That the Brauer group of a cocommutative coalgebra need not be equal to the Brauer group of the dual algebra is a good thing because it provides a certain richness to the theory. On the other hand, the relation between the coalgebraic theory and the Brauer group of a local ring is one of the main motivations to further the study in the coalgebra case.

Since a cocommutative coalgebra may be viewed as the direct sum of irreducible coalgebras its Brauer group reduces to a product of Brauer groups of irreducible cocommutative coalgebras. Hence it is enough to restrict our attention to the irreducible case. In this paper, for a cocartesian diagram of...
cocommutative irreducible coalgebras,

\[
\begin{array}{ccc}
C & \xrightarrow{f_1} & C_1 \\
\downarrow f_2 & & \downarrow g_1 \\
C_2 & \xleftarrow{g_2} & C
\end{array}
\]

we construct a Mayer-Vietoris exact sequence (Theorem 4.3)

\[
0 \rightarrow Br(C) \rightarrow Br(C_1) \oplus Br(C_2) \rightarrow Br(C).
\]

If we compare to the Mayer-Vietoris sequence in ring theory (see [1]), there is an interesting simplification presented in the coalgebra theory due to the fact (Proposition 4.1) that the Picard group of a cocommutative coalgebra vanishes. The sequence is derived from a dual version of the classical Milnor’s theorem (Theorem 3.6). This relates the category of finitely cogenerated injective comodules over the coalgebras appearing in the cocartesian diagram. More concretely, the category of finitely cogenerated \( C \)-comodules \( I(C) \) is equivalent to the fibre product category \( I(C_1) \times_{I(C)} I(C_2) \). Similar theorems are deduced for the category of finitely cogenerated injective cogenerators (Corollary 3.9) and Azumaya coalgebras (Corollary 3.11).

As application of the Mayer-Vietoris sequence, for a cocommutative irreducible coalgebra \( C \) over a field \( k \), its universal connected coalgebra \( R(C) \) can be viewed in a cocartesian diagram involving \( C \), its coradical \( C_0 \), and the ground field (Example 4.4). An exact sequence relating the Brauer group of the above coalgebras is derived (Corollary 4.5). When the ground field is finite, then \( Br(C) \cong Br(R(C)) \).

2 NOTATION AND PRELIMINARIES

Throughout \( k \) is a fixed field. All algebras, coalgebras, vector spaces and unadorned \( \otimes \) are over \( k \). We use the usual sigma notation for coalgebras and comodules. \( \mathcal{M}^C \) denotes the category of right \( C \)-comodules and for right \( C \)-comodules \( X,Y \), \( \text{Com}_{-C}(X,Y) \) denotes the vector space of all \( C \)-colinear maps from \( X \) to \( Y \).
Let $U: \mathcal{M}^C \to \mathcal{M}_k$ be the forgetful functor. $U$ has a right adjoint functor $- \otimes C$. For a comodule $X_C$, with structure map $\rho_X$, and a vector space $V$, the adjoint isomorphism

$$\Phi : \text{Com}_C(X, V \otimes C) \to \text{Hom}_k(X, V)$$

(1)

is given by $\Phi(F) = (1 \otimes \varepsilon)F$. The inverse $\Phi^{-1}$ is given by $\Phi^{-1}(f) = (f \otimes 1)\rho_X$. In particular, if $V = k$, then

$$\text{Com}_C(X, C) \cong X^*.$$

Let $\alpha : C \to D$ be a coalgebra map. Every right $C$-comodule $X$ may be viewed as a right $D$-comodule with the structure map:

$$(1 \otimes \alpha)\rho_X : X \to X \otimes C \to X \otimes D.$$

In this case, we will say $X_D$ is induced by $X_C$ via $\alpha$. A $(C-D)$-bicomodule is a left $C$-comodule and a right $D$-comodule $X$, denoted by $cX_D$, such that the $C$-comodule structure map $\rho_X : X \to X \otimes C$ is $D$-colinear.

**Cotensor product**: Let $M$ be a right $C$-comodule and $N$ a left $C$-comodule with structure maps $\rho_M$ and $\rho_N$ respectively. The cotensor product $M \Box_C N$ is the kernel of the map

$$\rho_M \otimes 1 - 1 \otimes \rho_N : M \otimes N \to M \otimes C \otimes N.$$

The functors $M \Box_C -$ and $- \Box_C N$ are left exact and preserve direct sums. If $cM_D$ and $dN_E$ are bicomodules, then $M \Box_C N$ is a $(C-E)$-bicomodule with comodule structures induced by those of $M$ and $N$.

Let $X \in \mathcal{M}^C$, we say that $X$ is finitely cogenerated if it is isomorphic to a submodule of $C^{(n)}$ for some $n \geq 1$, where $C^{(n)}$ denotes the direct sum of $C$ $n$ times. $X$ is free if $X$ is isomorphic to $C^{(I)}$ for some set $I$. $X$ is said to be a cogenerator if for any comodule $M \in \mathcal{M}^C$, $M \hookrightarrow X^{(I)}$, for some set $I$, as comodules. $X$ is injective if the functor $\text{Com}_C(-, X)$ is exact, or equivalently, the functor $X \Box_C -$ is exact.

**Co-hom functor**: A right $C$-comodule $X$ is quasi-finite if $\text{Com}_C(Y, X)$ is finite dimensional for any finite dimensional comodule $Y \in \mathcal{M}^C$. Now, we recall from [2] the definition of the co-hom functor:
Lemma 2.1 Let $C X_D$ be a bicomodule. Then $X_D$ is quasi-finite if and only if the functor $\square_C X : \mathcal{M}^C \rightarrow \mathcal{M}^D$ has a left adjoint functor, denoted by $h_D(X, -)$. That is, for comodules $Y_D$ and $Z_C$,

\[ \text{Com}_C(h_D(X, Y), Z) \cong \text{Com}_D(Y, Z \square_C X) \]  

where,

\[ h_D(X, Y) = \lim_{\mu} \text{Com}_D(Y_\mu, X) \]

is a right $C$-comodule and $\{Y_\mu\}$ is a directed family of finite dimensional subcomodules of $Y_D$ such that $Y_D = \bigcup_\mu Y_\mu$. We denote by $\theta$ the canonical $D$-colinear map $Y \rightarrow h_D(X, Y) \square_C X$ which corresponds to the identity map $h_D(X, Y) \rightarrow h_D(X, Y)$ in (2).

