Financial Laws As Algebraic Automata

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It is shown that the concept of financial law has the structure of an automaton. Furthermore, a financial law induces a group structure into the monoid of the automaton. The concepts of stationary, stationary of order n, and dynamic financial laws are deduced, proving two algebraic characterizations. Finally the concept of $\bar{a}$-stationary financial law and some applications are introduced. © 2001 John Wiley & Sons, Inc.

I. INTRODUCTION

It is shown that a certain similarity exists between economic activity and formal computation.[1] This parallelism can be seen in the Mathematics of Finance, whose predecessors are the Italian scholars (see, for instance, Refs. 2 through 5).

The semi-axiomatic stage in the Mathematics of Finance starts with Insolera and there is no doubt that its subsequent influence has been noticed. In his work, Insolera quotes Fisher[6] and states a difference between the *state* of an element of wealth, which is the *capital* and the *movement* of an element of wealth during a period of time, which is the *interest*.

According to Fisher,[6] the interest is an *abstract device*, a stream, an *action* from which the capital is affected, producing modifications to itself. Insolera[7] defines the capital as "all the money invested in a financial operation, understanding that each action determines a quantitative variation of capital." This point of view is shared by other authors (e.g., Bhaumik and Das).[8]

Let us mention that the measurable and quantifiable characters of profit have been criticized and discussed by various authors like Hicks and Allen, who elaborated their theory without considering that profit was measurable, on the basis that only the consumer is capable of establishing a scale of preferences, thus giving rise to a so-called ordinal approach.

We are interested in this starting point, i.e., the relational and algebraic setting of financial laws. This puts in evidence the temporal structure of any financial process and the actions of the agent. Of course such an approach is not

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INTERNATIONAL JOURNAL OF INTELLIGENT SYSTEMS, VOL. 16, 1085–1105 (2001)

only ordinal, but also algorithmic and, because of its general setting, applicable to a wide variety of financial systems.

Our aim is to exhibit financial applications and examples that parallel the introduction of the algebraic theoretical concepts. What put in evidence the relative lightness of the automaton modeling of financial concepts, and the unification of many different concepts.

We say that an economic agent prefers the vector x to the vector y, if it is necessary for him to choose x, provided that he is offered the alternatives x and y.

This relation fulfills a set of axioms[8] that lay the foundations of the Mathematics of Finance. Basically, we focus on the following two elements:

(1) The *quantity* C or expression in monetary units of a product or service.
(2) The *time* t or moment of availability or maturity of the product or service.

It seems rational to assume that the quantity C_0 considered in the actual moment t_0, in order that it remains "equivalent" within one year, has to increase its value giving rise to another quantity $C > C_0$ (impatience): that is why we will say that time is a negative economic good.[9,10]

It is also assumed that the relation of preference ($\preceq$) fulfills the reflexive, transitive, and complete or decisive properties, that provide us with the relation of induced indifference ($\sim$) which fulfills the reflexive, symmetrical, and transitive properties; that is to say it is an equivalence relation.[11] Thus the classes of equivalence in $\sim$ are curves of financial indifference and the quotient is the "map of indifference."

From this construction we can generate financial laws from a map of financial indifference and the opposite should also occur. In this approach to the concept of financial law let us consider first the time as a discrete variable.

In order to proceed to a formalization, we need a mathematical tool different from the ones used up until now (basically the concept of function).† This is due to the following considerations:

(1) The difference between the period of time and the maturity date in a same financial system.
(2) The concept of financial law is based on a set of properties, one of them is the homogeneity of the first degree concerning the quantity. However, financial practice is introducing operations, such as "highly remunerated current accounts,"‡ in which homogeneity is questionable.[12] In effect, the projection or substitute in an instant p from a "sufficiently high quantity" does not coincide with the product of that quantity by the projection of the monetary unit. So a

† A *financial system* is a function $F(C, t; p)$, which C is the quantity or capital, t is the initial instant, and p is the final instant, which verifies the following conditions: homogeneity of degree one with regard to the quantity $C, F(C, p; p) = C$, decrease according to t, and increase according to p. If p is fixed, $F(C; t, p)$ is called a *financial law*.

‡ A current account is called "highly remunerated" or "super-account" if it applies the compound interest with increasing rates of interest according to bands of quantities.

"super-account" supposes the application of several financial laws, each of which acts over a specified quantity.

This implies that we cannot calculate the equivalent of a financial capital by consecutive multiplications of C by $F(1,t;p)$, but that we have to compose the quantity C inside the law $F(C,t;p)$ consecutively.

(3) Mathematically, this procedure presents some difficulties. For instance, to handle expressions as

$$F[F(C,t,t'),t'),t'']$$

However, these difficulties can be solved using the *Algebraic Theory of Automata* (see Refs. 13 through 22). Indeed, we can separate quantities, for instance, and study properties and characterizations from additive and multiplicative systems.§

Therefore, we need to restrict the set of maturities to a discrete set, that, moreover, does not represent any lack of generality.

This paper is organized as follows: Section II introduces the model applied, presenting the concepts of semiautomaton, automaton, and series composition. Moreover, in Section III, axioms of preference in a rational choice are presented, which justifies the definition of financial law in Section IV. Section V includes the algebraic properties of financial law and, finally, a classification is reported with two theorems of characterization of stationary financial laws in Section VI.

II. SEMIAUTOMATA AND AUTOMATA

We recall some definitions and prerequisites. For these and further notions and results see the basic books of Lidl and Pilz[20] or Arbib.[14] A *semiautomaton* is a triple $\mathscr{S} = (Z, A, \delta)$ consisting of two nonempty sets Z and A and a function‖ $\delta\colon Z \times A \to Z$. Z is called the *set of states*, A the *input alphabet*, and δ the *next-state function* of $\mathscr{S}$.

An *automaton* is defined as a 5-ple $\mathscr{A} = (Z, A, B, \delta, \lambda)$, where (Z, A, δ) is a semiautomaton, B is a nonempty set called *output alphabet*, and $\lambda\colon Z \times A \to B$ is the *output function*.

If $z \in Z$ and $a \in A$, we then interpret $\delta(z,a) \in Z$ as the next state into which z is transformed by the input a. $\lambda(z,a) \in B$ is the output of z resulting from the input a. Thus, if the automaton is in state z and receives input a, then it changes into the state $\delta(z,a)$ with output $\lambda(z,a)$.

Let us consider the set $\overline{A} = F_A$ of the words (the empty word included) that we can write with the elements (letters) of a set A. Let $\bar{a} = a_1 a_2 \cdots a_p$,

§A system $F(C;t,p)$ is called *simply multiplicative* (resp. *additive*) if $F(1;t,p) \cdot F(1;p,p') = F(1;t,p')$ [resp. $I(1;t,p) + I(1;p,p') = I(1;t,p')$, being $I(1;t,p) = F(1;t,p) - 1$]. Moreover, simply multiplicative (resp. additive) + stationary = *amply multiplicative* (resp. *additive*).

‖In our study of automata, a *partial function* is a correspondence $f\colon A \to B$ such that $f(a) = b$ and $f(a) = c$ implies $b = c$, i.e., does not need that $\mathrm{Dom}(f) = A$.

$\bar{b} = b_1 b_2 \cdots b_q$ be words in $\bar{A}$. An operation $\top$ is defined in $\bar{A}$ as

$$\bar{a} \top \bar{b} = a_1 a_2 \cdots a_p b_1 b_2 \cdots b_q$$

Obviously, $\top$ is an internal operation in $\bar{A}$, is associative, and the empty word Λ is the identity element, $\bar{a} \top \Lambda = \Lambda \top \bar{a} = \bar{a}$, for every $\bar{a}$ in $\bar{A}$. The monoid $\bar{A}$ is called the *free monoid on* A.

In our study of automata we extend the input set A to the free monoid $\bar{A} = F_A$, with Λ as identity. We also extend δ and λ from $Z \times A$ to $Z \times \bar{A}$ (see Sect. IV B).

