Idempotents and Morita-Takeuchi Theory

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Introduction

Takeuchi's theory of equivalences between comodule categories [1] rests on the notion of co-hom functor and the always slippery construction of the co-endomorphism coalgebra of a quasi-finite comodule. In fact, only a few cases have been computed in [2], [3], [4] and [5], using smash coproducts. One of the most basic applications of Takeuchi's theory is the foundation of a well behaved notion of basic coalgebra [6]. However, this construction of the basic coalgebra as the co-endomorphism coalgebra of a minimal injective cogenerator for the category of all (right) comodules is not very handy, at least from the point of view of the original coalgebra. In this notes, with inspiration in the classical theory of idempotents for finite dimensional algebras, we offer an easier approach to basic coalgebras. To do this, we develop in Section 1 the theory of idempotents for coalgebras, initiated in [7], [8]. Our approach is based upon the remarkable fact that, given an idempotent $e \in C^*$, the functor $e(-) : \mathcal{M}^C \to \mathcal{M}^{eCe}$ which associates to any right comodule M the eCe–comodule eM is naturally equivalent to the co-hom functor $h_{-C}(Ce, -) : \mathcal{M}^C \to \mathcal{M}^{eCe}$ (Theorem 1.5). In Section 2 we give a description in terms of idempotents and matrix coalgebras of the coalgebras strongly equivalent [9] to a given coalgebra. In section 3 we introduce the notion of basic idempotent for a coalgebra and we show that the basic coalgebra in the sense of [6] of a coalgebra C, is given by eCe, where e is a basic idempotent for C. Basic idempotents may be lifted via the coradical. Using this new point of view, we may prove that if two coalgebras are

Morita-Takeuchi equivalent, their coradicals are so. Our approach is shown to be equivalent to the given in [6].

1 A theory of idempotents for coalgebras

Throughout this paper k is a fixed ground field and $\mathcal{M}_k$ stands for the category of k-vector spaces. More generally, for a k–algebra A, the category of all left (resp. right) A–modules is denoted by ${}_A\mathcal{M}$ (resp. $\mathcal{M}_A$). We will assume that the reader is familiarized with the theory of coalgebras. Basic references are [10] and [11]. All coalgebras, vector spaces and unadorned $\otimes$, Hom, etc., are over k. Every map is a k-linear map. For a coalgebra C, Δ and ε denote the comultiplication and the counit respectively. The category of right (resp. left) C-comodules is denoted by $\mathcal{M}^C$ (resp. ${}^C\mathcal{M}$); for M in $\mathcal{M}^C$, ρ_M is the comodule structure map. Given $M, N \in \mathcal{M}^C$ (resp. ${}^C\mathcal{M}$), $Com_{-C}(M,N)$ (resp. $Com_{C-}(M,N)$) denotes the space of right (resp. left) C-comodule maps from M to N. The fundamental properties of the categories of comodules can be found in several places, see e.g. [12], [1] and [13].

Let C be a coalgebra and C^* its dual algebra. As usual, we consider C as a left (resp. right) C^*-module via the actions:

$$c^*c = \sum_{(c)} \langle c^*, c_{(2)}\rangle c_{(1)}, \quad cc^* = \sum_{(c)} \langle c^*, c_{(1)}\rangle c_{(2)}, \qquad (c^* \in C^*, c \in C).$$

Definition 1.1 *An element $e \in C^*$ is said to be an idempotent for C if e is an idempotent in C^*. This means that $\langle e, c\rangle = \sum_{(c)} \langle e, c_{(1)}\rangle\langle e, c_{(2)}\rangle$ for all $c \in C$.*

For any idempotent e for C, it was proved in [7, Lemma 6] that eCe is a coalgebra with comultiplication and counit given by:

$$\Delta(ece) = \sum_{(c)} ec_{(1)}e \otimes ec_{(2)}e, \qquad \varepsilon(ece) = \langle e, c\rangle.$$

Moreover, the natural map $\eta : eC^*e \to (eCe)^*$ defined by $\langle \eta(ec^*e), ece\rangle = \langle ec^*e, c\rangle = \langle c^*, ece\rangle$ is an algebra isomorphism.

Let C, D be coalgebras. Any (C,D)-bicomodule M is a (D^*, C^*)-bimodule via the actions

$$d^*m = \sum_{(m)} \langle d^*, m_{(1)}\rangle m_{(0)}, \quad mc^* = \sum_{(m)} \langle c^*, m_{(-1)}\rangle m_{(0)} \qquad (c^* \in C^*, d^* \in D^*, m \in M).$$

If e, d are idempotents in C^* and D^* respectively, then we may consider the eC^*e-module Me and the dD^*d-module dM. Clearly, Me is a direct summand of M as D-comodule since $Me \oplus Me' = M$ where $e' = \varepsilon_C - e$. Similarly for dM.