If we assume that $X_D$ is a quasi-finite comodule, then $\epsilon_D(X) = h_D(X, X)$ is a coalgebra, called the co-endomorphism coalgebra of $X$. The comultiplication of $\epsilon_D(X)$ corresponds to $(1 \otimes \theta) \theta : X \rightarrow \epsilon_D(X) \otimes \epsilon_D(X) \otimes X$ in (2) when $C = k$ and the counit of $\epsilon_D(X)$ corresponds to the identity map $1_X$.

Let $C X_D$ be a bicomodule such that $X_D$ is quasi-finite. Then there exists a coalgebra map $\lambda : \epsilon_D(X) \rightarrow C$ such that the left $C$-comodule structure map equals $(\lambda \otimes 1) \theta$. Conversely, a coalgebra map $\lambda : \epsilon_D(X) \rightarrow C$ makes $X$ into a $(C - D)$-bicomodule.

In this paragraph we recall the notion of Azumaya coalgebra, the Brauer group of a cocommutative coalgebra and some of its properties. We refer the reader to [3] for the construction of the Brauer group and to [4] for the relative Brauer group and its cohomological interpretation.

**Brauer group:** A coalgebra map $f : D \rightarrow E$ is said to be cocentral if

\[ \sum_{(c)} f(c_1) \otimes c_2 = \sum_{(c)} f(c_2) \otimes c_1. \]

For a coalgebra $D$, there exists a cocommutative coalgebra $Z(D)$ with a surjective, cocentral coalgebra map $1_D^\psi : D \rightarrow Z(D)$ which satisfies the universal property: for any cocentral coalgebra map $f : D \rightarrow E$ there is a unique coalgebra map $g : Z(D) \rightarrow E$ such that $f = g 1_D^\psi$. $(Z(D), 1_D^\psi)$ is called the cocenter of $D$. In fact, $Z(D) = h_{D^\psi}(D, D) = \epsilon_{D^\psi}(D)$ where $D^\psi = D \otimes D^\text{op}$.

Let $C$ be cocommutative coalgebra. A $C$-coalgebra $D$ is a $k$-coalgebra with a cocentral coalgebra map $\epsilon_D : D \rightarrow C$, called the $C$-counit. A $k$-coalgebra map $f : D \rightarrow E$ is a $C$-coalgebra map if $\epsilon_E f = \epsilon_D$. A $C$-coalgebra $D$ is
said to be co-central if $Z(D) \cong C$ and $D$ is said to be $C$-coseparable if there is a $D$-bicomodule map $\pi : D \square_C D \to D$ such that $\pi \Delta = 1_D$. An Azumaya $C$-coalgebra is defined to be a $C$-co-central and $C$-coseparable coalgebra. If $P$ is an injective quasi-finite cogenerator then $e_{-C}(P)$ is an Azumaya coalgebra. Denote by $B(C)$ the set of the isomorphism classes of Azumaya $C$-coalgebras. An equivalence relation (indeed a Morita-Takeuchi equivalence relation) is introduced in $B(C)$ as follows: if $E, F \in B(C)$, then $E$ is equivalent to $F$, denoted by $E \sim F$, if there exist two quasi-finite injective cogenerators $M, N$ in $\mathcal{M}^C$ such that

$$E \square_C e_{-C}(M) \cong F \square_C e_{-C}(N).$$

The quotient set $B(C)/\sim$, denoted by $Br(C)$, is an abelian group with the multiplication $[E][F] = [E \square_C F]$, unit element $[C]$ and for $[E]$ the inverse is $[E^{op}]$. The group $Br(C)$ is called the Brauer group of the cocommutative coalgebra $C$.

Let $\eta : D \to C$ a map of cocommutative coalgebras, then $\eta$ induces a group homomorphism $\eta_* : Br(C) \to Br(D)$ given by $\eta_*([E]) = [E \square_C D]$ for all $[E] \in Br(C)$. If $C$ is of finite dimension, the Brauer group of $C$ is isomorphic to the Brauer group of the cocommutative algebra $C^\ast$. Since $C$ is cocommutative, $C$ can be expressed as $C = \oplus_{i \in I} C_i$ where each $C_i$ is an irreducible subcoalgebra and we have that $Br(C) \cong \prod_{i \in I} Br(C_i)$. This decomposition has two consequences:

1) In general $Br(C)$ is not torsion. Let $\mathbb{Q}$ be the rational number field and $C$ the group like coalgebra $C = \oplus_{n \in \mathbb{N}} \mathbb{Q}$. It is well-known that for any $n \in \mathbb{N}$ there is $[A_n] \in Br(\mathbb{Q})$ of order $n$. The coalgebra $A = \oplus_{n \in \mathbb{N}} A_n^\ast$ is $C$-Azumaya, cf. [3, Ex. 4.7], and $[A]$ does not have finite order in $Br(C)$.

2) To compute the Brauer group of a cocommutative coalgebra it is enough to compute the Brauer group of irreducible coalgebras.

If $C$ is irreducible, then the map $(-)^* : Br(C) \to Br(C^\ast), [D] \mapsto [D^\ast]$ is a group homomorphism. If in addition $C$ is coreflexive then $Br(C) \cong Br(C^\ast) \cong Br(C_0)$ where $C_0$ is the coradical of $C$ and $Br(C_0)$ is isomorphic to the classical Brauer group of some finite field extension.

3  MILNOR’S THEOREM

Let $C$ be a cocommutative coalgebra. We consider the following categories of $C$-comodules which are equipped with a product:
(1) $I(C)$, the category of finitely cogenerated injective left $C$-comodules and $C$-colinear maps with product the direct sum. Then $P \in I(C)$ if and only if $P$ is a direct summand of $C^{(n)}$ for some $n \geq 1$.

(2) $CI(C)$, the finitely cogenerated injective cogenerators $C$-comodules and $C$-isomorphisms, with product the cotensor product $\Box_C$. Then $M \in CI(C)$ if and only if $C$ is a direct summand of $P^{(n)}$ and $P$ is a direct summand of $C^{(m)}$ for some $m, n \geq 1$.

(3) $Az(C)$, the Azumaya $C$-coalgebras and $C$-coalgebra maps with product $\Box_C$. We recall that a $C$-coalgebra $A$ is Azumaya if $A$ is $C$-coseparable and $C$-cocentral. If $P \in CI(C)$ then $\epsilon_C(P) \in Az(C)$, cf. [3, Cor. 4.2].

Let $f : C \to D$ be a map of cocommutative coalgebras. If $P(C)$ is any of these three categories, then the functor $F(-) = -\Box_D C : P(D) \to P(C)$ is a product preserving functor.