Let $\mathscr{S} = (Z, A, \delta)$ be a semiautomaton and consider $\bar{\mathscr{S}} = (Z, \bar{A}, \bar{\delta})$. We use the following notation. For every $\bar{a} \in \bar{A}$, let $f_{\bar{a}}$: $Z \to Z$ such that $z \mapsto f_{\bar{a}}(z) = \bar{\delta}(z, \bar{a})$, i.e., $f_{\bar{a}}$ acts, transforming z into the next state $\bar{\delta}(z, \bar{a})$.

Let us recall the following result of Lidl and Pilz.[21]

THEOREM 1. $M_{\mathscr{S}} = (\{f_{\bar{a}} / \bar{a} \in \bar{A}\}, \circ)$ *is a monoid.*

An equivalence relation in $\bar{A}$ is defined as: $\bar{a} \equiv \bar{a}_2$ if and only if $f_{\bar{a}_1} = f_{\bar{a}_2}$.

In other words, $\bar{a}_1 \equiv \bar{a}_2$ if and only if, for every $z \in Z$, $f_{\bar{a}_1}(z) = f_{\bar{a}_2}(z)$, or, equivalently, for every $z \in Z$, $\bar{\delta}(z, \bar{a}_1) = \bar{\delta}(z, \bar{a}_2)$.

Let $\mathscr{A} = (Z, A, B, \delta, \lambda)$ be an automaton and $z, z' \in Z$. Then the following definition: $z \sim z'$ if and only if for every $\bar{a} \in \bar{A}$,

$$\bar{\lambda}(z, \bar{a}) = \bar{\lambda}(z', \bar{a})$$

yields an equivalence relation in Z.

DEFINITION 1 (*Series Composition*). *Let* $\mathscr{A}_i = (Z_i, A_i, B_i, \delta_i, \lambda_i)$ $(i \in \{1, 2\})$ *be automata, with the additional assumption* $A_2 = B_1$. *The* series composition $\mathscr{A}_1 \# \mathscr{A}_2$ *of* $\mathscr{A}_1$ *and* $\mathscr{A}_2$ *is defined as the automaton* $(Z_1 \times Z_2, A_1, B_2, \delta, \lambda)$ *with*

$$\delta((z_1, z_2), a_1) = (\delta_1(z_1, a_1), \delta_2(z_2, \lambda_1(z_1, a_1)))$$

and

$$\lambda((z_1, z_2), a_1) = \lambda_2(z_2, \lambda_1(z_1, a_1))$$

$((z_1, z_2) \in Z_1 \times Z_2, a_1 \in A_1)$.

This automaton operates as follows: An input $a_1 \in A_1$ *operates on* z_1 *and gives a state transition* $z_1' = \delta_1(z_1, a_1)$ *and an output* $b_1 = \lambda_1(z_1, a_2) \in B_1 = A_2$. *This output* b_1 *operates on* Z_2, *transforms a* $z_2 \in Z_2$ *into* $z_2' = \delta_2(z_2, b_1)$ *and produces the output* $\lambda_2(z_2, b_1)$. *Then* $\mathscr{A}_1 \# \mathscr{A}_2$ *is in the next state* (z_1', z_2').

Analogously, we can define the series composition of an automaton $\mathscr{A} = (Z_1, A_1, B, \delta_1, \lambda)$ and a semiautomaton $\mathscr{S} = (Z_2, B, \delta_2)$ as the semiautomaton $\mathscr{A} \# \mathscr{S} = (Z_1 \times Z_2, A_1, \delta)$, with next-state function δ.

III. RATIONAL CHOICE

Let (t, C) be the expression in monetary unities of an economic good referred to an instant t of time. Let $E = \mathscr{R} \times \mathscr{R}^+ = \{(t, C)/t \in \mathscr{R}, C \in \mathscr{R}^+\}$. In the following, an axiom system to be imposed to E is introduced:

AXIOM 1. *A total preorder relation can be defined in E, i.e., a binary relation $\preceq$ with reflexive, transitive, and complete properties.*

Consequences of Axiom 1. We consider the relation $\sim$ defined on E as: For every (t, C) and $(t', C') \in E$,

$$(t, C) \sim (t', C') \text{ if and only if } (t, C) \preceq (t', C') \text{ and } (t', C') \preceq (t, C)$$

(1) $\sim$ is an equivalence relation.
(2) $\preceq$ is compatible with $\sim$, in the sense that $(t, C) \preceq (t', C')$, if it is verified $(t, C) \sim (t_1, C_1)$ and $(t', C') \sim (t'_1, C'_1)$, then $(t_1, C_1) \preceq (t'_1, C'_1)$.
(3) On $E/\sim$ we define a relation $\preceq^*$ as

$$[(t, C)] \preceq^* [(t', C')] \Leftrightarrow (t, C) \preceq (t', C')$$

We can prove that $\preceq^*$ is an order relation on $E/\sim$, called *order relation associated with the relation* $\preceq$.

On the other hand, a *utility function* is a function

$$u \colon \mathscr{R} \times \mathscr{R}^+ \to \mathscr{R}$$

increasing with preferences. This means that if $(t, C) \preceq (t', C')$, then $u(t, C) \leq u(t', C')$ and if $(t, C) \sim (t', C')$, then $u(t, C) = u(t', C')$.

Therefore, each equivalence class of the relation $\sim$ is $[(t, C)] = \{(t', C') \in \mathscr{R} \times \mathscr{R}^+/u(t, C) = u(t', C')\}$. It is called *indifference curve* and is represented briefly by the level equation $u(t, C) = k$.

AXIOM 2. *If $C \leq C'$ and $t' \leq t$ then $(t, C) \preceq (t', C')$. Now, if $C < C'$ and $t = t'$ or $C = C'$ and $t > t'$, then $(t, C) \prec (t', C')$, where $\prec$ means $\preceq$ but not $=$.*

Consequences of Axiom 2. This axiom means that, given a financial capital (t_0, C_0), all points placed on the north–west of (t_0, C_0) are better than (t_0, C_0) and the points placed on the south–east are worse than (t_0, C_0). Therefore, according to axiom 2, the indifference curves must have a positive slope (> 0) and it is obvious that curves farther from the origin will represent greater *utility index* (k).

Interpretation of Axiom 2. There can be no doubt that for each rational economic subject, the measure in monetary unities (C) of an economic good increases with respect to the instant (t) of availability of this good, then the time can be considered as a “negative economic good.”

AXIOM 3. *The indifference curves are convex (increasing slope), i.e.,* $(t_0, C_0) \sim (t_1, C_1)$ *implies* $\alpha(t_0, C_0) + (1-\alpha)(t_1, C_1) \preceq (t_0, C_0)$, *for every* $\alpha \in \mathscr{R}$ *such that* $0 \leq \alpha \leq 1$.

An interpretation of axiom 3 can be illustrated by an inspection of Figure 1. At point B the slope is high: the economic subject is prepared to give up a lot of C by changing a unit of t decrease (because the maturity is far according to a fixed point t_0). Stated otherwise: the interest produced by a quantity C_0 placed at an instant t_0 becomes higher the farther we move from that temporal point. At point A the slope is less than that of B: the subject is prepared to give up less than C by changing a unit of t. At point C the slope is still smaller: the subject is prepared to give up very little of C, by changing a unit of t (now the maturity is immediate, i.e., t is very little).

Financial logic tells us that if a curve is over another one, its slope has to be bigger, to verify that the interest increases in relation to the quantity that is produced, that is:

AXIOM 4. *For all* $t \in \mathscr{R}$ *and for all* $a \in \mathscr{R}^+$, $C_0 < C_1$ *and* $(t, C_0) \sim (t+a, C_0+x)$ *implies* $(t+a, C_1+x) \prec (t, C_1)$.