Lemma 1.2 *Let M be a (C, D)-bicomodule, and e, d idempotents for C and D respectively.*

i) Me is an (eCe, D)-bicomodule via the maps:

$$\rho^+ : Me \to eCe \otimes Me,\ \ me \mapsto \sum_{(m)} em_{(-1)}e \otimes m_{(0)}e$$
$$\rho^- : Me \to Me \otimes D,\ \ me \mapsto \sum_{(m)} m_{(0)}e \otimes m_{(1)}$$

ii) dM is a (C, dDd)-bicomodule via the maps:

$$\omega^+ : dM \to C \otimes dM,\ \ dm \mapsto \sum_{(m)} m_{(-1)} \otimes dm_{(0)}$$
$$\omega^- : dM \to dM \otimes dDd,\ \ dm \mapsto \sum_{(m)} dm_{(0)} \otimes dm_{(1)}d$$

iii) $(Me)^* \cong eM^*$ and $(dM)^* \cong M^*d$ as (eC^*e, D^*) and (C^*, dD^*d)-bimodules respectively.

Proof: *i),ii)* Straightforward.

iii) $(Me)^*$ is a (eC^*e, D^*)-bimodule via the actions:

$$\langle (ec^*e)\phi, me\rangle = \sum_{(m)} \langle ec^*e, em_{(-1)}e\rangle \langle \phi, m_{(0)}e\rangle,$$
$$\langle \phi d^*, me\rangle = \sum_{(m)} \langle \phi, m_{(0)}e\rangle \langle d^*, m_{(1)}\rangle.$$

for all $\phi \in (Me)^*, c^* \in C^*, d^* \in D^*$, and $m \in M$. The isomorphism is given by $\Psi : eM^* \to (Me)^*$ defined as $\langle \Psi(em^*), me\rangle = \langle m^*, me\rangle = \langle em^*, m\rangle$ for all $m^* \in M^*, m \in M$. ∎

A comodule $X \in \mathcal{M}^C$ is called quasi-finite if $Com_{-C}(Y, X)$ is finite dimensional for every finite dimensional comodule $Y \in \mathcal{M}^C$. This is equivalent [1] to the existence of a left adjoint $h_{-C}(X, -)$, called co-hom functor, to $- \otimes X$. In the case that X is a (D, C)–bicomodule, the functor $h_{-C}(X, -)$ becomes a left adjoint to the cotensor product functor $-\Box_D X : \mathcal{M}^D \to \mathcal{M}^C$. We recall from [1] some basic facts concerning with the co-hom functors. Let $\theta_Y : Y \to h_{-C}(X, Y)\Box_D X$ denote the unit of the adjunction. If we assume that X_C is a quasi-finite comodule, then $e_{-C}(X) = h_{-C}(X, X)$ is a coalgebra, called the co-endomorphism coalgebra of X. The comultiplication of $e_{-C}(X)$

corresponds to $(1\otimes\theta_X)\theta_X : X \to e_{-C}(X)\otimes e_{-C}(X)\otimes X$ in the adjunction isomorphism when $D = k$, and the counit of $e_{-C}(X)$ corresponds to the identity map 1_X. X is a $(e_{-C}(X), C)$-bicomodule via $\theta_X : X \to e_{-C}(X)\otimes X$. Moreover, there is a coalgebra map $\lambda : D \to e_{-C}(X)$ such that $\rho_D = (\lambda\otimes 1)\theta_X$.

Morita-Takeuchi contexts were introduced in [1] with the name of pre-equivalence data (see also [14]). Next, we give a new example of Morita-Takeuchi context. An explicit example of co-endomorphism coalgebra is also computed.

Example 1.3 Let C be a coalgebra, and e an idempotent for C. In view of the above lemma, Ce is a (eCe, C)-bicomodule, and eC is a (C, eCe)-bicomodule. We define the maps

$$\begin{aligned} &f : eCe \to Ce\Box_C eC,\ ece \mapsto \textstyle\sum_{(c)} c_{(1)}e\otimes ec_{(2)}\\ &g : C \to eC\Box_{eCe} Ce,\ c \mapsto \textstyle\sum_{(c)} ec_{(1)}\otimes c_{(2)}e \end{aligned}$$

It is routine to check that f and g are bicomodule maps and (eCe, C, Ce, eC, f, g) is a Morita-Takeuchi context.

Proposition 1.4 *The map f is injective. Hence $e_{-C}(Ce)\cong eCe$.*

Proof: Suppose that $f(ece) = 0$. Then,

$$0 = \sum_{(c)} c_{(1)}e\otimes ec_{(2)} = \sum_{(c)} \langle e, c_{(1)}\rangle c_{(2)}\otimes c_{(3)}\langle e, c_{(4)}\rangle.$$

Applying $1\otimes e$,

$$0 = \sum_{(c)} \langle e, c_{(1)}\rangle c_{(2)}\langle e, c_{(3)}\rangle\langle e, c_{(4)}\rangle = \sum_{(c)} \langle e, c_{(1)}\rangle c_{(2)}\langle e, c_{(3)}\rangle = ece.$$

By [1, Theorem 2.5], f is an isomorphism, and $e_{-C}(Ce)\cong e_{C-}(eC)\cong eCe$. ∎

We will see later (see Corollary 1.8) that the above context is strict if and only if Ce is a cogenerator as C-comodule.