If we suppose that $C$ is an irreducible cocommutative coalgebra, then quasi-finite injective comodules are finitely cogenerated injective comodules.

**Proposition 3.1** Let $C$ be an irreducible cocommutative coalgebra and $P$ an injective right $C$-comodule. Then, $P$ is finitely cogenerated if and only if $P$ is quasi-finite.

**Proof.** If $P$ is finitely cogenerated then $P \hookrightarrow C^{(n)}$ for some $n \geq 1$. Let $F$ be a finite dimensional $C$-comodule, we have from (1)

$$Com_C(F, P) \subset Com_C(F, C^{(n)}) \cong Hom(F, k^{(n)}). \cong F^{*^{(n)}}$$

Hence $P$ is quasi-finite. This implication is always true, we do not need the injectivity of $P$ and irreducibility of the coalgebra.

Conversely, we suppose that $P$ is quasi-finite. As $P$ is injective and $C$ is irreducible it follows that $P$ is free, cf. [5, A.2.2]. But a comodule that is quasi-finite and free has to be finitely cogenerated. \[\rule{0.5em}{0.5em}\)

**Definition 3.2** A diagram of cocommutative coalgebras

\[\begin{array}{ccc}
C & \overset{f_1}{\longrightarrow} & C_1 \\
| & f_2 & | \\
\downarrow & & \downarrow g_1 \\
C_2 & \overset{g_2}{\longrightarrow} & C
\end{array}\] (3)
is a cocartesian diagram if \( C \cong \frac{C_i \oplus C_2}{\text{Im}(f_1 - f_2)} \), or equivalently, the following sequence is exact:

\[
\begin{array}{c}
\mathcal{C} \xrightarrow{f_1 - f_2} C_1 \oplus C_2 \xrightarrow{g_1 \oplus g_2} C \twoheadrightarrow 0. 
\end{array}
\]

Any \( \mathcal{C} \)-comodule may be viewed as a \( C_i \)-comodule and every \( C_i \)-comodule is a \( C \)-comodule, \( i = 1, 2 \). For \( \mathcal{P} \in I(C) \) we apply the functor \( \mathcal{P} \square \cdot \) to (3). So, we have:

\[
\begin{array}{c}
\mathcal{P} \square C \xrightarrow{f_i} \mathcal{P} \square C_1 \\
\quad \quad \downarrow \quad \quad \downarrow \quad \quad \quad \downarrow \\
\mathcal{P} \square C_2 \xrightarrow{g_i} \mathcal{P} \square C
\end{array}
\]

where \( f_i = 1 \square f_i \), \( g_i = 1 \square g_i \) for \( i = 1, 2 \). If \( P_i \) denotes \( \mathcal{P} \square C_i \), then \( P_i \in I(C_i) \) and we have an isomorphism of \( \mathcal{C} \)-comodules

\[
P_i \square C_i \mathcal{C} = \mathcal{P} \square C \square C_i \mathcal{C} \cong \mathcal{P} \square C.
\]

We put \( \mathcal{T}_i = P_i \square C_i \mathcal{C}, i = 1, 2 \). Then, \( \mathcal{T}_1 \cong \mathcal{T} \)(\( \mathcal{T}_2 \)). Let \( \psi : \mathcal{T}_1 \cong \mathcal{T}_2 \) be this isomorphism of \( \mathcal{C} \)-comodules. If we consider the \( P_i \) as \( C \)-comodules then \( \text{Im}(f_1 - f_2 \psi) \) is a \( C \)-subcomodule of \( P_1 \oplus P_2 \) and \( P \) is isomorphic to

\[
\frac{P_1 \oplus P_2}{\text{Im}(f_1 - f_2 \psi)}.
\]

Now, we consider the converse problem, i.e., given \( P_i \in I(C_i) \) \( i = 1, 2 \) such that \( \psi : P_1 \square C_1 \mathcal{C} \cong P_2 \square C_2 \mathcal{C} \) is an isomorphism of \( \mathcal{C} \)-comodules, is there a \( C \)-comodule \( P \in I(C) \) such that \( \mathcal{P} \square C \cong P_i \) for \( i = 1, 2 \)? If we impose that either \( f_1 \) or \( f_2 \) in (3) is injective, then the above question has a positive answer.

Let \( P_i \in I(C_i) \) \( i = 1, 2 \) such that \( \psi : P_1 \square C_1 \mathcal{C} \rightarrow P_2 \square C_2 \mathcal{C} \) is an isomorphism of \( \mathcal{C} \)-comodules. Let \( \varepsilon \) denote the counit of the coalgebras appearing in (3), \( \varepsilon = 1 \odot \varepsilon \) and \( \tilde{f}_i : P_i \square C_i \mathcal{C} \rightarrow P_i \square C_i C_i \) to be the map \( 1 \square f_i \), then \( \varepsilon = \varepsilon \tilde{f}_i \) for \( i = 1, 2 \). Moreover, if we identify \( P_i \square C_i C_i \) and \( P_i \) \( i = 1, 2 \) via \( \varepsilon \), \( \text{Im}(\tilde{f}_1 - \tilde{f}_2 \psi) = \text{Im}(\varepsilon - \varepsilon \psi) \). We define

\[
P = \frac{P_1 \oplus P_2}{\text{Im}(\varepsilon - \varepsilon \psi)}.
\]
$P$ is $C$-comodule. In fact, $P$ is nothing else but the pushout of

$$P_1 \sqcup_{C_1} C \xrightarrow{\psi} P_2 \sqcup_{C_2} C \xrightarrow{\bar{\varepsilon}} P_2$$

From now on, we denote $P_i \sqcup_{C_i} C$ by $\overline{P_i}$, $i = 1, 2$ and $P$ by $(P_1, P_2, \psi)$. We construct the fibred product category $\mathcal{C} = I(C_1) \times_{I(C)} I(C_2)$:

**Obj($\mathcal{C}$):** $C$-comodules of the form $(P_1, P_2, \psi)$ with $P_i \in I(C_i)$, $i = 1, 2$ and $\psi : \overline{P_1} \to \overline{P_2}$ an isomorphism of $\overline{C}$-comodules.

**Hom($\mathcal{C}$):** Given $(P_1, P_2, \psi), (Q_1, Q_2, \phi) \in \mathcal{C}$ a morphism $f : (P_1, P_2, \psi) \to (Q_1, Q_2, \phi)$ is a pair of $C$-comodule maps $f_i : P_i \to Q_i$, $i = 1, 2$ making the following diagram commutative:

$$\begin{array}{ccc}
\overline{P}_1 & \xrightarrow{\psi} & \overline{P}_2 \\
\downarrow{f_1} & & \downarrow{f_2} \\
\overline{Q}_1 & \xrightarrow{\phi} & \overline{Q}_2
\end{array}$$

With these conventions, we have the following facts:

1) $C \cong (C_1, C_2, 1_C)$, where $1_C$ denotes the identity map in $\overline{C}$.