IV. DEFINITION OF FINANCIAL LAW

(1) We consider the automaton $\mathscr{A} = (Z, A, FC_B, \delta_1, \lambda_1)$, where:
   (a) Z is a subset of $\mathscr{Z}$ (integer numbers set): $Z \subseteq \mathscr{Z}$.
   (b) A is a subgroup of $(\mathscr{Z}, +)$.
   (c) FC_B is the set of strictly increasing bijections of a set B, subset of $\mathscr{R}$, onto itself.
   (d) δ_1: $Z \times A \to Z$ such that $(t, a) \mapsto \delta_1(t, a) = t + a$ is an action of the group $(A, +)$ on the set Z.
   (e) λ_1: $Z \times A \to FC_B$ such that $(t, a) \mapsto \lambda_1(t, a)$ is a function which verifies the following conditions: for every $t \in Z$, $a, a' \in A$ and $x, y \in A^+$, $x \neq 0$:
      (i) $\lambda_1(t+a, a') \circ \lambda_1(t, a) = \lambda_1(t, a+a')$.
      (ii) $\lambda_1(t, a) < \lambda_1(t, a+x)$.
      (iii) $\lambda_1(t, a+x) - \lambda_1(t, a) \in FC_B$.
      (iv) $\lambda_1(t, a+x) - \lambda_1(t, a) \leq \lambda_1(t, a+x+y) - \lambda_1(t, a+y)$.

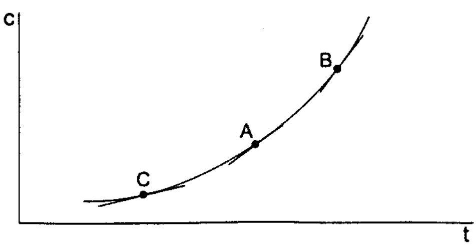

Figure 1. Different relations of substitution between C and t.

(2) Moreover the semiautomaton $\mathscr{S} = (B, FC_B, \delta_2)$, is introduced, where:
   (a) B is a subset of $\mathscr{R}$: $B \subset \mathscr{R}$.
   (b) δ_2: $B \times FC_B \to B$ such that $(C, f) \mapsto \delta_2(C, f) = f(C)$ is an action of the group $(FC_B, \circ)$ on B.

Now, a *financial law* is the series composition $\mathscr{L} = \mathscr{A} \# \mathscr{S}$ and $\mathscr{S}$ defined by the semiautomaton $(Z \times B, A, \delta)$, with $\delta((t, C), a) = (\delta_1(t, a), \delta_2(c, \lambda_1(t, a)))$; i.e., $\delta((t, C), a) = (t + a, \lambda_1(t, a)(C))$.

We list some consequences of the definition above. It is easy to see, for every $t \in Z, a, a' \in A$ and $x \in A^+$, $x \neq 0$:

(1) $\lambda_1(t, 0) = Id_B$.
(2) $\lambda_1^{-1}(t, a) = \lambda_1(t + a, -a)$.
(3) $\lambda_1(t, a) < \lambda_1(t, a + x)$ if and only if $\lambda_1(t, 0) < \lambda_1(t, x)$.
(4) $\lambda_1(t, a) > \lambda_1(t, a - x)$ if and only if $\lambda_1(t, 0) > \lambda_1(t, -x)$.
(5) $\lambda_1(t + x, a) < \lambda_1(t, a + x)$. In effect, because of the first condition of financial law, $\lambda_1(t + x, a) \circ \lambda_1(t, x) = \lambda_1(t, a + x)$ that implies $\lambda_1(t + x, a) = \lambda_1(t, a + x) \circ \lambda_1(t + x, -x) < \lambda_1(t, a + x)$.
(6) $\lambda_1(t + x, a - x) < \lambda_1(t, a)$. In effect, because of consequence 4, $\lambda_1(t + x, a - x) < \lambda_1(t, a - x + x) = \lambda_1(t, a)$.

A. Financial Laws of Capitalization and Discount

If we restrict δ_1 and λ_1 in the definition of financial law to the monoids $A^+ = \{a \in A / a \geq 0\}$ and $A^- = \{a \in A / a \leq 0\}$, we obtain the concept of *financial law of capitalization* and *financial law of discount*, respectively.

Let us classify financial laws according to criteria: time intervals and actions:

(1) Classification *according to the set* $Z \times B$ (time). Let $\mathscr{L} = (Z \times B, A, \delta)$ and $\mathscr{L}' = (Z' \times B', A, \delta')$ be financial laws. We say that $\mathscr{L} \leq \mathscr{L}'$ if $Z \times B \subseteq Z' \times B'$ and δ and λ are the restrictions of δ' and λ' on $Z \times B$, respectively. It is about two financial laws that differ only in that the first is applied to some days or some capitals to which the second is applied.
(2) Classification *according to the group A* (actions). We suppose that $Z = \mathscr{Z}$:
   (a) When $A = \mathscr{Z}$, we say that $\mathscr{L}$ is a financial law associated to an operation with liquidity.
   (b) When $A = n\mathscr{Z}$, we say that $\mathscr{L}$ is a financial law associated to a fixed term operation.
      (i) When $A = 360\mathscr{Z}$, it is about an annual term.
      (ii) When $A = 30\mathscr{Z}$, it is about a monthly term.

The graphic representation of a financial law is found in Figure 2.

B. Extension of the Concept of Financial Law

We extend the input set A to the free group $\overline{A} = \overline{F}_A$ and FC_B to the free group $\overline{FC_B}$, with identity Λ. We also extend δ_1 and λ_1 from $Z \times A$ to $Z \times \overline{A}$

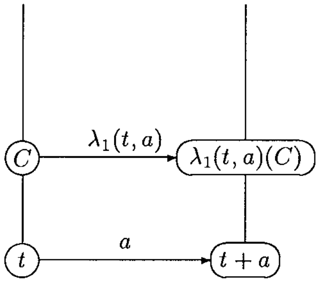

Figure 2. Representation of a financial law.

defining for $t \in Z$ and $a_1, a_2, \ldots, a_r \in A$:

$$\bar{\delta}_1(t, \Lambda) = t$$

$$\bar{\delta}_1(t, a_1) = \delta_1(t, a_1) = t + a_1$$

$$\bar{\delta}_1(t, a_1 a_2) = \delta_1\big(\bar{\delta}_1(t, a_1), a_2\big) = \delta_1(t + a_1, a_2) = t + (a_1 + a_2)$$

$$\vdots$$

$$\bar{\delta}_1(t, a_1 a_2 \cdots a_r) = \delta_1\big(\bar{\delta}_1(t, a_1 a_2 \cdots a_{r-1}), a_r\big) = t + (a_1 + a_2 + \cdots + a_r)$$

and

$$\bar{\lambda}_1(t, \Lambda) = \Lambda$$

$$\bar{\lambda}_1(t, a_1) = \lambda_1(t, a_1)$$

$$\bar{\lambda}(t, a_1 a_2) = \lambda_1(t, a_1)\bar{\lambda}_1(\delta_1(t, a_1), a_2) = \lambda_1(t, a_1)\lambda_1(t + a_1, a_2)$$

$$\vdots$$

$$\begin{aligned}\bar{\lambda}_1(t, a_1 a_2 \cdots a_r) &= \lambda_1(t, a_1)\bar{\lambda}_1(\delta_1(t, a_1), a_2 a_3 \cdots a_r) \\ &= \lambda_1(t, a_1)\bar{\lambda}_1(t + a_1, a_2 a_3 \cdots a_r) \\ &= \lambda_1(t, a_1)\lambda_1(t + a_1, a_2) \cdots \lambda_1(t + a_1 + a_2 + \cdots + a_{r-1}, a_r)\end{aligned}$$

Moreover, we extend δ_2 from $B \times FC_B$ to $B \times \overline{FC}_B$, defining for $C \in B$ and $f_1, f_2, \ldots, f_r \in FC_B$:

$$\bar{\delta}_2(C, \Lambda) = C,$$

$$\bar{\delta}_2(C, f_1) = \delta_2(C, f_1) = f_1(C)$$

$$\begin{aligned}\bar{\delta}_2(C, f_1 f_2) &= \delta_2\big(\bar{\delta}_2(C, f_1), f_2\big) \\ &= \delta_2(f_1(C), f_2) = f_2[f_1(C)] = (f_2 \circ f_1)(C)\end{aligned}$$

$$\vdots$$

$$\bar{\delta}_2(C, f_1 f_2, \ldots, f_r) = \delta_2\big(\bar{\delta}_2(c, f_1 f_2 \cdots f_{r-1}), f_r\big) = (f_r \circ \cdots \circ f_2 \circ f_1)(C)$$

We are now able to define, as a particular case, a classical financial law. In effect, let $f_1, f_2, \ldots, f_r$ be homogeneous of the degree one with respect to C:

$$\bar{\delta}_2(C, f_1 f_2 \cdots f_r) = (f_r \circ \cdots \circ f_2 \circ f_1)(C) = C \cdot f_1(1) \cdot f_2(1) \cdots f_r(1).$$

The financial law $\mathscr{L}$ (homogeneous of the degree one with respect to the quantity) is called a *classical financial law.*

Let

$$(t_1, C_1) = \delta((t, C), a_1)$$

$$(t_2, C_2) = \bar{\delta}((t, C), a_1 a_2) = \bar{\delta}(\delta((t, C), a_1), a_2)$$

$$\vdots$$

The input sequence $a_1 a_2 \cdots a_r \in \bar{A}$ acts on the capital (t, C) that transforms into (t_1, C_1), until the final capital (t_r, C_r) (see Fig. 3).