In light of Lemma 1.2, we have functors

$$\begin{aligned} &(-)e : {}^C\mathcal{M} \to {}^{eCe}\mathcal{M},\ M \mapsto Me,\ f \mapsto f\mid_{Me}\\ &e(-) : \mathcal{M}^C \to \mathcal{M}^{eCe},\ M \mapsto eM,\ f \mapsto f\mid_{eM} \end{aligned}$$

Since Ce is a direct summand of C as right C-comodule, Ce is a quasi-finite injective C-comodule. Note that C is quasi-finite and for a finite dimensional $Y\in\mathcal{M}^C$, $Com_{-C}(Y, Ce)$ is a quotient space of $Com_{-C}(Y, C)$.

Theorem 1.5 *The functor* $e(-) : \mathcal{M}^C \to \mathcal{M}^{eCe}$ *is naturally equivalent to* $h_{-C}(Ce, -) : \mathcal{M}^C \to \mathcal{M}^{eCe}$.

Proof: By the uniqueness of adjoints, we need just to prove that $e(-) : \mathcal{M}^C \to \mathcal{M}^{eCe}$ is a left adjoint to $-\square_{eCe}Ce : \mathcal{M}^{eCe} \to \mathcal{M}^C$. Let $Y \in \mathcal{M}^C$ and $Z \in \mathcal{M}^{eCe}$. We claim that the map

$$\Phi : Com_{-C}(Y, Z\square_{eCe}Ce) \to Com_{-eCe}(eY, Z)$$

defined by $\Phi(f) = (1 \otimes e)f$ is a natural isomorphism.

We first check that Φ is well-defined, that is, $\Phi(f)$ is a eCe-comodule map. In order to do this, we need a pair of facts. Let $y \in Y$, and set $f(y) = \sum_i z_i \otimes c_i e$. Since $f(y) \in Z\square_{eCe}Ce$, then

$$\sum_i \sum_{(z_i)} z_{i(0)} \otimes z_{i(1)} \otimes c_i e = \sum_i \sum_{(c_i)} z_i \otimes ec_{i(1)}e \otimes c_{(2)}e.$$

Applying $1 \otimes e$, we obtain the formula:

$$\sum_i \sum_{(z_i)} z_{i(0)} \otimes z_{i(1)}\langle e, c_i\rangle = \sum_i \sum_{(c_i)} z_i \otimes ec_{i(1)}e\langle e, c_{i(2)}\rangle = \sum_i z_i \otimes ec_i e. \qquad (1)$$

On the other hand, using that f is a C-comodule map, we deduce:

$$\begin{aligned} \textstyle\sum_{(y)} f(y_{(0)}) \otimes y_{(1)} &= \textstyle\sum_{(f(y))} f(y)_{(0)} \otimes f(y)_{(1)} \\ &= \textstyle\sum_i \sum_{(z_i)} z_{i(0)} \otimes z_{i(1)} \otimes c_i e \\ &= \textstyle\sum_i \sum_{(c_i)} z_i \otimes ec_{i(1)}e \otimes c_{(2)}e. \end{aligned} \qquad (2)$$

$$\begin{aligned} [\rho_Z\Phi(f)](ey) &= \rho_Z(\textstyle\sum_{(y)}\langle e, y_{(1)}\rangle(1 \otimes e)f(y_{(0)})) \\ &= \textstyle\sum_i \sum_{(c_i)} \rho_Z(\langle e, c_{i(2)}e\rangle z_i\langle e, ec_{i(1)}e\rangle) && \textit{by } (2) \\ &= \textstyle\sum_i \sum_{(z_i)}\langle e, c_i\rangle z_{i(0)} \otimes z_{i(1)} \\ &= \textstyle\sum_i z_i \otimes ec_i e && \textit{by } (1) \\ &= \textstyle\sum_i \sum_{(c_i)}\langle e, c_{i(1)}e\rangle z_i \otimes ec_{i(2)}e \\ &= \textstyle\sum_{(f(y))}(1 \otimes e)(f(y)_{(0)} \otimes ef(y)_{(1)}e) \\ &= \textstyle\sum_{(y)}(1 \otimes e)f(y_{(0)}) \otimes ey_{(1)}e \\ &= \textstyle\sum_{(y)}(1 \otimes e)f(ey_{(0)}) \otimes ey_{(1)}e \\ &= (\Phi(f) \otimes 1)\rho_{Ce}(ey). \end{aligned}$$

For $g \in Com_{-eCe}(eY, Z)$, the inverse to Φ is given by $\Psi(g)(y) = \sum_{(y)} g(ey_{(0)}) \otimes y_{(1)}e$ for all $y \in Y$. We next check that Ψ is well-defined, that is, $\Psi(g)$ is a C-comodule map, and $Im\Psi(g) \subseteq Z\square_{eCe}Ce$. The first fact may be easily checked. We only see the second one.

$$\begin{aligned}(\rho_Z \otimes 1)[\Psi(g)(y)] &= \textstyle\sum_{(y)} \sum_{(g(ey_{(0)}))} g(ey_{(0)})_{(0)} \otimes g(ey_{(0)})_{(1)} \otimes y_{(1)}e \\ &= \textstyle\sum_{(y)} g(ey_{(0)}) \otimes ey_{(1)}e \otimes y_{(2)}e \\ &\quad \textit{since } g \textit{ is a } eCe-\textit{comodule map} \\ &= (1 \otimes \rho^-_{Ce})[\Psi(g)(y)].\end{aligned}$$

