OF ITS SUCCESS

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## *SUMMARY*

In classical PERT, concrete values are assigned to the parameters determining the expected value and the variance (of a task, cash flow, etc.) based on the least possible, greatest possible and modal values provided by the expert. The use of these parameters has led to successful conclusions in numerous practical applications; nevertheless, their theoretical foundations have been debated from the start. This article attempts to find an explanation for the practical success of the classical PERT model, with an indepth study of its theoretical premises. A logical path towards the classical PERT parameters is presented; furthermore, this is shown to be the only constant variance model that has the same kurtosis as the normal one. A method is proposed for a better fit to the data provided by the expert, improving on that of the classical model.

*KEY-WORDS: PERT, kurtosis, constant variance.*

## 1. INTRODUCTION

Ever since PERT method (Program Evaluation and Review Technique) was first used in the 1950s to estimate the duration of tasks in a ballistic missile programme, this technique has been satisfactorily used by experts both for the theoretical treatment of the duration of tasks within a project and for the cash flow within an investment, as well as for other types of problem. However, during the 1960s and 1970s, various authors proposed a revision of the

hypotheses put forward by the creators of the PERT method: Heally (1961), Clark (1962), Grubbs (1962), MacCrimmon (1964), Pulido et al. (1964), Tomas (1964), Vazsonyi (1970), among others. In the 1980s, Sasieni (1986) reopened the debate, questioning the reasoning behind the classical values of the PERT parameters. Immediate replies came from Gallagher (1987) and Littlefield and Randolph (1987). Since then, numerous authors have participated in the same debate expressing similar doubts: Farnun and Stanton (1987), Berny (1989), Herrerías (1989), Troutt (1989), Chae and Kim (1990), Moitra (1990), Keefer and Verdini (1993) and many more.

Nevertheless, in practice the methodology has been used with great success in projects of diverse types and by different organizations. In the USA the methodology has been applied by companies such as Lockheed, General Electric, Douglas Aircraft Corporation, Texas Instruments, and General Motors etc.<sup>1</sup> With good results.

The present study consists of three elements: first, an investigation into the underlying reasons behind this practical success; secondly, it offers a new answer to the question posed by Sasieni (1986); finally, a procedure is proposed to improve the fit of the probabilistic model underlying the PERT method when the variance is assumed and equal to  $(b - a)^2 / 36$ .

It is well known that the basis for the methodology is the use of the first kind beta distribution as the underlying probabilistic model. Such a distribution has the following density function

$$f(x) = \frac{(x - a)^{p-1} (b - x)^{q-1}}{(b - a)^{p+q-1} \hat{a}(p, q)} \text{ if } a < x < b \text{ and } p > 1 \text{ and } q > 1, \quad (1)$$

where  $a$  is the minimum value,  $b$  the maximum value of the random variable and  $p$  and  $q$  are the characteristic parameters of the random variable.

The stochastic characteristics of this distribution (Dumas de Rauly, 1968) are as follows:

$$\text{Mode:} \quad m = \frac{p-1}{p+q-2} b + \frac{q-1}{p+q-2} a \quad (2)$$

$$\text{Mean:} \quad \bar{m} = \frac{p}{p+q} b + \frac{q}{p+q} a \quad (3)$$

$$\text{Variance:} \quad \sigma^2 = \frac{pq(b-a)^2}{(p+q-1)(p+q)^2} \quad (4)$$

<sup>1</sup> Cited by Pulido and others in their article: "Un método de la I.O.: Teoría de grafos". *Anales de Economía*, July-September 1964.

The PERT method requires the "expert" to assign three different values (time ( $T$ ) in the case of tasks, cash flow ( $Q$ ) in the case of investments, etc.) to identify  $T_0$  ( $Q_0$ ) the optimistic value,  $T_m$  ( $Q_m$ ) the modal value and  $T_p$  ( $Q_p$ ) the pessimistic value. Each of these estimates relates directly to the problem and enables us to identify  $a$  and  $b$  directly, making them coincide in each case with the pessimistic and optimistic values. Furthermore, if the most probable value is identified with the mode (see equation 2) then we have the following equation, relating  $p$  and  $q$  :

$$T_m = \frac{p-1}{p+q-2}b + \frac{q-1}{p+q-2}a \quad (5)$$

Obviously, this expression does not exactly determine the values of  $p$  and  $q$ . Classical PERT sacrifices rigour at this point in favour of simplicity, with the implicit acceptance that

$$p = 3 + \sqrt{2} \quad \text{and} \quad q = 3 - \sqrt{2}, \quad \text{if } m > (a+b)/2 \quad (6)$$

$$\text{or} \quad p = 3 - \sqrt{2} \quad \text{and} \quad q = 3 + \sqrt{2}, \quad \text{if } m < (a+b)/2 \quad (7)$$