2) $f$ is an isomorphism if and only if $f_i$ is an isomorphism for $i = 1, 2$.

3) $(P_1, P_2, \psi) \oplus (Q_1, Q_2, \phi) \cong (P_1 \oplus Q_1, P_2 \oplus Q_2, \psi \oplus \phi)$.

Next we define elementary maps for coalgebras. Let $C$ be a coalgebra. If $f : C^{(n)} \to C^{(n)}$ is a $C$-colinear map then $f$ induces $n^2$ $C$-colinear maps $f_{ij} : C \to C$ for $i, j = 1, \ldots, n$ defined by the composition:

$$C \xrightarrow{l_i} C^{(n)} \xrightarrow{f} C^{(n)} \xrightarrow{\pi_j} C$$

where $l_i$ is the injection at position $i$ and $\pi_j$ is the projection at position $j$ for $i, j = 1, \ldots, n$. So, $f$ can be represented as a matrix of $C$-colinear maps $f = (f_{ij})_{i,j}$. Let $End_C(C^{(n)})$ denote the space of right $C$-colinear maps from
Proof. The isomorphism is the map induced by \( \psi_1 \oplus \psi_2^{-1} : C_{1}^{(n)} \oplus C_{2}^{(n)} \rightarrow C_{1}^{(n)} \oplus C_{2}^{(n)} \). This map is an isomorphism since \( \psi_1, \psi_2^{-1} \) are so.
Lemma 3.5 We assume that in the cocartesian diagram (3) either $f_1$ or $f_2$ is injective. Then,

i) For all $\psi \in E_n(\overline{C})$, $(C_1^{(n)}, C_2^{(n)}, \psi) \cong C^{(n)}$.

ii) For all $\psi \in \text{Aut}_C(C^{(n)})$ we have

$$(C_1^{(n)}, C_2^{(n)}, \psi) \oplus (C_1^{(n)}, C_2^{(n)}, \psi^{-1}) \cong C^{(2n)}.$$ 

Hence, $(C_1^{(n)}, C_2^{(n)}, \psi)$ is a finitely cogenerated injective $C$-comodule, that is, $(C_1^{(n)}, C_2^{(n)}, \psi) \in I(C)$.

Proof. i) We may suppose that $f_1 : \overline{C} \to C_1$ is injective, then $f_1^* : C_1^* \to \overline{C}$ is surjective. Hence, if $\Phi$ is the map defined in (1), the composition

$$\text{Com}_{-C_1}(C_1, C_1) \xrightarrow{\Phi} C_1^* \xrightarrow{f_1^*} \overline{C} \xrightarrow{\Phi^{-1}} \text{Com}_{-C}(\overline{C}, \overline{C})$$

is surjective. Put $h = \Phi^{-1} f_1^* \Phi$; $h$ is defined by $\langle h(F), c \rangle = \sum_{(c)} \langle \varepsilon, F f_1(c) \rangle c_2$ for all $F \in \text{Com}_{-C_1}(C_1, C_1), c \in \overline{C}$.

Let $\phi = (\phi_{ij})_{i,j}$ be an elementary map in $\text{Aut}_{-C}(\overline{C}^{(n)})$ with distinguished entry $\phi_{uv}$, $u \neq v$. Since $h$ is surjective, there is $g_{uv} \in \text{Com}_{-C_1}(C_1, C_1)$ such that $h(g_{uv}) = \phi_{uv}$. Let $g = (g_{ij})_{i,j} \in \text{Aut}_{-C_1}(C_1^{(n)})$ defined by $g_{ii} = 1_{C_1}, g_{uv}$ at position $(u, v)$ and the rest zero. Then, we have:

$$\langle f_1 \phi_{uv}, c \rangle = \langle f_1 h(g_{uv}), c \rangle = \sum_{(c)} \langle \varepsilon, g_{uv} f_1(c) \rangle f_1(c_2) = \sum_{(g_{uv} f_1(c))} \langle \varepsilon, g_{uv} f_1(c) \rangle g_{uv} f_1(c) = g_{uv} f_1(c) = \langle g_{uv} f_1, c \rangle$$

where we have used the fact that $g_{uv}$ is a $C_1$-comodule map. Using this equality, we obtain that $\phi = \sum f_1 g_{uv} 1_{C_1^{(n)}}$ and from the preceding lemma,

$$(C_1^{(n)}, C_2^{(n)}, \phi) \cong (C_1^{(n)}, C_2^{(n)}, 1_{C_1^{(n)}}) \cong C^{(n)}.$$ 

ii) We have $(C_1^{(n)}, C_2^{(n)}, \phi) \oplus (C_1^{(n)}, C_2^{(n)}, \phi^{-1}) \cong (C_1^{(2n)}, C_2^{(2n)}, \phi \oplus \phi^{-1})$. By the identity preceding Lemma 3.4, $\phi \oplus \phi^{-1} \in E_2(\overline{C}^{(n)})$ and from i) we have,

$$(C_1^{(2n)}, C_2^{(2n)}, \phi \oplus \phi^{-1}) \cong C^{(2n)}.$$ 

Therefore $(C_1^{(n)}, C_2^{(n)}, \phi)$ is a finitely cogenerated injective $C$-comodule. □
Theorem 3.6 Suppose that in the cocartesian diagram (3), either $f_1$ or $f_2$ is injective. Let $P_i \in I(C_i)$ $i = 1, 2$ and let $\phi : \overline{P_1} \cong \overline{P_2}$ be a $\overline{C}$-isomorphism where $\overline{P_i} = P_i \square C\overline{C}$. Then the $C$-comodule

$$(P_1, P_2, \overline{\phi}) = \frac{P_1 \oplus P_2}{Im(\overline{\varepsilon} - \overline{\varepsilon}\overline{\phi})}$$

belongs to $I(C)$. Moreover,

$$(P_1, P_2, \overline{\phi}) \square C C_i \cong P_i \quad i = 1, 2.$$

Finally, let $Q_i \in I(C_i)$ $i = 1, 2$ and let $\phi' : \overline{Q_1} \cong \overline{Q_2}$ be a $\overline{C}$-isomorphism. Then $(P_1, P_2, \overline{\phi}) \cong (Q_1, Q_2, \phi')$ as $C$-comodules if and only if $\phi' = \overline{\phi_1} \overline{\phi_2}^{-1}$ for some isomorphisms $\phi_1 : P_1 \cong Q_1$ and $\phi_2 : P_2 \cong Q_2$.