Finally, we see that the maps Φ and Ψ are inverse to each other.

$$\begin{aligned}\Psi\Phi(f)(y) &= \Psi[(1\otimes e)f](y) = \textstyle\sum_{(y)}(1\otimes e)f(ey_{(0)}) \otimes y_{(1)}e \\ &= \textstyle\sum_{(y)}\langle e, y_{(1)}\rangle\langle e, y_{(2)}\rangle(1\otimes e)f(y_{(0)}) \otimes y_{(3)} \\ &= \textstyle\sum_{(y)}(1\otimes e)f(y_{(0)}) \otimes y_{(1)}e \\ &= \textstyle\sum_{(y)}(1\otimes e)f(y)_{(0)} \otimes f(y)_{(1)}e \\ &\quad \textit{since } f \textit{ is a } C-\textit{comodule map} \\ &= \textstyle\sum_i \langle e, c_{i(1)}e\rangle z_i \otimes c_{i(2)}e \\ &= \textstyle\sum_i z_i \otimes c_i e = f(y) \Rightarrow \Psi\Phi(f) = f.\end{aligned}$$

$$\begin{aligned}\Phi\Psi(g)(ey) &= (1\otimes e)(\textstyle\sum_{(y)} g(ey_{(0)}) \otimes y_{(1)}e) \\ &= \textstyle\sum_{(y)}\langle e, y_{(1)}\rangle\langle e, y_{(2)}\rangle g(ey_{(0)}) \\ &= g(ey) \Rightarrow \Phi\Psi(g) = g. \qquad \blacksquare\end{aligned}$$

Corollary 1.6 *Let M be a (D,C)-bicomodule and e, e' idempotents for C.*

i) $eM \cong h_{-C}(Ce, M)$ as (D, eCe)-bicomodules.

ii) $h_{-C}(Ce, Ce') \cong eCe'$ as $(e'Ce', eCe)$-bicomodules.

iii) $e(-) : \mathcal{M}^C \to \mathcal{M}^{eCe}$ is naturally equivalent to $-\square_C eC : \mathcal{M}^C \to \mathcal{M}^{eCe}$.

Proof: *i)* We already know that they are isomorphic as eCe-comodules. Let $\rho^- : M \to D \otimes M$ be the structure map. The D-comodule structure map of eM may be viewed as the induced map $e(\rho^-) = \rho^- |_{eM}: eM \to e(D \otimes M) = D \otimes eM$. Similarly, $h_{-C}(Ce, M)$ is a D-comodule via the map $h_{-C}(Ce, \rho^-) : h_{-C}(Ce, M) \to h_{-C}(Ce, D \otimes M) \cong D \otimes h_{-C}(Ce, M)$ (see [1, 1.7]). The claim follows now since the isomorphisms are natural.

ii) This is a particular case of *i)* for $M = Ce'$.

iii) Since Ce is quasi-finite injective as C-comodule, by [1, Proposition 1.14], $-\square_C h_{-C}(Ce, C)$ is naturally equivalent to $h_{-C}(Ce, -)$. By *ii)*, $eC \cong h_{-C}(Ce, C)$ as bicomodules. Then the functor $-\square_C eC$ and $-\square_C h_{-C}(Ce, C)$ are naturally equivalent, [1, Lemma 2.2]. These facts and Theorem 1.5 yield the result. $\blacksquare$

Remark 1.7 *A symmetric version of all the above results may be done for $h_{C-}(eC,-)$. In particular, the functors $(-)e, h_{C-}(eC,-)$ and $Ce\Box_C- : {}^C\mathcal{M} \to {}^{eCe}\mathcal{M}$ are naturally equivalent.*

Consider the functors

$$\begin{tikzcd} \mathcal{M}_k \arrow[r, shift left, "- \otimes eCe"] & \mathcal{M}^{eCe} \arrow[l, shift left, "\mathcal{U}_{eCe}"] \arrow[r, shift left, "-\Box_{eCe}Ce"] & \mathcal{M}^C \arrow[l, shift left, "e(-)"] \end{tikzcd}$$

where $\mathcal{U}_{eCe}$ is the forgetful functor. Notice that $(- \otimes eCe)\Box_{eCe}Ce \cong - \otimes Ce$. Then the functor $\mathcal{U}_{eCe} \circ e(-)$ is a left adjoint of $- \otimes Ce$. For W in $\mathcal{M}_k$, Y in $\mathcal{M}^C$, let Θ be the composition of the isomorphisms,

$$\begin{aligned} Com_{-C}(Y, W \otimes Ce) &\cong Com_{-C}(Y, (W \otimes eCe)\Box_{eCe}Ce) \\ &\cong Com_{-eCe}(eY, W \otimes eCe) \\ &\cong Hom_k((\mathcal{U}_{eCe} \circ e)(Y), W) \end{aligned}$$

and Π its inverse. Analyzing the definition of each isomorphism, we may deduce that $\Theta(f) = (1 \otimes e)f$ and $\Pi(g)$ is defined as $\Pi(g)(y) = \sum_{(y)} g(ey_{(0)}) \otimes y_{(1)}e$ for all $f \in Com_{-C}(Y, W \otimes Ce)$ and $g \in Hom_k((\mathcal{U}_{eCe} \circ e)(Y), W)$. Under these identifications the context of Example 1.3 is nothing but the Morita-Takeuchi context associated to the right quasi-finite C-comodule Ce, see [1, Section 3]. By [1, Corollary 3.4] we get:

Corollary 1.8 *The Morita-Takeuchi context (eCe, C, Ce, eC, f, g) of Example 1.3 is strict if and only if Ce is a cogenerator as C-comodule.*

Next, we will show that the decomposition theory of quasi-finite comodule is equivalent to the theory of idempotents of its coendomorphims coalgebra. Our exposition parallelizes the standard treatment of this topic for finite dimensional modules over algebras of finite dimension. For $M \in \mathcal{M}^C$, $E(M)$ denotes the injective hull of M.

