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Short Communication

Markowitz's model with Euclidean vector spaces

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ABSTRACT

In this paper a new approach of the Markowitz's model is presented. Indeed, using an inner product, a quantitative and explicit solution for optimal portfolio selection is given. To do this, a scalar product is defined in $\mathfrak{R}^n$ which allows us to calculate the composition of the optimal portfolio and the variance for a given expected return by means of the distance between the subspace of feasible solutions and the origin of the affine space.

1. Introduction

The publication in 1952 of the work *Portfolio Selection* by Harry Markowitz in *The Journal of Finance* marked the beginning of the modern portfolio theory. For the first time, the relation between return and risk was included in a financial model together with the concept of rational behaviour of the investors. Several authors, e.g. Lintner (1964), Sharpe (1963, 1970), further developed the portfolio theory using the ideas proposed by Markowitz (1959, 1991), giving rise to the *Diagonal Model* and the *Capital Market Line* (*CML*) (see Bodie and Merton, 2000, for example). In the context of this article it is worth to notice that the original Markowitz model had the "no short sales" constraint. Later this constraint was relaxed (see, for example, Hillier and Lieberman, 2005). A detailed description of the procedure for calculating the optimal portfolio using the classical Markowitz's model can be found in Ødegaard (2006).

Assuming an economy consisting of a set of risky assets together with a single riskless asset, the portfolios along the CML are superior to the efficient frontier portfolios containing risky assets only. There exist several procedures to derive the CML. All of them are based on the Lagrange multipliers method (cf. Merton, 1972, Elton et al., 1976, Elton and Gruber, 1995, Sharpe et al., 1999, Bodie et al., 2000, Ingersoll, 1987, Huang and Litzenberger, 1988, Feldman and Reisman, 2003, Bick, 2004).

Our approach differs from the existing analytical solutions in several aspects:

1. We start with a scalar product defined in the space $\mathfrak{R}^n$ of the portfolio weights for n assets; the so-called inner product is induced by the variance–covariance matrix of the risky assets (a similar approach can be found in Becker (2003)). Such formalization allows obtaining a solution of the Markowitz's model using geometric tools only, by calculating a vector with minimum norm. The main classical results of the Markowitz theory can be derived by simplified calculus comparing to the existing solutions.
2. The scalar product also allows to compute an orthogonal basis, a set of portfolios whose pairwise covariances are zeros, using the Gram-Schmidt algorithm. Therefore, we can always consider a diagonal variance–covariance matrix of security rates which notably simplifies the problem. (A similar idea can be found in Bouchaud and Potters (2004, chapter 12).) This basis has a remarkable importance because the efficient frontier can be derived using a portfolio belonging to the basis and a zero-net investment vector only.
3. We can extend our analysis to the case of risky assets only, as well as to the case when we invest less than 100% of the amount of capital, thus obtaining a superior portfolio.

The organization of this paper is as follows: in Section 2, a scalar product is defined in $\mathfrak{R}^n$ allowing us to calculate the composition of

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the optimal portfolio for a given level of expected return using the distance between the subspace of feasible solutions and the origin of the affine space. Thus, a formula linking mean and variance is deduced. The main result in this section is that the efficient frontier is a one-dimensional affine subspace of $\Re^n$, where a basis of uncorrelated portfolios can be always constructed. Finally, Section 3 summarizes our findings and concludes.

2. Markowitz's model in a Euclidean vector space

Let us consider a portfolio composed by n assets for which it is possible short sales and purchases without any limit of credit. Thus, the vector space $\Re^n$ can be considered as the set

$$\{(x_1, \ldots, x_n) \in \Re^n : x_i \text{ is the proportion invested in the asset } i\}.$$

In what follows, every vector x (resp. point X) will be represented by means of a $1 \times n$ matrix, while its transpose, x^t (resp. X^t), will be a $n \times 1$ matrix.

In the previous space it is defined the following scalar product (the so-called inner product induced by the variance–covariance volatilities matrix of the risky assets):

$$\langle x, y \rangle = x\mathbf{V}y^t,$$

for all $x, y \in \Re^n$, where $\mathbf{V}$ is the matrix of variances–covariances corresponding to the n assets. In effect, taking into account that $\langle x, x \rangle = x\mathbf{V}x^t$ is the variance associated with the composition x of a given portfolio then $\mathbf{V}$ is a symmetric and positive semidefinite matrix. If there exists no composition with null variance, i.e. the common distribution is not concentrated on a hyperplane (practically, this stipulation should be no impediment), it can be deduced that $\langle \cdot, \cdot \rangle$ is a scalar product.