In the following $\equiv_1$ (resp. $\equiv_2$) denotes the equivalence relation on $\bar{A}$ (resp. $\overline{FC_B}$) associated to the semiautomaton (Z, A, δ_1) [resp. (B, FC_B, δ_2)]. Moreover $\sim_1$ denotes the equivalence relation on Z associated to the automaton $(Z, A, FC_B, \delta_1, \lambda_1)$ (see Sect. II).

V. ALGEBRAIC PROPERTIES

THEOREM 2. *Let $\mathscr{L}$ be a financial law. Then $M_{\mathscr{L}}$ is an abelian group and $A \cong M_{\mathscr{L}}$.*

Proof. We know that $M_{\mathscr{L}}$ is a monoid. It is easy to see that $M_{\mathscr{L}}$ is an abelian group, indeed: $-\bar{a} = (-a_1)(-a_2)\cdots(-a_n)$, $f_{\bar{a}} \circ f_{-\bar{a}} = f_{-\bar{a}} \circ f_{\bar{a}} = \mathrm{Id}_{Z\times B}$, and $f_{-\bar{a}} \circ f_{\bar{a}} = \mathrm{Id}_{Z\times B}$. Moreover, the correspondence φ: $\bar{A}/\equiv \to M_{\mathscr{L}}$ such that: $[\bar{a}] \mapsto \varphi([\bar{a}]) = f_{\bar{a}}$ is a homomorphism between the groups A and $M_{\mathscr{L}}$ (Ref. 21). Finally, as $\bar{A}/\equiv_1 \cong A$, $\bar{A}/\equiv$ coincides with $A/\equiv_1$ and $A/\equiv \cong M_{\mathscr{L}}$, we have $A \cong M_{\mathscr{L}}$. ∎

A semigroup S_1 is said to *divide* a semigroup S_2, if S_1 is a homomorphic image of a subsemigroup of S_2. In symbols: $S_1 | S_2$.

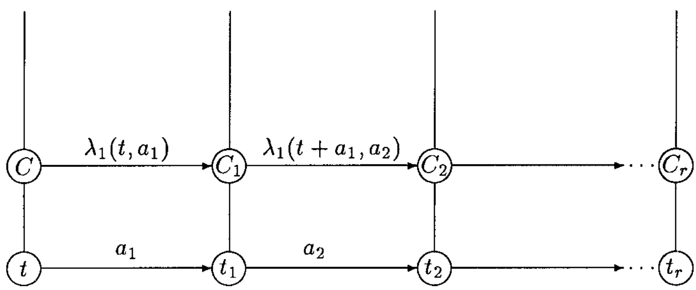

Figure 3. Financial law with a multiple input.

Let $\mathscr{A}_1 = (Z_1, A_1, B_1, \delta_1, \lambda_1)$ and $\mathscr{A}_2 = (Z_2, A_2, B_2, \delta_2, \lambda_2)$ be automata. An (*automata-*) *homomorphism* $\Phi: \mathscr{A}_1 \to \mathscr{A}_2$ is a triple $\Phi = (\zeta, \alpha, \beta)$, $\zeta: Z_1 \to Z_2$, $\alpha: A_1 \to A_2$, $\beta: B_1 \to B_2$ with the property

$$\zeta(\delta_1(z,a)) = \delta_2(\zeta(x), \alpha(a))$$

$$\beta(\delta_1(z,a)) = \lambda_2(\zeta(z), \alpha(a))$$

Φ is called a *monomorphism* (*epimorphism*, *isomorphism*) if all functions ζ, α, and β are injective (surjective, bijective).

An automaton $\mathscr{A}_1 = (Z_1, A, B, \delta_1, \lambda_1)$ [resp., a semiautomaton $\mathscr{S}_1 = (Z_1, A, \delta_1)$] is said to *divide* an automaton $\mathscr{A}_2 = (Z_2, A, B, \delta_2, \lambda_2)$] [resp., a semiautomaton $\mathscr{S}_2 = (Z_2, A, \delta_2)$] (equal input and output alphabets) if $\mathscr{A}_1$ (resp., $\mathscr{S}_1$) is a homomorphic image of a subautomaton of $\mathscr{A}_2$ (resp., of a subsemiautomaton of $\mathscr{S}_2$). In symbols: $\mathscr{A}_1 | \mathscr{A}_2$ (resp., $\mathscr{S}_1 | \mathscr{S}_2$).

Two semigroups, automata or semiautomata are called *equivalent* if they divide each other. In symbols $S_1 \sim S_2$, $\mathscr{A}_1 \sim \mathscr{A}_2$, or $\mathscr{S}_1 \sim \mathscr{S}_2$.

THEOREM 3. *Let $\mathscr{A}_1 = (Z_1, A, B, \delta_1, \lambda_1)$ be an automaton. We suppose that the set Z_1 is isomorphic to the set Z_2. Let f: $Z_1 \to Z_2$ be a bijection. We define*:

(1) δ_2: $Z_2 \times A \to Z_2$ *such that* $(z_2, a) \mapsto \delta_2(z_2, a) = f[\delta_1(f^{-1}(z_2), a)]$:

$$\begin{tikzcd} Z_2 \times A \arrow[r, "\delta_2"] \arrow[d, "f^{-1} \times Id_A"'] & Z_2 \\ Z_1 \times A \arrow[r, "\delta_1"'] & Z_1 \arrow[u, "f"'] \end{tikzcd}$$

(2) λ_2: $Z_2 \times A \to B$ *such that* $(z_2, a) \mapsto \lambda_2(z_2, a) = \lambda_2(f^{-1}(z_2), a)$:

$$\begin{tikzcd} Z_2 \times A \arrow[r, "\lambda_2"] \arrow[d, "f^{-1} \times Id_A"'] & B \\ Z_1 \times A \arrow[ur, "\lambda_1"'] & \end{tikzcd}$$

Then it verifies that $\mathscr{A}_1 \sim \mathscr{A}_2$, being $\mathscr{A}_2 = (Z_2, A, B, \delta_2, \lambda_2)$.

The following is an important practical consequence of the theorem above.

Let $\mathscr{L} = (Z \times B, A, \delta)$ be a financial law where the set B of quantities is expressed in a given monetary unit. Let B_m be the set of the previous quantities expressed in another monetary unit m. In these conditions, there exists a financial law $\mathscr{L}_m = (Z \times B_m, A, \delta_m)$ which is equivalent to $\mathscr{L}$.

In this way, every financial law $\mathscr{L}$ will be applied to any money: peseta, dollar, mark, etc.

In effect, if we use as a unit of measure, the money m, given a financial law $\mathscr{L}$, we will have a bijection

$$f_m: B \to B_m$$

from which we will obtain a financial law $\mathscr{L}_m$, such that $\mathscr{L}_m \sim \mathscr{L}$.

Analogously, Theorem 3 allows us to operate using thousands, millions, etc. of those money units.

A. Preference Relation between Financial Capitals

Let $\mathscr{L}$ be a financial law. On $Z \times B$ we define the following binary relation, in an extensive sense:

$$(t, C) \preceq (t', C) \text{ if and only if } t' - t \in A \text{ and } \lambda_1(t, t' - t)(C) \leq C'$$

The indifference relation associated to the preference relation is the following:

B. Equivalence between Financial Capitals

Let $\mathscr{L}$ be a financial law. The equivalence $(t, C) \sim (t', C')$ is defined as follows: there exists $a \in A$ such that $\delta((t, C), a) = (t', C')$.