In the case where the mode is  $(a+b)/2$  symmetric distribution  $\mathbf{b}$  (4, 4,  $a$ ,  $b$ ) is used.

It may easily be shown that from equation (5) and relations (6) and (7), the formulas for the parameters of classical PERT may be obtained directly, i.e.

$$\mathbf{m} = \frac{a+4m+b}{6} \quad \text{and} \quad \sigma^2 = \frac{(b-a)^2}{36} \quad (8)$$

without needing any kind of additional hypothesis, simplifying relations or logical reasoning.

Introducing the parameter  $k = p + q - 2$ , Golenko~Ginzburg (1988) obtained

$$p = 1 + k \frac{m-a}{b-a}, \quad q = 1 + k \frac{b-m}{b-a} \quad (9)$$

and from these, they deduced expressions that are more general than those of (8) for the parameters of the probabilistic PERT model:

$$\text{Mean:} \quad \mathbf{m} = \frac{a + km + b}{k+2} \quad (10)$$

$$\text{Variance:} \quad \sigma^2 = \frac{k^2(m-a)(b-m) + (k+1)(b-a)^2}{(k+3)(k+2)^2} \quad (11)$$

Note that these probabilistic models give a variable weighting to the modal value determined by the expert. Subsequently, Herréras (1989) established the following relation using equations (10) and (11):

$$\sigma^2 = \frac{(m-a)(b-m)}{k+3} \quad (12)$$

As  $p > 1$  and  $q > 1$ , it is clear that  $k$  varies within the interval  $(0, \infty)$  and, therefore, for given values of  $a$ ,  $b$  and  $m$ , there exists a different beta distribution for each value of  $k$  that is greater than 0.

As can be seen on comparing the mean expressions from (8) and (10), the classical PERT case corresponds to the value of  $k = 4$ .

However, as Sasieni has stated, it is always possible to consider the most general cases according to equations (10) and (11) instead of equations (8). On the other hand, if one substitutes

$$t_x = \frac{x-a}{b-a} \quad (13)$$

(where  $t_a = 0$ ,  $t_b = 1$  and only the standardized mode, between 0 and 1, will vary), the above equations (1), (2), (3) and (4) are reduced to the following expressions:

$$f(t) = \frac{t^{p-1}(1-t)^{q-1}}{\hat{a}(p,q)} \text{ if } 0 < t < 1 \text{ and } p > 1 \text{ and } q > 1, \quad (14)$$

$$\text{Mode:} \quad m = \frac{p-1}{p+q-2} \quad (15)$$

$$\text{Mean:} \quad m = \frac{p}{p+q} \quad (16)$$

$$\text{Variance:} \quad \sigma^2 = \frac{pq}{(p+q-1)(p+q)^2} \quad (17)$$

and the expressions (9), (10), (11) and (12) are reduced to:

$$p = 1 + km, \quad q = 1 + k(1-m) \quad (18)$$

$$\text{Mean:} \quad m = \frac{1+km}{k+2} \quad (19)$$

$$\text{Variance:} \quad \sigma^2 = \frac{k^2 m(1-m) + (k+1)}{(k+3)(k+2)^2} \quad (20)$$

$$\sigma^2 = \frac{m(1-m)}{(k+3)} \quad (21)$$

Note that if  $p$  and  $q$  fulfil relation (6) or (7), then the mode  $m$  (which is the only one that can vary, in the standardized case) is perfectly well determined, and there is no need to enquire of the expert, as it is possible to obtain from (15), according to the asymmetry of beta distribution, that:

$$m = \frac{1}{2} \pm \frac{\sqrt{2}}{4} \quad (22)$$

Clearly if either of the values for  $m$  given in (22) is substituted in (20), and assuming  $k = 4$ , then we have  $\sigma^2 = 1/36$  a value that coincides with the standardized one obtained from (8). Conversely, assuming  $k = 4$  and that (20) is  $1/36$  then (22) is obtained.