Proof. Let $P_i \in I(C_i)$, then there is $Q_i \in I(C_i)$ such that $P_i \oplus Q_i \cong C_i^{(n)}$ for $i = 1, 2$. $\phi : \overline{P_1} \cong \overline{P_2}$ induces an isomorphism of $\overline{C}$-comodules $\psi$ given by the composition:

$$C_i^{(n)} \oplus Q_1 \cong P_2 \oplus Q_2 \cong Q_1 \cong P_1 \oplus Q_1 \oplus Q_2 \cong C_i^{(n)} \oplus Q_2.$$

We have $(P_1, P_2, \phi) \oplus (C_i^{(n)} \oplus Q_1, C_i^{(n)} \oplus Q_2, \psi) \cong (C_i^{(2n)}, C_i^{(2n)}, \phi \oplus \psi)$ and $(C_i^{(2n)}, C_i^{(2n)}, \phi \oplus \psi) \in I(C)$ by the foregoing lemma. Hence, $(P_1, P_2, \phi) \in I(C)$.

For the second claim, first we see that for all $\phi \in Aut_{\overline{C}}(C_i^{(n)})$ we have:

$$(C_i^{(n)}, C_i^{(n)}, \phi) \square C C_i \cong C_i^{(n)} \quad i = 1, 2.$$

Let $I = Im(\overline{\varepsilon} - \overline{\varepsilon}\overline{\phi})$ and we suppose $i = 1$. We define,

$$f : (C_i^{(n)}, C_i^{(n)}, \phi) \square C C_i \longrightarrow C_i^{(n)} \quad \sum_i(x_{i} \oplus y_{i} + I) \otimes c_i \mapsto \sum_i x_i \varepsilon(c_i),$$

$$g : (C_i^{(n)}, C_i^{(n)}, \phi^{-1}) \square C C_i \longrightarrow C_i^{(n)} \quad \sum_i(x_{i} \oplus y_{i} + I) \otimes c_i \mapsto \sum_i x_i \varepsilon(c_i).$$

From the preceding lemma $(C_i^{(n)}, C_i^{(n)}, \phi) \oplus (C_i^{(n)}, C_i^{(n)}, \phi^{-1}) \cong C_i^{(2n)}$ then,

$$[(C_i^{(n)}, C_i^{(n)}, \phi) \square C C_i] \oplus [(C_i^{(n)}, C_i^{(n)}, \phi^{-1}) \square C C_i] \cong C_i^{(2n)}.$$

$$f \quad \quad \quad \quad g \quad \quad \quad \quad f \oplus g$$

$$C_i^{(n)} \oplus C_i^{(n)} \quad \cong \quad C_i^{(2n)}$$
and \( f \oplus g \) is an isomorphism. Hence \( f \) and \( g \) are isomorphisms. Now, we consider the identity,

\[
(P_1, P_2, \phi) \oplus (C_1^{(n)} \oplus Q_1, C_2^{(n)} \oplus Q_2, \psi) \cong (C_1^{(2n)}, C_2^{(2n)}, \phi \oplus \psi).
\]

Cotensoring this identity with \( - \square_C C_i \), defining maps similar to the above and then using the foregoing fact we obtain that \((P_1, P_2, \phi) \square_C C_i \cong P_i\).

Finally, let \( P = (P_1, P_2, \phi) \) and \( Q = (Q_1, Q_2, \phi') \) and \( \rho : P \cong Q \) an isomorphism of \( C \)-comodules. Let

\[
\sigma_i : P \square_C C_i \cong P_i \quad \tau_i : Q \square_C C_i \cong Q_i \quad i = 1, 2,
\]

be the isomorphisms of \( C_i \)-comodules. We consider the following commutative diagram:

\[
\begin{array}{ccc}
P_1 & \xrightarrow{\sigma_1^{-1}} & (P \square_C C_1) \square_C C_1 \\
\phi & & \phi \\
P_2 & \xrightarrow{\sigma_2^{-1}} & (P \square_C C_2) \square_C C_2 \\
\end{array}
\]

\[
\begin{array}{ccc}
(Q \square_C C_1) \square_C C_1 & \xrightarrow{\rho} & (Q \square_C C_1) \square_C C_1 \\
\tau_1 \square_C C_1 & & \tau_1 \square_C C_1 \\
(Q \square_C C_2) \square_C C_2 & \xrightarrow{\rho} & (Q \square_C C_2) \square_C C_2 \\
\tau_2 \square_C C_2 & & \tau_2 \square_C C_2 \\

\end{array}
\]

We define

\[
\mu = \tau_1(\rho \square_C 1)\sigma_1^{-1} : P_1 \rightarrow Q_1 \quad \nu = \tau_2(\rho \square_C 1)\sigma_2^{-1} : P_2 \rightarrow Q_2.
\]

Then \( \mu, \nu \) are isomorphisms such that \( \phi' \mu = \nu \phi \).

**Corollary 3.7** \( I(C) \) and \( I(C_1) \times_{[C]} I(C_2) \) are equivalent categories.

**Theorem 3.8** Assumptions are as in the above theorem. If \( P_i \in CI(C_i) \) \( i = 1, 2 \) and \( \phi : P_1 \cong P_2 \) is an isomorphism of \( C \)-comodules, then \((P_1, P_2, \phi) \in CI(C)\). Moreover, if \( Q_i \in I(C_i) \) \( i = 1, 2 \) and \( \phi' : Q_1 \cong Q_2 \) is an \( \overline{C} \)-isomorphism, then

\[
(P_1, P_2, \phi) \square_C (Q_1, Q_2, \phi') \cong (P_1 \square_C Q_1, P_2 \square_C Q_2, \phi \square \phi').
\]

**Proof.** If \( P_i \in CI(C_i) \) then there are \( Q_i, M_i \in P(C_i) \) such that \( P_i \oplus Q_i \cong C_i^{(n)} \) and \( C_i \oplus N_i \cong P_i^{(m)} \) for some \( m, n \geq 1, i = 1, 2 \). We have \( \overline{Q_1} \cong \overline{Q_2} \) and if \( \phi' \)
denotes this isomorphism, \((Q_1, Q_2, \phi') \in I(C)\) by the preceding theorem, and 
\((P_1, P_2, \phi) \oplus (Q_1, Q_2, \phi') \cong C^{(m)}\). Moreover, we have

\[
\phi^{(m)} \\
\begin{array}{c}
\downarrow \\
P_1^{(m)} \cong P_1^{(m)} \\
\cong C_1 + M_1 \cong C + M_1
\end{array}
\]

\[
\phi^{(m)} \\
\begin{array}{c}
\downarrow \\
P_2^{(m)} \cong P_2^{(m)} \\
\cong C_2 + M_2 \cong C + M_2
\end{array}
\]