Lemma 1.9 *Let e be an idempotent for C and consider $\bar{e} = e \mid_{C_0}$.*

i) $E(C_0\bar{e}) \cong Ce$.

ii) $soc(Ce) \cong C_0\bar{e}$

Proof: *i)* It is well-known that C is the injective hull of C_0, see [12]. The inclusion map $i : C_0 \to C$ is essential. Consider the inclusion map $j : C_0\bar{e} \to Ce$. Since Ce is injective, it suffices to see that j is essential. Let N be a subcomodule of Ce such that $Im(j) \cap N = \{0\}$. Pick $x \in Im(i) \cap N$, then $x \in C_0$ and $x = ce$ for some $c \in C$. The element $x = ce = (ce)e = xe \in N \cap C_0\bar{e}$. By hypothesis, $x = 0$. Then $Im(i) \cap N = \{0\}$, and since i is essential, $N = \{0\}$.

ii) For any right C-comodule M, $soc(M) \cong M\square_C C_0$. Then $soc(Ce) \cong Ce\square_C C_0 \cong C_0\bar{e}$. ∎

Proposition 1.10 *Let e, e' be idempotents for C. The following assertions are equivalent: i) $Ce \cong Ce'$; ii) $C_0\bar{e} \cong C_0\bar{e'}$; iii) $eC \cong e'C$; iv) $\bar{e}C_0 \cong \bar{e'}C_0$.*

Proof: *i)* $\Leftrightarrow$ *ii)* If $Ce \cong Ce'$, then $C_0\bar{e} \cong soc(Ce) \cong soc(Ce') \cong C_0\bar{e'}$. Conversely, if $C_0\bar{e} \cong C_0\bar{e'}$, then $Ce \cong E(C_0\bar{e}) \cong E(C_0\bar{e'}) \cong Ce'$.

iii) $\Leftrightarrow$ *iv)* Symmetric version of the above.

i) $\Leftrightarrow$ *iii)* If $Ce \cong Ce'$, then $eC \cong h_{-C}(Ce, C) \cong h_{-C}(Ce', C) \cong e'C$. The converse is similar using $h_{C-}(-, C)$. ∎

Let $M \in \mathcal{M}^C$ be quasi-finite. Since $- \otimes M$ is right adjoint to $h_{-C}(M, -)$ we get an isomorphism $\Phi : Com_{-C}(M, M) \to e_{-C}(M)^*$ given by

$$Com_{-C}(M, M) \cong Com_{-C}(M, k \otimes M) \cong Hom_k(h_{-C}(M, M), k) \cong e_{-C}(M)^*.$$

Lemma 1.11 *The isomorphism Φ gives an algebra isomorphism*

$$e_{-C}(M)^* \cong End_{-C}(M)^{op} = End_{C^*}(M)^{op}.$$

Moreover, the $(C^*, e_{-C}(M)^*)$-bimodule structure induced on M by its $(e_{-C}(M), C)$-bicomodule structure is compatible, via Φ, with the canonical $(C^*, End_{C^*}(M)^{op})$-bimodule structure of ${}_{C^*}M$.

Proof: Since $h_{-C}(M, -)$ is left adjoint to $- \otimes M$, we can consider the unit of the adjunction $\theta_M : 1 \to h_{-C}(M, -) \otimes M$. We may reconstruct Φ and Φ^{-1} from θ_M since it verifies the following universal property: for any $f \in Com_{-C}(M, M)$ there is a unique map $u \in e_{-C}(M)^*$ such that $f = (u \otimes 1)\theta_M$. Then $\Phi(f) = u$ and $\Phi^{-1}(u) = (u \otimes 1)\theta_M$.

We check that Φ^{-1} is indeed an algebra map. We denote by $*$ the convolution product in $e_{-C}(M)^*$. Let $u, v \in e_{-C}(M)^*$ and $f = (u \otimes 1)\theta_M, g = (v \otimes 1)\theta_M$. Then,

$$
\begin{aligned}
[(u*v)\otimes 1]\theta_M &= [(u*v)\Delta\otimes 1]\theta_M \\
&= (u\otimes v\otimes 1)(\Delta\otimes 1)\theta_M \\
&= (u\otimes v\otimes 1)(1\otimes\theta_M)\theta_M \\
&= (u\otimes 1\otimes 1)(1\otimes(v\otimes 1)\theta_M)\theta_M \\
&= (u\otimes 1)(1\otimes g)\theta_M \\
&= g(u\otimes 1)\theta_M = gf.
\end{aligned}
$$

Hence $\Phi^{-1}(uv) = \Phi^{-1}(v)\Phi^{-1}(u)$. Clearly $(\varepsilon\otimes 1)\theta_M = 1_M$ since M is a left $e_{-C}(M)$-comodule via θ_M.