## 2. SOME HYPOTHESES TO JUSTIFY THE VALUES OF P, Q AND K IN CLASSICAL PERT

Historically, the PERT method has been linked to the normal distribution, as made plain in the initial report on the PERT method (Appendix A, p. 2)<sup>2</sup>: "The duration of each activity is a normally distributed random variable..."

Nevertheless, on page six of Appendix B, the beta distribution is analyzed: "As a distribution model for the duration of an activity we shall introduce the beta distribution ...". Indeed, this link provides a justification frequently used for the classical PERT values, since in the normal distribution 997 thousandths of the whole lies between  $m \pm 3s$ . Also, since it is impossible to determine  $a$ ,  $b$ ,  $p$  and  $q$  with the information from the expert's three estimates, a fourth hypothesis is added<sup>3</sup>: a beta distribution is chosen, in which  $b - a = 6\sigma$  (see Yu Chuen Tao (1980)). Thus, using expression (17), the following equation can be established:

$$\frac{pq}{(p+q-1)(p+q)^2} = \frac{1}{36} \quad (23)$$

which, together with (15), constitutes a two-equation system for the unknowns  $p$  and  $q$ . This system leads to a cubic equation for which, without attempting to solve it due to the difficulty of the task, Romero (1991) proposed the values (6) and (7) as a solution, following Kauffman and Dezbazeille (1965).

Other authors, such as Suárez (1988), have used simplifying relations:

$$p + q = 6 \text{ and } \frac{pq}{p + q - 1} = 1 \quad (24)$$

<sup>2</sup> PERT, Program Evaluation Research Task, Phase 1 Summary Report, Special Projects Office, Bureau of Ordnance, Department of the Navy, Washington D. C., July 1958 (cited by MacCrimmon, K. R. and Ryavec, C. A., 1964).

<sup>3</sup> Widely used, even in manuals, such as those by Hillier, I. and Lieberman, G. J. (1980), or that by Daellenbach, H. G. George, G. A. and McNickle, D. C. (1987), etc..

which have been strongly criticized for lacking a convincing theoretical justification and for having as their sole basis the ease of computation provided; whilst others, such as Hillier and Lieberman (1982) simply rely on logical reasoning.

A more recent solution consists in disregarding the variance, and testing the coherence of the classical values by giving the beta of the PERT the same kurtosis as normal distribution. In other words, the coefficient of kurtosis of the beta must be zero, as it is in the following (see Canavos (1984)):

$$g_2 = \frac{\mu_2}{\mu_1^2} - 3 = 6 \frac{p(p+1)(p-2q) + q(q+1)(q-2p)}{pq(p+q+2)(p+q+3)} = 0, \quad (25)$$

where  $\mu_2$  is the moment of the second order of the distribution.

It can easily be shown that the classical values (6) and (7) are solutions to the equation:

$$p(p+1)(p-2q) + q(q+1)(q-2p) = 0 \quad (26)$$

and that for these

$$pq(p+q+2)(p+q+3) = 504 \neq 0, \text{ and so } g_2 = 0.$$

### 3. THE CONSTANT VARIANCE MODELS

From the study of the constant variance models for the PERT the following result can be obtained: subjectively estimating  $a$ ,  $b$  and  $m$ , and if the variance is fixed at  $(b-a)^2/36$  ( $1/36$  if the substitution is made in equation (13)):

- a) There exists only one solution for  $k$  ( $k = p + q - 2$ ), that fulfils (19) and (20), and
- 2b)  $k \in (2.872..., 6)$ .