Hence \(M_1 \cong M_2\). If \(\psi\) denotes this isomorphism, then \((M_1, M_2, \psi) \in I(C)\) by the above theorem and

\[
(C_1, C_2, 1_\mathbb{C}) \oplus (M_1, M_2, \psi) \cong (C_1 \oplus M_1, C_2 \oplus M_2, 1_\mathbb{C} \oplus \psi) \cong (P_1, P_2, \phi)^{(m)}.
\]

Thus \(C \cong (C_1, C_2, 1_\mathbb{C})\) is a direct summand of \((P_1, P_2, \phi)^{(m)}\), so that \((P_1, P_2, \phi) \in CI(C)\). For the second claim, first we prove that if \(\phi, \psi \in \text{Aut}(C)\) then

\[
(C_1^{(n)}, C_2^{(n)}, \psi) \cong (C_1^{(m)} \oplus C_1^{(n)}, C_2^{(m)} \oplus C_2^{(n)}, \psi) \cong (C_1^{(m)} \boxplus C_1^{(n)}, C_2^{(m)} \boxplus C_2^{(n)}, \psi) \cong (C_1^{(n)} \boxplus C_1^{(n)}, C_2^{(m)} \boxplus C_2^{(m)}, \psi).
\]

In order to prove this, we cotensor the equalities below with each other

\[
(C_1^{(n)}, C_2^{(n)}, \phi) \oplus (C_1^{(n)}, C_2^{(n)}, \phi^{-1}) \cong C^{(2n)},
\]

\[
(C_1^{(m)}, C_2^{(m)}, \phi) \oplus (C_1^{(m)}, C_2^{(m)}, \phi^{-1}) \cong C^{(2m)},
\]

and define maps as in the proof of the above theorem. Since the claim is true for triples of this form, it is also true for direct summand of such triples by an argument similar to the one given before. 

**Corollary 3.9** \(CI(C)\) and \(CI(C_1) \times_{CI(C)} CI(C_2)\) are equivalent categories.

The foregoing result is also true for the category \(AZ(C)\) when all coalgebras are irreducible. This condition is not restrictive since the computation of the Brauer group of a coalgebra only requires the computation of the Brauer group of its irreducible subcoalgebras, cf. [3, Th. 4.9]. If \(D_i\) are \(C_i\)-coalgebras \(i = 1, 2\) such that \(\phi : D_1 \boxplus C_1 \cong D_2 \boxplus C_2\) is an isomorphism of \(C\)-coalgebras, it is routine to check that \((D_1, D_2, \phi)\) is a \(C\)-coalgebra.

**Lemma 3.10** Suppose that in the cartesian diagram (3) \(C_1\) and \(C_2\) are irreducible and either \(f_1\) or \(f_2\) is injective. If \(P = (P_1, P_2, \phi) \in I(C_1) \times_{I(C)} I(C_2)\) then \(e_{-C}(P) \cong (e_{-C_1}(P_1), e_{-C_2}(P_2), \tilde{\phi})\).

Moreover, \(e_{-C_1}(P) \boxplus C_1 \cong e_{-C_2}(P) \boxplus C_2\) for \(i = 1, 2\).
Proof. From [6, Prop. 3.1] we know that for a finitely cogenerated free $C$-comodule $Q = C^{(n)}$, $e_{-C}(Q) \cong M^{i}(C, n)$ the $n \times n$ comatrix coalgebra over $C$, i.e., $C \otimes k_{n}^{\ast}$. Since $P_{i} \in I(C_{i})$, $i = 1, 2$, and the coalgebras are irreducible we have $P_{1} \cong C_{1}^{(n)}$, $P_{2} \cong C_{2}^{(m)}$. As $\overline{P_{1}} = \overline{P_{2}}$, $\overline{C^{(n)}} = \overline{C^{(m)}}$ and then $n = m$. Thus, we may suppose that $\phi = 1_{C_{(s)}}$. Hence $P \cong C^{(n)}$ and

$$e_{-C}(P) \cong C \otimes k_{n}^{\ast} \cong (C_{1} \otimes C_{2} \otimes 1_{C_{(s)}}) \otimes k_{n}^{\ast} \cong (C \otimes k_{n} \otimes k_{n}^{\ast}, 1_{C_{(s)}}) \cong (e_{-C}(P_{1}), e_{-C}(P_{2}), 1_{C_{(s)}}).$$

Finally,

$$e_{-C}(P_{i} \square_{C_{i}} C) \cong \overline{C} \otimes k_{n}^{\ast} \cong (C_{i} \square_{C_{i}} 1_{C}) \otimes k_{n}^{\ast} \cong (C_{i} \otimes k_{n}^{\ast}) \square_{C_{i}} C \cong e_{-C}(P_{i}) \square_{C_{i}} C \quad i = 1, 2.$$

We recall from [3, Prop. 3.2] the following characterization of coseparable coalgebras. Let $D$ be a $C$-coalgebra and let $D^{e}$ be the $C$-enveloping coalgebra $D \square_{C} D^{op}$, then $D$ is $C$-coseparable if and only if $D$ is injective as a right $D^{e}$-comodule.

**Corollary 3.11** With the same hypotheses as in the foregoing lemma, the categories $A_{z}(C)$ and $A_{z}(C_{1}) \times A_{z}(C_{2})$ are equivalent.