For the second part, we recall that M is a $(C^*, e_{-C}(M)^*)$-bimodule via the maps

$$
c^*m = \sum_{(m)}\langle c^*, m_{(1)}\rangle m_{(0)}, \qquad md^* = \sum_{(m)}\langle d^*, m_{(-1)}\rangle m_{(0)},
$$

where $\theta_M(m) = \sum_{(m)} m_{(-1)}\otimes m_{(0)}$. On the other hand, if we consider M as a left C^*-module, then M is a $(C^*, End_{C^*}(M)^{op})$-bimodule. Both structure are compatible with Φ^{-1}. Let $d^*\in e_{-C}(M)^*$,

$$
md^* = \sum_{(m)}\langle d^*, m_{(-1)}\rangle m_{(0)} = \Phi^{-1}(d^*)(m) = m\Phi^{-1}(d^*). \quad \blacksquare
$$

Lemma 1.11 entails that the decomposition theory of a quasi-finite right C–comodule M is encoded by the idempotents of its coendomorphism coalgebra $e_{-C}(M)$. We summarize the most relevant facts in the following two propositions.

Proposition 1.12 *Let $M\in\mathcal{M}^C$ be quasi-finite and $\{M_i\}_{i\in I}$ a family of subcomodules of M. Then, $M = \oplus_{i\in I}M_i$ if and only if there is a family of orthogonal idempotents $\{e_i\}_{i\in I}$ for $e_{-C}(M)$ such that $M_i = Me_i$.*

Proof: $M = \oplus_{i\in I}M_i$ as right C-comodules if and only if $M = \oplus_{i\in I}M_i$ as left C^*-modules. By [15, Proposition 6.18], there exists a family of orthogonal idempotents $\{e_i\}_{i\in I}$ in $End_{C^*}(M)^{op}$ such that $M_i = Me_i$. Now it is enough to take into account Lemma 1.11. $\blacksquare$

Proposition 1.13 *Let $M\in\mathcal{M}^C$ be quasi-finite.*

a) The following assertions are equivalent:

i) M is indecomposable.

ii) 0 *and* ε *are the only idempotents for* $e_{-C}(M)$.
iii) ε *is a primitive idempotent for* $e_{-C}(M)$.

b) Let d *be an idempotent for* $e_{-C}(M)$. *Then* Md *is indecomposable if and only if* d *is a primitive idempotent for* $e_{-C}(M)$.

c) M *is indecomposable injective if and only if there is a primitive idempotent* e *for* C *such that* $M \cong Ce$ *as right* C*-comodules.*

Proof: Note that M is indecomposable as right C-comodule if and only if it is indecomposable as left C^*-module. Taking in mind Lemma 1.11, *a),b)* follow from [15, Proposition 5.10] and [15, Corollary 5.11] respectively.

c) If M is indecomposable injective, then M is isomorphic to a direct summand of C. By Proposition 1.12, there is an idempotent e for $e_{-C}(C) \cong C$ such that $M \cong Ce$. Since M is indecomposable, e is primitive. The converse follows from b). ∎

Remark 1.14 If the family $\{M_i\}_{i\in I}$ appearing in Proposition 1.12 consists of indecomposable comodules, then the idempotents $\{e_i\}_{i\in I}$ are primitive.

Example 1.15 (see [16, Ex. 1.6]) Let C be the Sweedler coalgebra, that is, the vector space generated by the set $\{g_n, s_n : n \in I\!N\}$ with comultiplication and counit given by:

$$\Delta(g_n) = g_n \otimes g_n, \quad \varepsilon(g_n) = 1, \quad \Delta(s_n) = g_n \otimes s_n + s_n \otimes g_{n+1}, \quad \varepsilon(s_n) = 0.$$

for all $n \in I\!N$. Consider the elements in C^* defined as:

$$g_n^*(g_i) = \delta_{n,i}, \quad g_n^*(s_i) = 0, \quad s_n^*(g_i) = 0, \quad s_n^*(s_i) = \delta_{n,i},$$

where $\delta_{i,j}$ is the Kronecker delta. The family $\{g_n^* : n \in I\!N\}$ is a family of orthogonal idempotents of C^* such that $\varepsilon = \sum_{n\in I\!N} g_n^*$. The right C-comodule $Cg_n^* = k\langle g_n, s_n\rangle$ is indecomposable. Thus the family $\{g_n^*\}$ consits of primitive idempotents. The coradical $C_0 = \oplus_{n\in I\!N} kg_n$, and the simple comodules of C are of the form $kg_n = C_0\bar{g}_n^*$ for $n \in I\!N$. Hence the family $\{Cg_n^* : n \in I\!N\}$ is a full set of indecomposable injective right C-comodules.

As a consequence of the locally finite property in coalgebras and Zorn's Lemma, it is shown in [8, Section 2.2] that idempotents may be lifted in the dual algebra of a coalgebra. Let C be a coalgebra and e an idempotent in C^*.