As $\delta((t, C), a) = (t', C')$ and $\delta((t, C), a) = (\delta_1(t, a), \delta_2(C, \lambda_1(t, a)))$, then $(\delta_1(t, a), \delta_2(C, \lambda_1(t, a))) = (t', C')$, or, analogously:

(1) $\delta_1(t, a) = t' \Leftrightarrow t + a = t' \Leftrightarrow a = t' - t$
(2) $\delta_2(C, \lambda_1(t, a)) = C' \Leftrightarrow \lambda_1(t, a)(C) = C'$

Obviously $\sim$ is an equivalence relation.

C. Meaning of Relations $\equiv$ and $\sim$ in a Financial Law $\mathscr{L}$

Some considerations about the relation $\equiv_1$ follow. We have:

$$\bar{a} \equiv_1 \bar{a}' \Leftrightarrow f_{\bar{a}} = f_{\bar{a}'} \Leftrightarrow \forall t \in Z: \qquad f_{\bar{a}}(t) = f_{\bar{a}'}(t) \Leftrightarrow \forall t \in Z$$

$$\bar{\delta}_1(t, \bar{a}) = \bar{\delta}_1(t, \bar{a}') \Leftrightarrow \left(\text{if } \bar{a} = a_1 a_2 \cdots a_n \text{ and } \bar{a}' = a'_1 a'_2 \cdots a'_m\right)$$
$$\Leftrightarrow t + a_1 + a_2 + \cdots + a_n = t + a'_1 + a'_2 + \cdots + a'_m$$
$$\Leftrightarrow a_1 + a_2 + \cdots + a_n = a'_1 + a'_2 + \cdots + a'_m$$

We define the following mapping:

$$\varphi: \bar{A}/\equiv_1 \to A$$

by

$$[\bar{a}] = [a_1 a_2 \cdots a_n] \mapsto \varphi([\bar{a}]) = a_1 + a_2 + \cdots + a_n$$

Of course φ is well defined. Furthermore, φ is a homomorphism of groups:

$$\varphi: \left(\bar{A}/\equiv_1, \text{concatenation}\right) \to (A, +)$$

being $[\bar{a}][\bar{a}'] = [\overline{aa'}]$. Indeed $\varphi([\bar{a}][\bar{a}']) = \varphi[\overline{aa'}]) = \varphi([\bar{a}]) + \varphi([\bar{a}'])$.

Therefore $(\overline{A}/\equiv_1, \text{concatenation}) \cong (A, +)$.

Here are some considerations about $\sim_1$. We have:

$$
\begin{aligned}
t \sim_1 t' &\Leftrightarrow \forall \bar{a} \in \overline{A}\colon \bar{\lambda}_1(t, \bar{a}) = \bar{\lambda}_1(t', \bar{a}) \Leftrightarrow (\text{if } \bar{a} = a_1 a_2 \cdots a_n) \\
&\Leftrightarrow \lambda_1(t, a_1)\lambda_1(t + a_1, a_2) \cdots \lambda_1(t + a_1 + \cdots + a_{n-1}, a_n) \\
&= \lambda_1(t', a_1)\lambda_1(t' + a_1, a_2) \cdots \lambda_1(t' + a_1 + \cdots + a_{n-1}, a_n) \\
&\Leftrightarrow \left\{ \begin{array}{rcl} \lambda_1(t, a_1) & = & \lambda_1(t', a_1) \\ \lambda_1(t + a_1, a_2) & = & \lambda_1(t' + a_1, a_2) \\ \vdots & = & \vdots \\ \lambda_1(t + a_1 + \cdots + a_{n-1}, a_n) & = & \lambda_1(t' + a_1 + \cdots + a_{n-1}, a_n) \end{array} \right. \\
&\Leftrightarrow t + a \sim_1 t' + a, \qquad \forall a \in A
\end{aligned}
$$

This allows to define the following automaton:

$$\delta_1^*: Z/\sim_1 \times A \to Z/\sim_1$$

where

$$([t], a) \mapsto \delta_1^*([t], a) = [t + a]$$

Of course δ_1^* is well defined.

$$\lambda_1^*: Z/\sim_1 \times A \to FC_B$$

where

$$([t], a) \mapsto \lambda_1^*([t], a) = \lambda_1(t, a)$$

$$\delta_2^*: B \times FC_B \to B$$

defined by

$$(C, f) \mapsto \delta_2^*(C, f) = f(C)$$

The semiautomaton $\mathscr{A}^* \# \mathscr{S}^*$ being $\mathscr{A}^* = (Z/\sim_1, A, FC_B, \delta_1^*, \lambda_1^*)$ and $\mathscr{S}^* = (B, FC_B, \delta_2^*)$ is called the equivalent minimal semiautomaton of $\mathscr{A}$.

More particularly,

$$\mathscr{L}^* = \mathscr{A}^* \# \mathscr{S}^* = (Z/\sim_1 \times B, A, \delta^*)$$

is called the *equivalent minimal financial law of* $\mathscr{A}$.

VI. STATIONARY FINANCIAL LAWS

Last construction justifies the following definition.

DEFINITION 2. *A financial law is called a* stationary financial law *if* $Card(Z/\sim_1) = 1$, *i.e.*, $Z/\sim_1 = \{[t_0]\}$.

In this case, δ^* and λ^* only depend on A and B.

More generally, the previous equivalence relation $\sim_1$ allows the following.

A. Classification of Financial Laws

(1) *Stationary or stationary financial laws of order* 1 if $Card(Z/\sim_1) = 1$, i.e., $Z/\sim_1 = \{[t_0]\}$ or, analogously,

$$\forall t \in Z, \qquad \forall \bar{a} \in \bar{A}, \qquad \bar{\lambda}_1(t, \bar{a}) = \bar{\lambda}_1(t_0, \bar{a})$$

(2) *Stationary financial laws of order* n if $Card(Z/\sim_1) = n$, i.e., $Z/\sim_1 = \{[t_1], [t_2], \ldots, [t_n]\}$ or, analogously,

$$\forall t \in Z, \qquad \forall \bar{a} \in \bar{A}, \qquad \exists t_i (i = 1, 2, \ldots, n) / \bar{\lambda}_1(t, \bar{a}) = \bar{\lambda}_1(t_i, \bar{a})$$

(3) *Dynamic financial laws* if $Card(Z/\sim_1) = Card(\mathscr{Z})$, i.e., $Z/\sim_1 = Z$ or, analogously,

$$\forall t, t' \in Z, \qquad t \sim_1 t' \Rightarrow t = t'$$

B. Examples

(A) Let us consider the following financial law in which $Z = A = \mathscr{Z}$ and $B = \mathscr{R}^+$: δ_1: $\mathscr{Z} \times \mathscr{Z} \to \mathscr{Z}$ defined by $(t, a) \mapsto \delta_1(t, a) = t + a$, λ_1: $\mathscr{Z} \times \mathscr{Z} \to FC_B$, such that $(t, a) \mapsto \lambda_1(t, a)$: $\mathscr{R}^+ \to \mathscr{R}$ defined, at a given time t, by $\forall C \in B$, $\lambda_1(t, a)(C) = C \cdot e^{ka}$, $k > 0$.

In this case, for every $t \in Z$, and for every $\bar{a} \in \bar{A}$ such that $\bar{\lambda}_1(t, \bar{a}) = \bar{\lambda}_1(t_0, \bar{a})$, the property holds: if $Z/\sim_1 = \{[t_0]\}$ then $\mathscr{L}$ is stationary or stationary of order 1.