**Proof.** Let $D = (D_{1}, D_{2}, \phi) \in A_{z}(C_{1}) \times A_{z}(C_{2})$, then $D$ is a $C$-coalgebra. We have to check that $D \in A_{z}(C)$. We have the following cocartesian diagram:

$$\begin{array}{ccc}
D = D \square_{C} C & \xrightarrow{\bar{f}_{1}} & D \square_{C} C_{1} = D_{1} \\
\downarrow \bar{f}_{2} & & \downarrow \bar{g}_{1} \\
D_{2} = D \square_{C_{2}} C_{2} & \xrightarrow{\bar{f}_{2}} & D_{2} \square_{C_{2}} C_{1} = D_{2} \xrightarrow{\phi} D_{1} \square_{C_{1}} C_{1} = D_{1}
\end{array}$$

Also, we have a similar square for the opposite coalgebras. Hence we obtain the following cocartesian diagram for the enveloping coalgebras.

$$\begin{array}{ccc}
D^{e} = D \square_{C} D^{op} & \xrightarrow{\bar{f}_{1} \square_{C} f_{1}^{op}} & D_{1} \square_{C_{1}} D_{1}^{op} = D_{1}^{e} \\
\downarrow \bar{f}_{2} \square_{C} f_{2}^{op} & & \downarrow \bar{g}_{1} \square_{C_{1}} g_{1}^{op} \\
D_{2}^{e} = D_{2} \square_{C_{2}} D_{2}^{op} & \xrightarrow{\bar{g}_{1} \square_{C_{1}} g_{1}^{op}} & D_{1} \square_{C_{1}} D_{1}^{op} = D_{1}^{e}
\end{array}$$
Since either $f_1$ or $f_2$ is injective and the cotensor product is left exact either $f_1 \Box f_1^op$ or $f_2 \Box f_2^op$ is injective. On the other hand, $D_i$ is always quasi-finite as a $D_i^c$-comodule. Since $C_i$ is irreducible and $D_i$ is a $C_i$-Azumaya coalgebra it follows that $D_i^c$ is irreducible, cf [3, Prop. 4.10]. Since $D_i$ is $C_i$-coseparable, then $D_i$ is injective as a $D_i^c$-comodule $i = 1, 2$. Hence, $D_i$ is a finitely cogenerated injective $D_i^c$-comodule $i = 1, 2$. From Theorem 3.6 applied to the square above we obtain that $D = (D_1, D_2, \phi)$ is finitely cogenerated injective as a $D^c$-comodule, therefore $D$ is $C$-coseparable. To see that $D$ is cocommutative we use the foregoing lemma, $Z(D) = \epsilon_{-D_i^c}(D) \cong (\epsilon_{-D_1^c}(D_1), \epsilon_{-D_2^c}(D_2), \tilde{\phi})$ and since $D_i$ is $C_i$-Azumaya it follow that $Z(D_i) = \epsilon_{-D_i^c}(D_i) \cong C_i$ $i = 1, 2$. So, $Z(D) \cong C$ and $D$ is cocommutative. Consequently, $D$ is $C$-Azumaya.

Conversely, if $D$ is $C$-Azumaya we always have $D \cong (D_1, D_2, \phi)$ where $D_i = D \Box C_i$ $i = 1, 2$ and $\phi$ is the canonical isomorphism $D_1 \Box C_1 C \cong D_2 \Box C_2 C$. 

\[\square\]

4 A MAYER-VIETORIS TYPE EXACT SEQUENCE

We recall the definition of the Picard group of a cocommutative coalgebra $C$, cf. [3, page 558], [7]. The Picard group of $C$, denoted by $\text{Pic}(C)$, is defined as the group of all isomorphism classes of invertible $C$-comodules, where $M$ is an invertible $C$-comodule if there exists a $C$-comodule $N$ such that $M \Box C N \cong C$ as $C$-comodules. Multiplication is induced by cotensor product, i.e., $[M][N] = [M \Box C N]$. The class $[C]$ is the identity element, and the inverse is given by $[M]^{-1} = [M^{-1}]$, with $M^{-1} = h_C(M, C)$.

PROPOSITION 4.1 Let $C$ be a cocommutative coalgebra, then $\text{Pic}(C)$ is trivial.

Proof. If $P$ is an invertible $C$-comodule, then $P$ defines an equivalence from $M^C$ to itself. From [2, Th. 3.5] we obtain that $P$ is a quasi-finite injective cogenerator. When $C$ is irreducible, from Proposition 3.1 we retain that quasi-finite injective implies finitely cogenerated injective, hence $P$ is finitely cogenerated injective and invertible, so $P$ is isomorphic to $C$. If $C$ is cocommutative then $C = \bigoplus_{i \in I} C_i$ with $C_i$ irreducible for all $i \in I$. Every $C$-comodule $P$ is isomorphic to $\bigoplus_{i \in I} P_i$ with $P_i = P \Box C_i$ and $P$ is invertible if and only if $P_i$ is invertible as $C_i$-comodule. Since $C_i$ is irreducible then $P_i \cong C_i$ and $P \cong \bigoplus_{i \in I} P_i \cong \bigoplus_{i \in I} C_i = C$. 

\[\square\]
**Proposition 4.2** Let $C$ be a cocommutative coalgebra, $P, Q \in CI(C)$ and $\alpha : \epsilon_C(P) \to \epsilon_C(Q)$ an isomorphism of $C$-coalgebras. Then $\alpha$ is induced by an isomorphism of $C$-comodules $f : P \to Q$.

**Proof.** Put $D = \epsilon_C(P)$, then $Q$ is a left $D$-comodule with structure map,

$$Q \xrightarrow{\theta} \epsilon_C(Q) \otimes Q \xrightarrow{\alpha \otimes 1} D \otimes Q.$$

If $P^*$ denotes $h_C(P, C)$, from [2, Th. 3.5] we have $Q \cong P \otimes_C (P^* \otimes_D Q)$ as left $D$-comodules. Let $I = P^* \otimes_D Q$ and $Q^* = h_C(Q, C)$ then $I \in Pic(C)$ since,

$$(Q^* \otimes_D P) \otimes_C (P^* \otimes_D Q) \cong Q^* \otimes_D (P \otimes_C P^*) \otimes_D Q 
\cong Q^* \otimes_D D \otimes_D Q \cong Q^* \otimes_D Q \cong C.$$

Since $Pic(C)$ is trivial, $I \cong C$ and we have an isomorphism of left $D$-comodules $f : Q \to P$, i.e., the diagram

$$\begin{array}{ccc}
Q & \xrightarrow{f} & P \\
\downarrow{(\alpha \otimes 1)\theta} & & \downarrow{\theta} \\
D \otimes Q & \xrightarrow{1 \otimes f} & D \otimes P
\end{array}$$

is commutative. So, $(1 \otimes f^{-1})(\theta \otimes 1)f = (\alpha \otimes 1)\theta$. Hence $Q$ is a $D$-comodule with structure map $(1 \otimes f^{-1})(\theta \otimes 1)f$ and from [2, 1.18] it follows that $\alpha$ is induced by this map. 