We denote by $\bar{e}$ the restriction of e to C_0. It is known that $J = C_0^{\perp}$ is the Jacobson radical of C^*. Let $i : C_0 \to C$ denotes the inclusion map, and identify C_0^* with C^*/J, then $i^* : C^* \to C_0^* \cong C^*/J$ is the canonical projection. Then $\bar{e} = i^*(e)$. Now, [8, Proposition 2.2.2] states that if $\{u_i\}_{i\in I}$ is a family of orthogonal idempotents in C_0^*, then there exists a family of orthogonal idempotents $\{e_i\}_{i\in I}$ in C^* such that $\bar{e}_i = u_i$ for all $i \in I$. If, in addition, $\bar{\varepsilon} = \sum_{i\in I} u_i$, then $\varepsilon = \sum_{i\in I} e_i$ (the sum $e = \sum_{i\in I} e_i$ makes sense since for a given $c \in C$, only finitely many of the $e_i(c)'s$ are not zero).

Lemma 1.16 *Let C be a cosemisimple coalgebra, and e an idempotent for C. Then e is primitive if and only if $(eCe)^*$ is a division algebra.*

Proof: If e is primitive, then Ce is a simple C-comodule. Hence $(eCe)^* \cong eC^*e \cong e_{-C}(Ce)^* \cong End_{C^*}(Ce)$ is a division algebra, by Schur's Lemma. Conversely, suppose that $(eCe)^*$ is a division algebra, then e is the unique idempotent for $eCe \cong e_{-C}(Ce)$. Thus e is a primitive idempotent for $e_{-C}(Ce)$. By Proposition 1.13 a), Ce is indecomposable, and by b), e is primitive. ∎

Proposition 1.17 *Let e be an idempotent for C. Then e is primitive if and only if $\bar{e} = e\mid_{C_0}$ is primitive.*

Proof: Suppose that e is primitive, then by Proposition 1.13 b), Ce is indecomposable. It is well-known that the endomorphism ring of an indecomposable comodule is local. Hence $eC^*e \cong e_{-C}(Ce)^* \cong End_{C^*}(Ce)^{op}$ is local. Then $eC^*e/Rad(eC^*e) = eC^*e/eC_0^{\perp}e \cong \bar{e}C_0^*\bar{e} \cong (\bar{e}C_0\bar{e})^*$ is a division algebra. By Lemma 1.16, $\bar{e}$ is primitive.

Conversely, assume that $\bar{e}$ is primitive, then $C_0\bar{e}$ is simple. Hence $Ce \cong E(C_0\bar{e})$ is indecomposable injective. By Proposition 1.13 c), e is primitive. ∎

Remark 1.18 The foregoing proposition in conjunction with the property of lifting of idempotents shows that primitive idempotents may be lifted.

Definition 1.19 *A coalgebra is said to be colocal if its coradical is the dual of a division algebra. Equivalently, its dual algebra is a local algebra.*

Reasoning as in the proof of Proposition 1.17, we arrive to:

Proposition 1.20 *Let e be an idempotent for a coalgebra C. Then Ce is an indecomposable right C-comodule if and only if eCe is colocal.*

Corollary 1.21 *A coalgebra C is indecomposable as a C-comodule if and only if C is colocal.*

Proposition 1.22 *i) If e is a central idempotent for a coalgebra C, then eCe is a subcoalgebra of C.*

ii) Let $\{e_i\}_{i\in I}$ be a family of orthogonal idempotents for C such that $C = \oplus_{i\in I} Ce_i$. Then $\varepsilon = \sum_{i\in I} e_i$.

iii) Let $\{C_i\}$ be a family of subcoalgebras of C such that $C = \oplus_{i\in I} C_i$. Then there is a family of orthogonal central idempotents $\{e_i\}_{i\in I}$ for C such that $\varepsilon = \sum_{i\in I} e_i$ and $C_i = e_iCe_i$.

Proof: *i)* Suppose that e is central, and let $c^* \in C^*$. Then, $\langle c^*, ece\rangle = \langle ec^*e, c\rangle = \langle ec^*, c\rangle = \langle c^*e, c\rangle$. The set $(eCe)^{\perp(C^*)} = \{c^* \in C^* : \langle c^*, ece\rangle = 0, \ \forall c \in C\} = \{c^* \in C^* : ec^*e = ec^* = c^*e = 0\}$ is a two-sided ideal of C^*. Hence eCe is a subcoalgebra of C.

ii) Given $c \in C$, $c = \sum_{j=1}^n c_j e_{i_j} = \sum_{j=1}^n \sum_{(c_j)} \langle e_{i_j}, c_{j(1)}\rangle c_{j(2)}$ for some $i_1, ..., i_n \in I$ and $c_j \in C_{i_j}$ for all $j = 1, .., n$. If $l \neq i_1, ..., i_n$, then $e_l(c) = \sum_{j=1}^n \sum_{(c_j)} \langle e_{i_j}, c_{j(1)}\rangle \langle e_l, c_{j(2)}\rangle = 0$. Now, $\varepsilon(c) = \sum_{j=1}^n \langle e_{i_j}, c_j\rangle = \langle \sum_{j=1}^n e_{i_j}, c\rangle = \langle \sum_{i\in I} e_i, c\rangle$.

iii) For every $i \in I$, let ε_i be the counit of each C_i. The family $\{\varepsilon_i\}_{i\in I}$ verifies the required condition. ∎

Remark 1.23 As a consequence of the proposition 1.22, we give an alternative proof of the following fact: If C is a cocommutative coalgebra, then C is indecomposable (as coalgebra) implies that C is irreducible. A proof appears in [17, Remark 2.3].