To prove this, the standardized model formulas are used; taking into account expressions (19) and (21), then

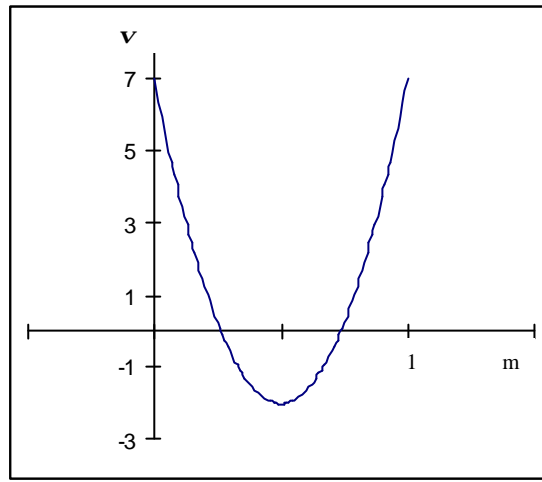
$$\sigma^2 = \frac{1}{36} = \frac{\frac{1+km}{k+2} \left( 1 - \frac{1+km}{k+2} \right)}{k+3} = \frac{(1+km)(k+1-km)}{(k+2)(k+3)} \quad (27)$$

from which is derived the cubic expression

$$k^3 + k^2 [7 - 36(m - m^2)] - 20k - 24 = 0 \quad (28)$$

This equation is similar to the one given by Littlefield and Randolph (1987), but is simpler, with the final two terms of the sum being unaffected by the square and cube powers, respectively, of  $m$ .

It is now the intention of this work to prove that equation (28) has one and only one positive solution for  $k$ . First, it can be shown that, for all  $m \in [0,1]$ , the polynomial  $\mathbf{w} = 7 - 36(m - m^2)$  takes its values at  $(-2,7)$  (see graph 1).



Graph 1. Graph of polynomial  $\mathbf{w} = 7 - 36(m - m^2)$

If  $\mathbf{w} = -2$ , (28) has a single positive root, that is  $k = 6$ ; and if  $\mathbf{w} = 7$ , then (28) has a single positive root, that is  $k = 2.872\dots$ . Furthermore, independently of the value of  $w$ , equation (28) presents just one variation in the signs of the coefficients; the rule of Descartes<sup>4</sup> allows us to state that when there is a positive root then there is only one such root. Finally, it is intended to prove that, for all  $m \in (0,1)$  (which implies  $\mathbf{v} \in (-2,7)$ ), the equation (28) has one and only one positive root, and that this root lies between 2.872... and 6.

It can be shown that if  $k_1, k_2$  and  $k_3$  are the roots of equation (28), then by using the fundamental theorem of Algebra and the sign of the independent term, the only possibilities for the roots are:

- a) The three roots are real, in which case:
  - a.1) The three are positive.
  - a.2) Two roots are negative and one is positive.
- b) Two of the roots are complex and conjugated and the third is real and positive.

<sup>4</sup> Descartes' well-known rule of signs: The number of positive roots of a polynomial  $f$  with real coefficients coincides with the number of alternances or is lower by an even number.

As a) cannot exist, since it contradicts the Descartes' rule of signs, there must be just one positive root. Furthermore, if  $\alpha$  is the positive root of the polynomial  $k^3 + k^2 w - 20k - 24 = 0$ , when  $w = 7 - 36(m - m^2)$  and  $m \in (0,1)$ ,  $\mathbf{a}$  is decreasing function of  $\omega$ . Indeed, as  $\mathbf{a} > 0$ , from  $\mathbf{a}^3 + w\mathbf{a}^2 - 20\mathbf{a} - 24 = 0$ , isolating  $\dot{u} = \frac{-\dot{a}^3 + 20\dot{a} + 24}{\dot{a}^2}$  and then differentiating, we obtain:

$$\frac{d\dot{u}}{d\dot{a}} = \frac{-(\dot{a} + 2)(\dot{a}^2 - 2\dot{a} + 24)}{\dot{a}^3} < 0, \text{ for all } \mathbf{a} > 0,$$

from which 
$$\frac{d\mathbf{a}}{d\dot{u}} = \frac{1}{\frac{d\dot{u}}{d\dot{a}}} < 0, \text{ for all } \mathbf{a} > 0.$$

Therefore, it can be concluded that the positive root of equation (28) always lies between 6, corresponding to the lower value of  $\omega$  ( $\omega = -2$ ), and 2.872..., corresponding to the upper value of  $\omega$  ( $\omega = 7$ ). Note that this is a restriction which does not arise in the mesokurtic models. Thus, assuming the constant variance hypothesis and that it is equal to  $1/36$ , the correct procedure to adjust the beta is as follows: Given  $a$ ,  $b$  and  $m$ , normalized according to (13), the value of  $t_m$  is substituted for  $m$  in equation (28), the cubic equation is solved, and the value of  $k$  obtained; from this value,  $p$  and  $q$  are obtained using (18).