Markowitz's model establishes that, fixing an average return m, the optimal portfolio will be the one with the smallest variance among the portfolios having such average profitability. In other words, once fixed a yield m, the problem is to find a $x \in \Re^n$, such that

- $x\mathbf{V}x^t$, that is to say, the variance, is minimum,
- $\mathbf{R}x^t = m$ or, equivalently, the average return is m, and
- $\mathbf{1}x^t = 1$; in other words, the total investment is 100%,

where $\mathbf{R} = (r_1, \ldots, r_n)$, with r_i the expected return of the asset i $(i = 1, \ldots, n)$, and $\mathbf{1} = (1, \ldots, 1)$.

In order to do this, let us consider the $(n-1)$-dimensional affine subspaces

$$H_1 = \{Y \in \Re^n : \mathbf{R}Y^t = Y\mathbf{R}^t = m\},$$

with direction linear subspace $U_1 = \{\overrightarrow{AB} : A, B \in H_1\} = \{y : \mathbf{R}y^t = y\mathbf{R}^t = 0\}$, and

$$H_2 = \{Y \in \Re^n : \mathbf{1}Y^t = Y\mathbf{1}^t = 1\},$$

with direction linear subspace $U_2 = \{\overrightarrow{AB} : A, B \in H_2\} = \{y : \mathbf{1}y^t = y\mathbf{1}^t = 0\}$.

The intersection of these hyperplanes is the $(n-2)$-dimensional affine subspace (we assume that $\mathbf{R}$ and $\mathbf{1}$ are linearly independent, i.e. not all assets have the same expected return. Otherwise, the problem is more simple and the reasoning is analogous):

$$S = H_1 \cap H_2$$

which will be called the *subspace of feasible solutions*.

So, taking into account that $x = \overrightarrow{OX}$, the problem is reduced to find a point $X \in S$ such that the vector x has a minimum norm $\langle x, x \rangle = x\mathbf{V}x^t$, or equivalently, a point $X \in S$ which represents the minimum distance between 0 and S (see Fig. 1).

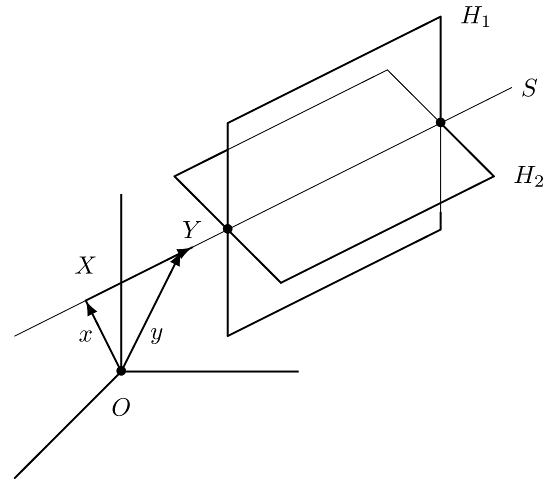

Fig. 1. Minimum distance between O and S.

It is well known that it suffices to find a $X \in S$ such that

$$\langle \overrightarrow{OX}, \overrightarrow{XY} \rangle = 0$$

for all $Y \in S$ or, equivalently,

$$\langle x, y - x \rangle = 0.$$

Since $y - x$ ranges the vector subspace underlying S, the problem involves finding a point $X \in S$, such that x is orthogonal to S. Such a point is the intersection of S and the plane spanned by two vectors orthogonal to H_1 and H_2, respectively. A vector orthogonal to H_1 is $\mathbf{R}\mathbf{V}^{-1}$, since

$$\langle z, \mathbf{R}\mathbf{V}^{-1} \rangle = z\mathbf{V}\mathbf{V}^{-1}\mathbf{R}^t = z\mathbf{R}^t = 0$$

for all vector z in the linear subspace U_1, because $\mathbf{R}$ is a vector orthogonal to H_1 according to the Euclidean scalar product. Analogously, a vector orthogonal to H_2 is $\mathbf{1}\mathbf{V}^{-1}$. So, vector x is a linear combination of $\mathbf{R}\mathbf{V}^{-1}$ and $\mathbf{1}\mathbf{V}^{-1}$:

$$x = \alpha_1 \mathbf{R}\mathbf{V}^{-1} + \alpha_2 \mathbf{1}\mathbf{V}^{-1}, \tag{1}$$

where α_1 and α_2 can be obtained starting from the condition $X \in S$. Hence it is necessary to solve the following system of equations:

$$\begin{cases} x\mathbf{R}^t = \alpha_1 \mathbf{R}\mathbf{V}^{-1}\mathbf{R}^t + \alpha_2 \mathbf{1}\mathbf{V}^{-1}\mathbf{R}^t = m, \\ x\mathbf{1}^t = \alpha_1 \mathbf{R}\mathbf{V}^{-1}\mathbf{1}^t + \alpha_2 \mathbf{1}\mathbf{V}^{-1}\mathbf{1}^t = 1, \end{cases}$$

whose solution is

$$\alpha_1 = \frac{m\mathbf{1}\mathbf{V}^{-1}\mathbf{1}^t - \mathbf{R}\mathbf{V}^{-1}\mathbf{1}^t}{(\mathbf{R}\mathbf{V}^{-1}\mathbf{R}^t)(\mathbf{1}\mathbf{V}^{-1}\mathbf{1}^t) - (\mathbf{R}\mathbf{V}^{-1}\mathbf{1}^t)^2}$$

and

$$\alpha_2 = \frac{\mathbf{R}\mathbf{V}^{-1}\mathbf{R}^t - m\mathbf{1}\mathbf{V}^{-1}\mathbf{R}^t}{(\mathbf{R}\mathbf{V}^{-1}\mathbf{R}^t)(\mathbf{1}\mathbf{V}^{-1}\mathbf{1}^t) - (\mathbf{R}\mathbf{V}^{-1}\mathbf{1}^t)^2}, \tag{2}$$

and so

$$x = \frac{(m\mathbf{1}\mathbf{V}^{-1}\mathbf{1}^t - \mathbf{R}\mathbf{V}^{-1}\mathbf{1}^t)\mathbf{R}\mathbf{V}^{-1} + (\mathbf{R}\mathbf{V}^{-1}\mathbf{R}^t - m\mathbf{1}\mathbf{V}^{-1}\mathbf{R}^t)\mathbf{1}\mathbf{V}^{-1}}{(\mathbf{R}\mathbf{V}^{-1}\mathbf{R}^t)(\mathbf{1}\mathbf{V}^{-1}\mathbf{1}^t) - (\mathbf{R}\mathbf{V}^{-1}\mathbf{1}^t)^2}, \tag{3}$$

which is the composition of the portfolio with average return m and minimum variance. Now, let us calculate an expression for σ^2 according to m. In effect,

$$x\mathbf{V} = \alpha_1 \mathbf{R} + \alpha_2 \mathbf{1},$$

from which

$$\sigma^2 = x\mathbf{V}x^t = \alpha_1 \mathbf{R}x^t + \alpha_2 \mathbf{1}x^t$$

and, taking into account that $X \in S$, then

$$\sigma^2 = \alpha_1 m + \alpha_2. \tag{4}$$

By substituting the values of α_1 and α_2 (see Eq. (2)) in expression (4) and using the simplifications:

$$a = \frac{(\mathbf{R}\mathbf{V}^{-1}\mathbf{R}^t)(\mathbf{1}\mathbf{V}^{-1}\mathbf{1}^t) - (\mathbf{R}\mathbf{V}^{-1}\mathbf{1}^t)^2}{\mathbf{1}\mathbf{V}^{-1}\mathbf{1}^t}, \tag{5}$$

$$b = \frac{1}{\mathbf{1}\mathbf{V}^{-1}\mathbf{1}^t} \tag{6}$$

and

$$c = \frac{\mathbf{R}\mathbf{V}^{-1}\mathbf{1}^t}{\mathbf{1}\mathbf{V}^{-1}\mathbf{1}^t}, \tag{7}$$

it can be obtained

$$a(\sigma^2 - b) = (m - c)^2, \tag{8}$$

that is the equation of a parabola in the (m, σ^2)-plane which provides the relationship between the minimum variance σ^2 and the fixed return m. Moreover, this is the equation of the efficient frontier in Markowitz's model.

In what follows, we will identify a portfolio P with its vector composition $(x_1, x_2, \ldots, x_n)$:

$P \equiv (x_1, x_2, \ldots, x_n)$.

The following lemma and its proof provide an algorithm to construct a basket composed by uncorrelated portfolios, which can help us to obtain a better diversification. On the other hand and using that basket as a basis in the Euclidean space, we can obtain a diagonal variance–covariance matrix of securities which notably simplifies the problem.