(B) Let us consider the following financial law in which $Z = \mathscr{Z}$, $A = 2\mathscr{Z}$, and $B = \mathscr{R}^+$: δ_1: $\mathscr{Z} \times 2\mathscr{Z} \to \mathscr{Z}$ defined by $(t, a) \mapsto \delta_1(t, a) = t + a$, λ_1: $\mathscr{Z} \times 2\mathscr{Z} \to FC_B$ such that $(t, a) \mapsto \lambda_1(t, a)$: $\mathscr{R}^+ \to \mathscr{R}$ defined, at a given time t, by $\forall C \in B$, $\lambda_1(t, a)(C) = C \cdot e^{ka(|\sin \pi/2(t-1)|+1)}$, being $k > 0$, or, analogously,

$$\lambda_1(t, a)(C) = \begin{cases} C \cdot e^{ka} & \text{if } t \text{ is odd} \\ C \cdot e^{2ka} & \text{if } t \text{ is even} \end{cases}$$

because

$$\left| \sin \frac{\pi}{2}(t-1) \right| = \begin{cases} 0 & \text{if } t \text{ is odd} \\ 1 & \text{if } t \text{ is even} \end{cases}$$

In this case, if $Z/\sim_1 = \{[t_0], [t_0 + 1]\}$ the $\mathscr{L}$ is stationary of order 2.

(C) Generally, if $Z = \mathscr{Z}$, $A = n\mathscr{Z}$, $B = \mathscr{R}^+$, and

$$\lambda_1(t, a)(C) = Ce^{ka(|\sin((n-1)\pi/n)(t-1)|+1)}, \qquad k > 0$$

the property holds: if $Z/\sim_1 = \{[t_0], [t_0 + 1], \ldots, [t_0 + n - 1]\}$ then $\mathscr{L}$ is stationary of order n.

(D) Let us consider the following financial law in which $Z = A = \mathscr{Z}$ and $B = \mathscr{R}^+$: δ_1: $\mathscr{Z} \times \mathscr{Z} \to \mathscr{Z}$ defined by $(t, a) \mapsto \delta_1(t, a) = t + a$, λ_1: $\mathscr{Z} \times \mathscr{Z} \to FC_B$ such that: $(t, a) \mapsto \lambda_1(t, a)$: $\mathscr{R}^+ \to \mathscr{R}$ defined, at a given time t, by $\forall C \in B$, $\lambda_1(t, a)(C) = C(t + a + k)/(t + k)$, $k > 0$.

We suppose that if $t \sim_1 t'$ then $\forall \bar{a} \in \bar{A}$, $\lambda_1(t, \bar{a}) = \lambda_1(t', \bar{a})$. Thus, if for every $a \in A$ and $C \in B$, $\lambda_1(t, a)(C) = \lambda_1(t', a)(C)$ then $C \cdot (t + a + k)/(t + k) = C \cdot (t' + a + k)/(t' + k)$ and then $t' = t$. Therefore, $\mathscr{L}$ is dynamic.

(E) Let us consider the following financial law in which $Z = \mathscr{Z}$, $A = n\mathscr{Z}$, and $B = \mathscr{R}^+$: δ_1: $\mathscr{Z} \times n\mathscr{Z} \to \mathscr{Z}$ defined by $(t, a) \mapsto \delta_1(t, a) = t + a$, λ_1: $\mathscr{Z} \times n\mathscr{Z} \to FC_B$, such that $(t, a) \mapsto \lambda_1(t, a)$: $\mathscr{R}^+ \to \mathscr{R}$, defined by: for every $C \in B$, $\lambda_1(t, a)(C) = C \cdot e^{ka[t]_n + 1}$, being $k > 0$, $n \in \mathscr{N}^*$, and $[t]_n$ the class of t, mod n.

Thus,

$$\lambda_1(t, a)(C) = \begin{cases} C \cdot e^{ka.0} & \text{if } t = \dot{n}, \\ C \cdot e^{ka.1} & \text{if } t = \dot{n} + 1, \\ \vdots & \vdots \\ C \cdot e^{ka(n-1)} & \text{if } t = \dot{n} + n - 1 \end{cases}$$

In this case, if $Z/\sim_1 = \{[t_0], [t_0 + 1], \ldots, [t_0 + n - 1]\}$ then $\mathscr{L}$ is stationary of order n.

In order to relate the concepts of stationary law of order n, we can enunciate the following:

THEOREM 4. *Let us suppose that $A = n\mathscr{Z}$. In these conditions, $\mathscr{L} = (Z \times B, A, \delta)$ is a stationary financial law of order n if and only if there exist n stationary financial laws $\mathscr{L}_1, \mathscr{L}_2, \ldots, \mathscr{L}_n$, being $\mathscr{L}_1 = (Z_1 \times B, A, \delta_1)$, $\mathscr{L}_2 = (Z_2 \times B, A, \delta_2), \ldots, \mathscr{L}_n = (Z_n \times B, A, \delta_n)$, such that*

(1) *Z is the disjoint union of $\{Z_i, i = 1, \ldots, n\}$.*
(2) *$\mathscr{L}_1, \mathscr{L}_2, \ldots, \mathscr{L}_n \leq \mathscr{L}$.*

Proof. (i) $\Rightarrow$ Let $\mathscr{L} = (Z \times B, A, \delta)$ be a stationary financial law of order n. This implies $Z/\sim_1 = \{[t_1], [t_2], \ldots, [t_n]\}$. If we denote $Z_1 = [t_1]$, $Z_2 = [t_2], \ldots, Z_n = [t_n]$, Z is the disjoint union of Z_i, $i = 1, \ldots, n$.

If δ_i is the restriction of δ to $Z_i \times B$, then $\mathscr{L}_1, \mathscr{L}_2, \ldots, \mathscr{L}_n \leq \mathscr{L}$. Obviously, $\mathscr{L}_i$ is a stationary financial law.

(ii) $\Leftarrow$ It is obvious, because in this case, $Z/\sim_1 = \{Z_1, Z_2, \ldots, Z_n\}$. ∎

C. Interpretation of Stationary Laws of Order *n*

Stationary financial laws of order n must not be considered as a pure abstraction because it could be thought, in a financial entity, that using a stationary financial law of order 30 in which $Z = \mathscr{Z}$, $A = 30\mathscr{Z}$, and $\lambda_i(t_i, 30) < \lambda_j(t_j, 30)$, for $i < j$, at which it would remunerate plus a deposit in the space of a month, made on the first of each month, if that same quantity was invested on the second day, and so on, with the object of raising the colocation in the space of capital volumes punctually important, as, e.g., the payrolls cashed at the beginning of the month or the treasureship excedents in determined dates.

Next, we are going to establish the first algebraic characterization of the concept of stationary financial law.

Let $\mathscr{L}$ be a financial law and let us suppose that $(Z, +)$ is a subgroup of $(\mathscr{Z}, +)$. In these conditions, the following is verified:

First Theorem of Characterization. *$\mathscr{L}$ is stationary if and only if λ_1 is a homomorphism of the groups $(Z \times A, +)$ and $(FC_B, \circ)$, i.e.,*

$$\lambda_1((t,a) + (t',a')) = \lambda_1(t,a) \circ \lambda_1(t',a')$$

or, equivalently,

$$\lambda_1(t + t', a + a') = \lambda_1(t,a) \circ \lambda_1(t',a')$$

Proof. (i) $\Rightarrow$ First let us suppose that the financial law $\mathscr{L}$ is stationary. We will prove that λ_1 is an homomorphism of groups, i.e., that

$$\lambda_1((t,a) + (t',a')) = \lambda_1(t,a) \circ \lambda_1(t',a'), \forall t, t' \in Z; \forall a, a' \in A$$

In effect,

$$\lambda_1((t,a) + (t',a')) = \lambda_1(t + t', a + a') = \lambda_1(t,a) \circ \lambda_1(t',a')$$

(ii) $\Leftarrow$ Now let us suppose that λ_1 is a homomorphism of groups. We will prove that $\mathscr{L}$ is stationary, i.e., that $\overline{\lambda}_1(t, \bar{a}) = \overline{\lambda}_1(t', \bar{a})$. In effect, let $\bar{a} = a_1 a_2 \ldots a_n$, then

$$\begin{aligned}
\overline{\lambda}_1(t, \bar{a}) &= \lambda_1(t, a_1)\lambda_1(t + a_1, a_2) \cdots \lambda_1(t + a_1 + \cdots + a_{n-1}, a_n) \\
&= \lambda_1((t,0) + (0, a_1))\lambda_1((t + a_1, 0) + (0, a_2)) \\
&\quad \cdots \lambda_1((t + a_1 + \cdots + a_{n-1}, 0) + (0, a_n)) \\
&= (\lambda_1(t',0) \circ \lambda_1(0, a_1))(\lambda_1(t' + a_1, 0) \circ \lambda_1(0, a_2)) \\
&\quad \cdots (\lambda_1(t' + a_1 + \cdots + a_{n-1}, 0) \circ \lambda_1(0, a_n)) \\
&= \overline{\lambda}_1(t', a_1 a_2 \cdots a_n) = \overline{\lambda}_1(t', \bar{a}) \Rightarrow \overline{\lambda}_1(t, \bar{a}) = \overline{\lambda}_1(t', \bar{a}).
\end{aligned}$$

Therefore, $\mathscr{L}$ is stationary. ■

Next, let us expose the second characterization of the concept of stationary financial law.