Nevertheless, this procedure may become excessively tedious when the project to be evaluated contains a large number of tasks (more than 50.000 in the case of the ballistic missile programme) and, consequently, a shortcut may be sought. Table 1 illustrates the differences and similarities between PERT models with constant variance and the classical model.

By assuming  $\omega = 2.5$  (i. e. the midpoint of the possible values of  $\omega$ ) in all cases, the possible overall error when numerous tasks are estimated is minimized, and we obtain:

$$2.5 = 7 - 36(m - m^2) \quad (29)$$

that implies  $m = \frac{1 \pm \sqrt{2}}{2}$  and so the cubic equation becomes

$$k^3 + 2.5k^2 - 20k - 24 = 0, \quad (30)$$

from which  $k_1 = 4$ ,  $k_2 < 0$  and  $k_3 < 0$ , and using (18),

$$p = 3 \mp \sqrt{2} \text{ and } q = 3 \pm \sqrt{2}, \quad (31)$$

$p < q$  for positively skewed distribution, and  $p > q$  for other cases.

#### 4. OBTAINING THE CLASSICAL MODEL

If equation (28) is solved for  $m$ , then:

$$m = \frac{1}{2} \mp \frac{\sqrt{(6-k)(k+2)^2}}{6k} \quad (32)$$

and so  $k < 6$  it is the same in the case of symmetric distribution, and when (18) is taken into account, it can be concluded that  $p = q = 4$ . Note that if we take  $k = 4$ , this leads to (22), and that  $m \in [0,1]$ , if  $2.872... \leq k \leq 6$ .

Furthermore, if the model is required to have the same kurtosis as the normal distribution, i. e., equation (26) is satisfied, then (26) together with (15) provides a system which, by making the (18) substitution, gives the cubic equation:

$$k^3(5m^2 - 5m + 1) + k^2(16m^2 - 16m + 2) - 5k - 4 = 0 \quad (33)$$

which characterizes the mesokurtic models. Solving this equation for  $m$ , we obtain:

$$m = \frac{1}{2} \pm \frac{k+2}{2k} \sqrt{\frac{k+4}{5k+16}} \quad (34)$$

It can be shown that the only solutions for system (32) and (34) are  $k = 4$ ,  $k = -2$  and  $k = -3$ . As the last two are unacceptable and  $k = 4$  is the parameter value in the classical model, it can be concluded that the only model which has a constant variance and is, moreover, mesokurtic, the classical PERT model. In consequence, the classical PERT model is the intersection between the constant variance and the mesokurtic models.

#### 5. PRACTICAL CASE

The practical case that is presented here is the reduced version of a real case, about the valuation of the loss of profits of a company, held in the Court of Madrid (Spain) in 1999. The loss of profits' years were from 1995 to 1999.

In this case, the experts accepted that the least possible, the modal and the great possible expected profits during these years would be a percentage of the total volumes of sales for each one of the years. More specifically the pessimistic, the more probable and the optimistic profits of the year  $x$  ( $x = 1995, \dots, 1999$ ) would be, respectively, the 4.2%, the 4.4% and the 6.4% of the contracted volume of sales of this company during the year  $x$ . So, the following table shows the minimum ( $a$ ), more probable ( $m$ ) and maximum ( $b$ ) values of the expected profit for each year, from 1995 to 1999 starting from their contracted volume of sales (in millions of pesetas).