Lemma 1. *Starting from n linearly independent portfolios $P_1, P_2, \ldots, P_n$, n uncorrelated portfolios can be constructed.*

Proof 1. In effect, given n linearly independent portfolios, to obtain n uncorrelated portfolios, we can apply Gram-Schmidt Orthogonalization Algorithm, since, in this case, uncorrelated portfolios means orthogonal portfolios:

- $P_1^* = P_1$.
- $P_2^* = P_2 - \frac{\langle P_2, P_1^* \rangle}{\langle P_1^*, P_1^* \rangle} P_1^* = P_2 - \alpha_{2,1} P_1^*$, being $\alpha_{2,1} = \frac{\mathrm{cov}(P_2, P_1^*)}{\mathrm{var}(P_1^*)}$.

$\vdots$

- $P_k^* = P_k - \sum_{j=1}^{k-1} \alpha_{k,j} P_j^*$, where $\alpha_{k,j} = \frac{\mathrm{cov}(P_k, P_j^*)}{\mathrm{var}(P_j^*)}$, $j = 1, 2, \ldots, k-1$. □

The proof of the following theorem describes a methodology to obtain any portfolio in the efficient frontier starting from two other portfolios, one of them also in the frontier.

Theorem 1. *The efficient frontier is a one-dimensional affine subspace of $\Re^n$ and so it can be generated by any portfolio in the frontier and a suitable zero-net investment vector.*

Proof 2. Taking $\alpha_1 = 0$ in Eq. (1):

$x = \alpha_2 \mathbf{1}\mathbf{V}^{-1}$,

the condition $X \in S$ necessarily implies that $\alpha_2 \mathbf{1}\mathbf{V}^{-1}\mathbf{1}^t = \alpha_2 \frac{1}{b} = 1$, from which $\alpha_2 = b$. Thus, the portfolio $P_1 = b\mathbf{1}\mathbf{V}^{-1}$ is in the efficient frontier. In effect, the mean m_1 of P_1 is c and its variance σ_1^2 is b. Thus, because of Eq. (8), P_1 is in the efficient frontier. Indeed, it is the minimum of the parabola.

Analogously, taking $\alpha_2 = 0$ in Eq. (1):

$x = \alpha_1 \mathbf{R}\mathbf{V}^{-1}$,

the condition $X \in S$ necessarily implies that $\alpha_1 \mathbf{R}\mathbf{V}^{-1}\mathbf{1}^t = \alpha_1 \frac{c}{b} = 1$, from which $\alpha_1 = \frac{b}{c}$. Thus, the portfolio $P_2 = \frac{b}{c}\mathbf{R}\mathbf{V}^{-1}$ belongs to the efficient frontier. In effect, the mean m_2 of P_2 is $c + \frac{ab}{c}$ and its variance σ_2^2 is $b + \frac{ab^2}{c^2}$. Thus, because of Eq. (8), P_2 is also in the efficient frontier.

Therefore, we can construct the vector

$$p = \overrightarrow{P_1 P_2} = \frac{b}{c}\mathbf{R}\mathbf{V}^{-1} - b\mathbf{1}\mathbf{V}^{-1},$$

with the following characteristics:

- The total investment of p is $p\mathbf{1}^t = 0$.
- The mean of p is $\frac{ab}{c}$.
- The variance of p is $\frac{ab^2}{c^2}$.

Observe that p can be considered as a zero-net investment portfolio. But, in order to make the calculations easier, we choose p with the same variance as P_1. So, we define $p^* = \frac{c}{\sqrt{ab}} p$. Thus, the mean of p^* is $m^* = \sqrt{ab}$ and its variance is b.

Given an average return m, the portfolio in the efficient frontier (that is, with minimum variance) is $X = P_1 + \frac{m - m_1}{m^*} p^*$ or equivalently $X = P_1 + \frac{m - c}{\sqrt{ab}} p^*$. Reciprocally, given a portfolio $X = P_1 + \alpha p^*$, it belongs to the efficient frontier because its mean is given by $m_1 + \alpha m^* = c + \alpha\sqrt{ab}$ and the variance by $\sigma_1^2 + \alpha^2 \sigma_2^2 = b(1 + \alpha^2)$. We will like to remark that this means that the set of portfolios in the efficient frontier is exactly a line in $\Re^n$. □

Remark 1. The portfolios P_1 and p^* could be constructed starting from $\mathbf{1}\mathbf{V}^{-1}$ and $\mathbf{R}\mathbf{V}^{-1}$ and applying Lemma 1 in order to obtain two uncorrelated portfolios:

- $P_1^* = \mathbf{1}\mathbf{V}^{-1}$.
- $P_2^* = \mathbf{R}\mathbf{V}^{-1} - c\mathbf{1}\mathbf{V}^{-1}$.