Let $\mathscr{L}$ be a financial law and let us suppose that $(Z, +)$ is a subgroup of $(\mathscr{Z}, +)$. In these conditions, the following is verified:

SECOND THEOREM OF CHARACTERIZATION. *$\mathscr{L}$ is stationary if and only if $(\{\lambda_1(t,a)/t \in Z, a \in A\}, \circ)$ is a cyclic group such that $\lambda_1(t,a) = [\lambda_1(0,\alpha)]^{a/\alpha}$, $\forall t \in Z$, $\forall a \in A$, being α the generator of A.*

Proof. (i) $\Rightarrow$ Let us suppose that $\mathscr{L}$ is a stationary financial law. Then it is verified that

$$\forall t \in Z, \forall a \in A,\ \lambda_1(t,a) = \lambda_1\left(t, \alpha + \alpha + \cdots \underbrace{\frac{a}{\alpha}\ \textit{times}} \cdots + \alpha\right)$$

$$= \lambda_1(t + \alpha + \cdots + \alpha, \alpha) \circ \cdots \circ \lambda_1(t+\alpha, \alpha) \circ \lambda_1(t, \alpha)$$

$$= (\text{as } \mathscr{L} \text{ is a stationary financial law})$$

$$= \lambda_1(0,\alpha) \circ \cdots \underbrace{\frac{a}{\alpha}\ \textit{times}} \cdots \circ \lambda_1(0,\alpha) \circ \lambda_1(0,\alpha) = \left[\lambda_1(0,\alpha)\right]^{a/\alpha}$$

being α the generator of A. This reasoning is valid when a and α have the same sign. Otherwise we have

$$\lambda_1(t,a) = \lambda_1\left(t, -\alpha - \alpha - \cdots \underbrace{-\frac{a}{\alpha}\ \textit{times}} \cdots - \alpha\right),$$

and therefore, $(\{\lambda_1(t,a)/t \in Z, a \in A\}, \circ)$ is a cyclic group. ∎

A consequence of last theorem is the general expression of a stationary financial law: $\lambda_1(t,a) = K^{a/\alpha}$, with $K > Id_B$.

Example 1. Let us consider the following financial law in which $Z = A = \mathscr{Z}$, $B =]1, +\infty[$, δ_1: $\mathscr{Z} \times \mathscr{Z} \to \mathscr{Z}$ defined by $(t,a) \mapsto \delta_1(t,a) = t + a$, λ_1: $\mathscr{Z} \times \mathscr{Z} \to FC_B$ such that $(t,a) \mapsto \lambda_1(t,a)$: $[1, +\infty[\to [1, +\infty[$ defined, at a time, by $\forall C \in B$, $\lambda_1(t,a)(C) = C^{(k^a)}$, being k a real number > 1. Obviously, $[\lambda_1(0,1)]^a(C) = C^{(k^a)}$.

DEFINITION 3 (*$\bar{a}$-equivalence on Z*). *Let $\mathscr{L} = (Z \times B, A, \delta)$ be a financial law and $\bar{a} \in \bar{A}$. $\forall t, t' \in Z$, t and t' are called* $\bar{a}$-equivalent *and it is denoted $t \sim_{\bar{a}} t'$ if $\bar{\lambda}_1(t,\bar{a}) \equiv_2 \bar{\lambda}_1(t',\bar{a})$.*

DEFINITION 4 (*$\bar{a}$-stationary financial law*). *Let $\mathscr{L} = (Z \times B, A\ \delta)$ be a financial law and $\bar{a} \in \bar{A}$. $\mathscr{L}$ is called* $\bar{a}$-stationary *if $\forall t, t' \in Z$, $\bar{\lambda}_1(t,\bar{a}) \equiv_2 \bar{\lambda}_1(t',\bar{a})$, e.g., $\forall t, t' \in Z$, $t \sim_{\bar{a}} t'$.*

Let us consider the following subset of $\overline{A}$, that we will denote as G:

$$G = \left\{\bar{a} \in \overline{A}/\mathscr{L} \text{ is } \bar{a}\text{-stationary}\right\}$$

THEOREM 5. *Let* $\top$ *be the concatenation operation in* A. *Then* $(G, \top)$ *is an abelian group.*

Proof.

(1) $G \neq \emptyset$, because Λ (empty word) $\in G$:

$$\overline{\lambda}_1(t, \Lambda) = Id_B = \overline{\lambda}_1(t', \Lambda), \forall t, t' \in Z$$

(2) If $\bar{a}, \bar{a}' \in G$, then $\overline{aa'} = \bar{a} \top \bar{a}' \in G$.
(3) In $(G, \top)$ it is verified the associative property, because it is verified in $(A, \top)$.
(4) In $(G, \top)$, the identity element is Λ.
(5) $\forall \bar{a} = a_1 a_2 \cdots a_n \in G$, let us consider $-\bar{a} = (-a_1)(-a_2)\cdots(-a_n)$.

$$\begin{aligned}
&\forall t, t' \in Z, \overline{\lambda}_1(t, -\bar{a}) \\
&\quad = \left[\lambda_1(t - a_1, a_1)\right]^{-1}\left[\lambda_1(t - a_1 - a_2, a_2)\right]^{-1} \\
&\quad\quad \cdots \left[\lambda_1(t - a_1 - \cdots - a_n, a_n)\right]^{-1} \\
&\quad \equiv_2 \lambda_1(t', -a_1)\lambda_1(t' - a_1, -a_2) \cdots \lambda_1(t' - a_1 - \cdots - a_{n-1}, a_n) \\
&\quad = \overline{\lambda}_1(t', (-a_1)(-a_2)\cdots(-a_n)) = \overline{\lambda}_1(t' - \bar{a})
\end{aligned}$$

Therefore, $-\bar{a} \in G$. ∎

The following theorem relates stationary financial laws and a-stationary financial laws.

THEOREM 6. *$\mathscr{L}$ is stationary if and only if $\mathscr{L}$ is x-stationary, for all $x \in A$.*

Example 2. Let $\mathscr{L} = (Z \times B, A, \delta)$ a financial law such that $\forall t \in Z$, $\forall a \in A$, $\forall C \in B$, $\lambda_1(t, a)(C) = C \cdot e^{\sum_{s=t}^{t+a-1} f(s)}$ and $\lambda_1(t, 0)(C) = C$, being $f(t)$ the function which verifies the following condition:

$$f(t) + f(t+1) + \cdots + f(t+n) = k$$

where $f(t) > 0$ and $k > 0$ are constant.