**Theorem 4.3** Suppose that in the cocartesian diagram (3) all coalgebras are irreducible and either $f_1$ or $f_2$ is injective. If we consider the induced maps on the Brauer group level, the following sequence is exact:

$$0 \longrightarrow Br(C) \longrightarrow Br(C_1) \oplus Br(C_2) \longrightarrow Br(C).$$

**Proof.** Let $f, g$ denote the group homomorphisms $f : Br(C) \to Br(C_1) \oplus Br(C_2), g : Br(C_1) \oplus Br(C_2) \to Br(C)$. Explicitly, these maps are given by:

$$f([D]) = (g_{1*}, g_{2*})([D]) = ([D \otimes_C C_1], [D \otimes_C C_2]),$$

$$g(([D_1], [D_2])) = (f_{1*}, f_{2*}^o)([D_1], [D_2]) = [D_1 \otimes_C C] \otimes_C [D_2 \otimes_C C]^o,$$
for all $[D] \in Br(C)$, $([D_1],[D_2]) \in Br(C_1) \oplus Br(C_2)$. First we prove that $f$ is injective. By Corollary 3.11, if $D = (D_1,D_2,\phi) \in Az(C)$ is such that $[D] \in \text{Ker}(f)$, then $[D \square_C C_i] = [D_i] = [C_i]$ in $Br(C_i)$ for $i = 1, 2$. Hence there is a $P_i \in CI(C_i)$ such that $D_i \equiv e_{-C_i}(P_i)$ $i = 1, 2$. Using Lemma 3.10 we obtain,

$$
e_{-C}(P_1 \square_{C_1} C) \equiv e_{-C_1}(P_1) \square_{C_1} C \equiv e_{-C_2}(P_2) \square_{C_2} C \equiv e_{-C}(P_2 \square_{C_2} C).$$

Let $\phi' : e_{-C}(P_1 \square_{C_1} C) \to e_{-C}(P_2 \square_{C_2} C)$ denote this isomorphism. By the foregoing proposition, $\phi'$ is induced by a $C$-isomorphism $\psi : P_1 \square_{C_1} C \to P_2 \square_{C_2} C$. Therefore,

$$D = (D_1,D_2,\phi) \equiv (e_{-C_1}(P_1),e_{-C_2}(P_2),\phi') \equiv e_{-C}(P_1, P_2, \psi)).$$

By Theorem 3.8, $(P_1, P_2, \psi) \in CI(C)$. So $[D] = [C]$ in $Br(C)$.

Next, we check the exactness of the sequence. It is clear that $\text{Im}(f) \subseteq \text{Ker}(g)$. Let $([E_1],[E_2]) \in Br(C_1) \oplus Br(C_2)$ be such that $([E_1],[E_2]) \in \text{Ker}(g)$, i.e., $[E_1 \square_C C] = [E_2 \square_C C]$ in $Br(C)$. Then, there are $P, Q \in CI(C)$ such that

$$(E_1 \square_{C_1} C) \square_{C_1} e_{-C}(P) \equiv (E_2 \square_{C_2} C) \square_{C_2} e_{-C}(P).$$

Since $C$ is irreducible we may suppose $P \cong C^{(n)}$, $Q \cong C^{(m)}$ and in view of Lemma 3.10 the above equality transforms to:

$$E_1 \square_{C_1} M^{(n)}(C_1, n) \equiv E_2 \square_{C_2} M^{(n)}(C_2, m).$$

Put $P_1 = C_1^{(n)}$ and $P_2 = C_2^{(m)}$, then

$$(E_1 \square_{C_1} e_{-C_1}(P_1)) \square_{C_1} C \equiv (E_2 \square_{C_2} e_{-C_2}(P_2)) \square_{C_2} C \equiv E_1 \square_{C_1} M^{(n)}(C_1, n) \equiv E_2 \square_{C_2} M^{(m)}(C_2, m) \equiv (E_2 \square_{C_2} M^{(n)}(C_2, m)) \square_{C_2} C \equiv (E_2 \square_{C_2} e_{-C_2}(P_2)) \square_{C_2} C.$$

Let $\psi$ be this isomorphism and $E = (E_1 \square_{C_1} e_{-C_1}(P_1), E_2 \square_{C_2} e_{-C_2}(P_2), \psi)$. By Corollary 3.11, $E \in Az(C)$ and $[E \square_C C_i] = [E_i \square_C e_{-C_i}(P_i)] = [E_i]$ $i = 1, 2$. Therefore $([E_1],[E_2]) \in \text{Im}(f)$. $lacksquare$
Example 4.4 Let $C$ be a cocommutative irreducible coalgebra, and $C_0$ its coradical. The universal connected coalgebra associated to $C$, is defined as $R(C) = C/C_0^+\varepsilon$ where $C_0^+ = C_0 \cap \text{Ker}\varepsilon$. We are going to relate $Br(C)$ and $Br(R(C))$ via the Mayer-Vietoris sequence. Let $i : C_0 \rightarrow C$ the inclusion map, $I = \text{Im}(i, -\varepsilon)$ and $D = C \oplus k/I$. We define $q : k \rightarrow D$, $\lambda \mapsto (0, \lambda) + I$, and $p : C \rightarrow D$, $c \mapsto (c, 0) + I$ for all $\lambda \in k, c \in C$. The commutative diagram

\[ C_0 \xrightarrow{\varepsilon} k \]
\[ \downarrow i \quad \downarrow q \]
\[ C \quad \xrightarrow{p} D \]

is cocartesian. Since $\varepsilon$ is surjective, any element in $D$ can be taken of the form $(c, 0) + I$ for $c \in C$. It is trivial to prove that $q$ is injective, $p$ is surjective and the map $D \rightarrow R(C)$, $(c, 0) + I \mapsto c + C_0^+$ for any $c \in C$ is an isomorphism of coalgebras.

As a consequence of Theorem 4.3 we obtain,

Corollary 4.5 Let $C$ be a cocommutative irreducible coalgebra. There is an exact sequence:

\[ 0 \rightarrow Br(R(C)) \rightarrow Br(k) \oplus Br(C) \rightarrow Br(C_0). \]

Taking into account that the Brauer group of a finite field is trivial, we obtain:

Corollary 4.6 If $C$ is a cocommutative irreducible coalgebra over a finite field $k$, then $Br(C) \cong Br(R(C))$.

Acknowledgment

For this work, the authors have been supported by the grant CRG 971543 from NATO. Moreover, the first three authors benefitted from grant PB98-1005 from DGES. J. Cuadra is grateful to the Professors Y.H. Zhang and D. Stefan for some helpful discussions and to the research group Categorías, Computación y Teoría de anillos from PAI for its financial support while this work was done.
REFERENCES