Percs %	Profit values	Years				
		1995	1996	1997	1998	1999
4.2	<i>a</i>	93.895	160.444	136.728	172.598	190.660
4.4	<i>m</i>	97.900	167.288	142.560	179.960	198.792
6.4	<i>b</i>	143.290	244.849	208.656	263.396	290.959
Volumes of sales		2,225	3,802	3,240	4,090	4,518

*Table 2. Minimum, more probable and maximum values of the expected profit, from 1995 to 1999.*

Taking into account that we want go from a uncertainty situation to stochastic model, we can apply the PERT methodology with, at least, three models:

1. Classical.
2. Of constant variance.
3. Mesokurtic.

### 5.1 CLASSICAL MODEL

The following table shows the expected value and the variance for each one of the years, from 1995 to 1999, using the steps described in the right side of table 1.

Mean and Variance	Years				
	1995	1996	1997	1998	1999
<i>m</i>	104.798	179.074	152.604	192.639	212.798
$\sigma^2$	67.774	197.892	143.712	229.008	279.445

*Table 3. Expected values and variances, from 1995 to 1999.*

### 5.2 CONSTANT VARIANCE MODEL

Note that, in all years, from 1995 to 1999, the standardized value of the mode

$$t_m = \frac{m - a}{b - a}$$

is equal to 0.081 and that the value of polynomial  $\omega$  is  $7 - 36(t_m - t_m^2) = 4.318$ .

So, the solution of the cubicequation  $k^3 + 4.318k^2 - 20k - 24 = 0$  is  $k = 3.432$ .

The following table shows the expected value and the variance for each one of the years, from 1995 to 1999, using the steps described in the left side of table 1.

Mean and Variance	Years				
	1995	1996	1997	1998	1999
$\mu$	105.519	180.307	153.654	193.965	214.263
$\sigma^2$	67.764	197.864	143.692	228.975	279.405

*Table 4. Expected values and variances, from 1995 to 1999.*

### 5.3 MESOKURTIC MODEL

Solving the equation (33), we obtain the value  $k = 2.65$ . So the following table shows the expected value and the variance for each one of the years, from 1995 to 1999.

Mean and Variance	Years				
	1995	1996	1997	1998	1999
$\mu$	106.800	182.496	155.520	196.320	216.864
$\sigma^2$	83.316	243.272	176.668	281.524	343.527

*Table 5. Expected values and variances, from 1995 to 1999.*

Using the legal interest rates ( $r$ ) in Spain during the period from 1995 to 1999, we can deduce the capitalization factors for the mean ( $U$ ) and the variance ( $U^2$ ) for each one of these years

Rates and factors	Models				
	1995	1996	1997	1998	1999
$R$	0.090	0.090	0.075	0.055	0.043
$U$	1.090	1.188	1.277	1.347	1.347
$U^2$	1.188	1.412	1.631	1.816	1.816

*Table 6. Capitalization factors for the mean and the variance.*

in order to calculate one mean and one variance for each PERT model:

Mean and Variance	Models		
	Classical	Constant Variance	Mesokurtic
$\mu$	667.242	671.835	679.992
$\sigma^2$	565.377	565.297	695.029

*Table 7. Mean and variance for each PERT model.*

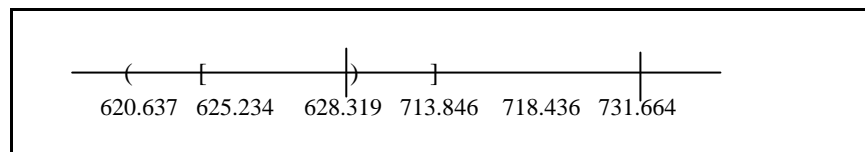
Finally, at 5% level of significance, we can obtain a confidence interval where, for each model, the profits values will be included.

Confidence intervals	Models		
	Classical	Constant Variance	Mesokurtic
$m \pm 1.96 \sigma$	(620.637, 713.846)	(625.234, 718.436)	(628.319, 731.664)

Table 8. Confidence intervals according to the models

Observe that:

- a) the value  $k$  decreases, the confidence interval moves around to the right:



2. The mean of the classical model is minor than the mean in the constant variance model and this is less than the mean of the mesokurtic model. Moreover, the variance of the mesokurtic model is greater than the variance of the other two models. This property is not general but it is true in this case. So we can work with more moderate models with respect to the mean or more conservative models with respect to the variance, choosing, in each case, the more suitable model. Because, when one works in an uncertainty situation, it is preferable to get right approximately than to make a mistake exactly.