Last equation is a finite difference equation. It is known that the solutions of equation $t^n + t^{n-1} + \cdots + t + 1 = 0$ are the nth roots of the unity, except 1:

$$\sqrt[n+1]{1} = \sqrt[n+1]{1_{0^0}} = 1_{k.360^0/n+1}, k = 0, 1, \ldots, n$$

We consider two cases:

(A) If n is even, we consider r_k, $k = 1, \ldots, (n/2)$. The solutions of equation $f(t) + f(t+1) + \cdots + f(t+n) = k$ are

$$\left.\begin{aligned} a_{kt}^{(1)} &= \rho^t \cdot \cos\big(t(k \cdot 360^0)/(n+1)\big) \\ a_{kt}^{(2)} &= \rho^t \cdot \sin\big(t(k \cdot 360^0)/(n+1)\big) \end{aligned}\right. \qquad k = 1, 2, \ldots, \frac{n}{2}$$

in particular

$$\left.\begin{aligned} a_{kt}^{(1)} &= \cos\big(t(k \cdot 360^0)/(n+1)\big) \\ a_{kt}^{(2)} &= \sin\big(t(k \cdot 360^0)/(n+1)\big) \end{aligned}\right. \qquad k = 1, 2, \ldots, \frac{n}{2}$$

Therefore the general solution of the equation is

$$\begin{aligned} f(t) &= C_1 \cdot a_{1t}^{(1)} + C_2 \cdot a_{1t}^{(2)} + C_3 \cdot a_{2t}^{(1)} + C_4 \cdot a_{2t}^{(2)} \\ &\quad + \cdots + C_{n-1} \cdot a_{(n/2)t}^{(1)} + C_n \cdot a_{(n/2)t}^{(2)} + C_{n+1} \end{aligned}$$

being $C_1, C_2, \ldots, C_n$ arbitrary constants and $C_{n+1} = k/(n+1)$.

It is verified that $\mathscr{L}$ is a $(n+1)$-stationary financial law in wide sense. However, $\mathscr{L}$ is not a stationary financial law:

(1)
$$\forall t \in Z;\ \forall a, a' \in A;\ \forall C \in B$$
$$\begin{aligned} &\big[\lambda_1(t+a, a') \circ \lambda_1(t,a)\big](C) \\ &\quad = \lambda_1(t+a, a')\big[\lambda_1(t,a)(C)\big] \\ &\quad = \lambda_1(t+a, a')\Big[C \cdot e^{\sum_{s=t}^{t+a-1} f(s)}\Big] = C \cdot e^{\sum_{s=t}^{t+a-1} f(s)} \cdot e^{\sum_{s=t+a}^{t+a+a'-1} f(s)} \\ &\quad = C \cdot e^{\sum_{s=t}^{t+a+a'-1} f(s)} = \lambda_1(t, a+a')(C) \end{aligned}$$

(2)
$$\forall t \in Z;\ \forall a \in A;\ \forall x \in A^+;\ x \neq 0;\ \forall C \in B$$
$$\lambda_1(t, a+x)(C) = C \cdot e^{\sum_{s=t}^{t+a+x-1} f(s)} > C \cdot e^{\sum_{s=t}^{t+a-1} f(s)} = \lambda_1(t,a)(C)$$

(3)
$$\forall t \in Z;\ \forall a \in A;\ \forall x \in A^+;\ x \neq 0;\ \forall C \in B$$
$$\begin{aligned} \big[\lambda_1(t, a+x) - \lambda_1(t,a)\big](C) &= C \cdot e^{\sum_{s=t}^{t+a+x-1} f(s)} - C \cdot e^{\sum_{s=t}^{t+a-1} f(s)} \\ &= C \cdot e^{\sum_{s=t}^{t+a-1} f(s)}\Big[e^{\sum_{s=t+a}^{t+a+x-1} f(s)} - 1\Big]. \end{aligned}$$

As the powers of e are bigger than 1, the product of the two factors of C is bigger than 0, from which:

$$\lambda_1(t, a+x) - \lambda_1(t,a) \in FC_B.$$

Moreover, $\forall t \in Z$,

$$\lambda_1(t, n+1)(C) = C \cdot e^{\sum_{s=t}^{t+n+1-1} f(s)} = C \cdot e^k$$

As $f(t)$ depends on n arbitrary constants, we can fix $f(t)$ choosing $k_1, k_2, \ldots, k_n$ such that:

$$k_1 + k_2 + \cdots + k_{n+1} = k$$

and calculating the values of $C_1, C_2, \ldots, C_n$ that verify the following system of equations:

$$\begin{array}{rcl} f(0) & = & k_1 \\ f(1) & = & k_2 \\ \vdots & & \vdots \\ f(n-1) & = & k_n \end{array}$$

that is possible, because the determinant:

$$\begin{vmatrix} \cos\left(0\dfrac{1\cdot 360^0}{n+1}\right) & \sin\left(0\dfrac{1\cdot 360^0}{n+1}\right) & \cdots & \sin\left(0\dfrac{\frac{n}{2}\cdot 360^0}{n+1}\right) \\ \cos\left(1\dfrac{1\cdot 360^0}{n+1}\right) & \sin\left(1\dfrac{1\cdot 360^0}{n+1}\right) & \cdots & \sin\left(1\dfrac{\frac{n}{2}\cdot 360^0}{n+1}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \cos\left((n-1)\dfrac{1\cdot 360^0}{n+1}\right) & \sin\left((n-1)\dfrac{1\cdot 360^0}{n+1}\right) & \cdots & \sin\left((n-1)\dfrac{\frac{n}{2}\cdot 360^0}{n+1}\right) \end{vmatrix}$$

is not zero.

(B) If n is odd, the general solution of the equation is:

$$\begin{aligned} f(t) &= C_1 \cdot a_{1t}^{(1)} + C_2 \cdot a_{1t}^{(2)} + C_3 \cdot a_{2t}^{(1)} + C_4 \cdot a_{2t}^{(2)} \\ &\quad + \cdots + C_{n-2} \cdot a_{\frac{n-1}{2}t}^{(1)} + C_{n-1} \cdot a_{\frac{n-1}{2}t}^{(2)} + C_n(-1)^t + C_{n+1} \end{aligned}$$

being $C_1, C_2, \ldots, C_n$ arbitrary constants and $C_{n+1} = \frac{k}{n+1}$.

The discussion of this case is analogous to the previous one.

D. Financial Applications

(1) Let us consider a financial entity that offers a $t\%$ as percentage of interest payable monthly, but variable.

Thus this entity can establish the interests of the first months lower and the interests of the last months higher, in order to secure a period of time more elevated in the imposition.

(2) Also this financial law can be applied to the financing of vehicles or other goods, establishing a percentage of interest very low or zero in the first months and an interest very high in the last ones.

VII. CONCLUSIONS

According to Levi,[5] a financial law is homogeneous of the first-degree with respect to the quantity within certain limits, which leads us to think about a homogeneity in terms of "intervals of quantity." However, this problem has not been considered in the literature.

Observe that the kernel of our financial law concept is the output function λ in which the temporal period (and not the maturity of the financial capitals) is fixed. This implies an algebraic development of financial law concept, which allows us to study the Financial Mathematics from a new point of view, generalizing classical concepts as the stationary financial systems.

We have defined a financial law as a "device" which transforms capitals into capitals as a consequence of the action of a temporal input on the capitals, giving rise to a "change of state."

In order to summarize we list the following properties of financial laws:

(1) Financial laws are strictly increasing with respect to the final moment.
(2) Financial laws are strictly decreasing with respect to the initial moment.
(3) Financial laws are strictly increasing with respect to the quantity.

Other axioms and properties are softened:

(1) The homogeneity of the first-degree with respect to the quantity is not necessary.
(2) The continuity and, therefore, the derivability with respect to the quantity are not necessary.
(3) The existence of a homeomorphism of the set of maturities into the set of quantities is not necessary.

From the algebraic definition of financial law, we deduce some properties as:

(1) The monoid of a financial law is a group, which, moreover, is isomorphic to the group of possible temporal inputs.
(2) If a financial law is not homogeneous of the first-degree with respect to the quantity, and if we want to operate with other monetary units, we can find another law equivalent to the first one which offers the same results when we cancel the change of unit.

Successively, we study the concepts of stationary and dynamic financial law, introducing an intermediate concept that is the stationary of order n financial law. This law has its origin in the changeable nature of some financial decisions on rates of interest, profitabilities, etc. We justify these laws as an incentive to place the capitals at certain moments of time.

Finally, we introduce the concept of $\bar{a}$-stationary financial law which also has its origin in a changeable nature of profitabilities, rates of interest, etc. It includes the changes not only at the origin, but in the whole lifetime of the operation. This is to encourage operations consisting of placing the capitals at certain fixed time periods.

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