By Eq. (6), $P_1^* \mathbf{1}^t = \frac{1}{b}$, from where $bP_1^* \mathbf{1}^t = 1$ and so $P_1 = bP_1^* \in H_1$. On the other hand, we have that $p^* = \sqrt{\frac{b}{a}} P_2^*$.

Remark 2. Vector p^* could have been obtained from Eq. (3), by considering the coefficient of parameter m.

Remark 3. The obtained efficient frontier coincides with the one given by Roy (1952).

Because the efficient frontier is a parabola, some properties can be deduced:

1. It can be calculated the portfolio with the smallest variance corresponding to the one with mean c. In this case, the variance of the portfolio is b and it is the smallest one among all the possible values of this parameter. Note that this portfolio is P_1.
2. c can be negative.
3. If a mean value less than c is fixed, a portfolio with a greater mean and the same variance can be obtained.

This approach to obtain the efficient frontier allows us to consider some particular cases in a natural way. Next, we sketch some of them.

1. There is a riskless asset. Graphically, the combination of the riskless asset with any point of the efficient frontier portfolios containing risky assets only is a line in the (m, σ)-plane, so that the efficient frontier will be a combination of the riskless asset with a point in the risky composition, such that the line connecting both points is tangent to the risky efficient frontier.

Let r_f be the rate of return of the riskless asset and X the portfolio composition in risky assets. In this case, the mean of the total portfolio will be

$$m = X\mathbf{R}^t + (1 - X\mathbf{1}^t)r_f = X(\mathbf{R}^t - \mathbf{1}^t r_f) + r_f \tag{9}$$

(note that $1 - X\mathbf{1}^t$ is the percentage that has not been invested in risky assets).
From Eq. (9), X must verify $X(\mathbf{R} - r_f\mathbf{1})^t = m - r_f$, and then we can follow a reasoning similar to the general case to obtain the optimal combination of risky assets, which is given by

$$P_0 = \frac{b}{c - r_f}(\mathbf{R} - r_f\mathbf{1})\mathbf{V}^{-1} = \frac{(\mathbf{R} - r_f\mathbf{1})\mathbf{V}^{-1}}{(\mathbf{R} - r_f\mathbf{1})\mathbf{V}^{-1}\mathbf{1}}.$$

Let $\omega = 1 - X\mathbf{1}^t$ be the percentage of investment in the riskless asset. Then, from Eq. (9), given $m \geqslant r_f$, one has

$$m = \omega r_f + (1 - \omega)m_0,$$

from which $\omega = \frac{m-m_0}{r_f-m_0}$ and $1 - \omega = \frac{r_f-m}{r_f-m_0}$. These last equations give the percentage of investment (ω) in the riskless asset and the proportion of investment $(1 - \omega)$ in risky assets (given by portfolio P_0).

2. Composition without investing 100% of the amount. The constraint of investing 100% of the amount seems no natural. The natural constraint should be to invest at most 100% of the amount. But, when we do not consider a riskless asset, we can take into account the possibility of investing less than 100% of the amount, since this strategy can give in some cases a better optimal portfolio. This situation is equivalent to the existence of a riskless asset with rate of return equal to zero and then we can proceed as in the previous item.

3. Conclusion

In this paper Markowitz's model is considered from a new point of view. More specifically, the frame of this paper is affine geometry because we introduce a symmetric bilinear form (inner product or scalar product) starting from the matrix of variances–covariances of the n assets composing the portfolio, which allows us to study some metric concepts as distance, orthogonality, etc.

The main results of this paper are:

1. The solution of Markowitz's model is deduced with purely geometric tools, by means of an orthogonal projection. Of course, searching for a minimum distance can be solved using the Lagrange constraint method, but our approach is much more intuitive and a matter of basic vector algebra. This approach makes possible to present this topic for an undergraduate audience.
2. The efficient portfolio is an affine subspace of dimension one, that is, a line in the space $\mathfrak{R}^n$ of the n-tuple representing the proportions of investment on n assets. Thus, the efficient portfolio is an affine subspace spanned by two (uncorrelated) points. Essentially, this is the "two fund separation theorem" which now can be understood in a more geometric way.
3. The application of the Gram-Schmidt Orthogonalization Algorithm allows us to work with a basis of orthogonal (perpendicular) portfolios and so with a diagonal variances–covariances matrix, which reduces the calculations and diversifies a possible "portfolio basket".
4. This approach allows for results being slightly sharper than corresponding ones given in the standard literature, and may be applied to more general situations. For example, when there is a riskless asset and when not the whole budget is invested, obtaining even a better optimal portfolio.

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