So we can propose, as interval of negotiation, the smallest among the lower extremes and the biggest of the upper extremes of the three intervals, obtaining the following interval: (620.637, 731.664) where, with more probability, the loss of profits will be located.

## 5. CONCLUSIONS

As previously described, the creators of the PERT method used a beta with given  $p$  and  $q$  parameters, (6) or (7), thereby determining a mode. Though some authors have claimed that the success of the PERT method is largely due to the fact that the times of many of the tasks analyzed for the ballistic missile construction programme fitted the proposed beta distribution, what in fact happened was that the overall error rate was reduced by always choosing the central point of the interval  $[-2, 7]$  for  $\omega$ . This method, almost certainly due to its simplicity, has been employed without variation and with equal success for numerous tasks and applications in industry, public administration, commerce, etc.. The explanation for this is that any other simplification which avoids

solving the cubic equation (28), or giving  $k$  a constant value different from  $k = 4$  produces, in repeated use, a greater degree of error than the  $k = 4$  option.

Furthermore, if the correctness of the constant variance hypothesis is accepted, the errors of fit of the classical PERT method can be still further reduced by solving the cubic equation (28) for each case, a procedure which today may easily be implemented with the aid of computers.

The final conclusions may be summarized as:

- 1<sup>a</sup> The classical PERT values come from the midpoint of the  $[2,7]$  interval for  $\omega$ , providing an answer to Sasieni's question (1986).
- 2<sup>a</sup> The selection of the midpoint of this interval explains why PERT has functioned so well in numerous and varied cases.
- 3<sup>a</sup> The parameters of the classical PERT model are obtained by the simultaneous application of the conditions that the variance is equal to  $(b-a)^2/36$  and that kurtosis, as in the normal distribution, is zero.
- 4<sup>a</sup> When the model is a beta with constant variance, the solution for each modal value  $m$  of the cubic equation (28) determines a beta which provides a better fit to the data provided by the expert than does the classical model.

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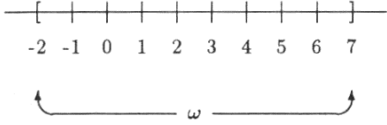
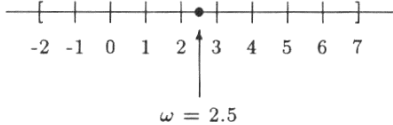
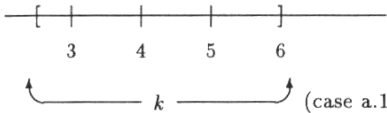
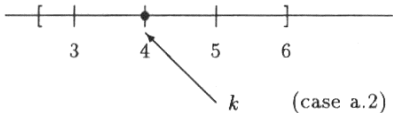
MODEL WITH CONSTANT VARIANCE, EQUAL TO $\frac{(b-a)^2}{36}$ AND UNDERLYING BETA	CLASSICAL PERT MODEL
The expert provides $a$ , $b$ and $m$ .	The expert provides $a$ , $b$ and $m$ .
By substituting $t_x = \frac{x-a}{b-a}$ , we obtain $t_m \in (0, 1)$ .	By substituting $t_x = \frac{x-a}{b-a}$ , we obtain $t_m \in (0, 1)$ .
<p>The numerical value of the polynomial <math>\omega</math> in <math>[-2, 7]</math> is obtained.</p> 	<p>2.5 is always taken as the numerical value of the polynomial <math>\omega</math> (the interval center)</p> 
<p>The cubic equation (28) is solved, obtaining <math>k \in [2.872 \dots, 6]</math>.</p> 	<p>The cubic equation is solved using the above value <math>\omega = 2.5</math>, obtaining <math>k = 4</math>.</p> 
<p>For the above <math>k</math>,  <math>p = 1 + kt_m</math> and <math>q = 1 + k(1 - t_m)</math>  are obtained.</p>	<p>For <math>k = 4</math>,  <math>p = 3 \pm \sqrt{2}</math> and <math>q = 3 \mp \sqrt{2}</math>  depending on whether the distribution is positively or negatively skewed.</p>

Table 1<sup>5</sup>. Differences and similarities between PERT models with constant variance and the classical model.

<sup>5</sup> In the classical PERT model the third step is not necessary, but is included for the similarity to be emphasized.