Assume that C is indecomposable. In view of Proposition 1.22 *iii)*, ε is the unique central idempotent in C^*. Since C is cocommutative, the lifting of idempotents property forces $\varepsilon \mid_{C_0}$ is the unique idempotent in C_0^*. Then C_0 is simple and thus C is irreducible.

Remark 1.24 The theory of idempotents just expounded allows to obtain easily Green's block decomposition for coalgebras [12].

2 Strong equivalences

A right comodule M over a coalgebra C is said to be an *ingenerator* if it is a finitely cogenerated injective cogenerator. The coalgebra C is said [9] to be *strongly equivalent* to D if $\mathcal{M}^C$ is equivalent to $\mathcal{M}^D$ via inverse equivalences $f : \mathcal{M}^C \to \mathcal{M}^D$ and $g : \mathcal{M}^D \to \mathcal{M}^C$ such that $f(C)$ is an ingenerator in $\mathcal{M}^D$ and $g(D)$ is an ingenerator in $\mathcal{M}^C$. In [9] it was proved that $\mathcal{M}^C$ is strongly equivalent to $\mathcal{M}^D$ if and only if ${}_{C^*}\mathcal{M}$ is Morita equivalent to ${}_{D^*}\mathcal{M}$ via functors $F : {}_{C^*}\mathcal{M} \to {}_{D^*}\mathcal{M}$ and $G : {}_{D^*}\mathcal{M} \to {}_{C^*}\mathcal{M}$ satisfying $F(\mathcal{M}^C) \subseteq \mathcal{M}^D$ and $G(\mathcal{M}^D) \subseteq \mathcal{M}^C$.

An example of this kind of equivalences is that between a coalgebra C and any matrix coalgebra of order n over C, $M^c(C, n)$. We recall the definition of the matrix coalgebra. Let $M^c(k, n)$ be the vector space generated by the set of symbols $\{x_{ij} : 1 \leq i, j \leq n\}$. The maps

$$\Delta(x_{ij}) = \sum_{l=1}^{n} x_{il} \otimes x_{lj}, \qquad \varepsilon(x_{ij}) = \delta_{ij},$$

endow to $M^c(k, n)$ with a coalgebra structure. The dual of this coalgebra is the matrix algebra of order n. Let s_{ij} be the matrix with 1 in the position (i, j) and zero elsewhere. Then $s_{ij}(x_{lm}) = \delta_{il}\delta_{jm}$.

For a coalgebra C, the matrix coalgebra of order n over C, denoted by $M^c(C, n)$ is defined as $C \otimes M^c(k, n)$. The dual of $M^c(C, n)$ is the matrix algebra $M_n(C^*)$. For a vector space W of dimension n, it was proved in [3] that, $e_{-C}(C \otimes W) \cong M^c(C, n)$.

Proposition 2.1 *Let $M \in \mathcal{M}^C$ be quasi-finite, and $D = e_{-C}(M)$.*

i) If d is an idempotent for D, then $e_{-C}(Md) \cong dDd$.

ii) If M is finitely cogenerated and injective, then there is a positive integer n and an idempotent $e \in M_n(C^*)$ such that $D \cong eM^c(C, n)e$.

Proof: *i)* In order to prove i), we first recall the following result which may be found in [15, Proposition 5.9]. Let R be a ring, M a left R-module and e an idempotent in $S = End_R(M)$. The map $\Psi : eSe \to End_R(Me), ese \mapsto \psi(ese) : xe \mapsto xese$ is a ring isomorphism.

We know that Md is a (dDd, C)-bicomodule where the left structure is given by the map $\rho^- : Md \to dDd \otimes Md, md \mapsto \sum_{(m)} dm_{(-1)}d \otimes m_{(0)}d$. By the universal property of $e_{-C}(Md)$ (see [1, 1.18]), there is a coalgebra map $\lambda : e_{-C}(Md) \to dDd$ such that $(\lambda \otimes 1)\rho^- = \theta_{Md}$. We identify $(dDd)^* \cong dD^*d$ and $e_{-C}(Md)^* \cong End_{C^*}(Md)$ via Φ^{-1} as in Lemma 1.11. Let $\varphi \in D^*$, then,

$$\begin{aligned}\langle \Phi^{-1}(\lambda^*(d\varphi d)), md\rangle &= \langle (\lambda^*(d\varphi d) \otimes 1)\theta_{Md}, md\rangle \\ &= \langle (((d\varphi d) \otimes 1)(\lambda \otimes 1)\theta_{Md}, md\rangle \\ &= \langle (((d\varphi d) \otimes 1)\rho^-, md\rangle \\ &= (md)(d\varphi d).\end{aligned}$$

Hence λ^* is an isomorphism, and thus, λ is an isomorphism.

ii) Since M is finitely cogenerated and injective, there is a vector space W of dimension n such that M is a direct summand of $C \otimes W$. There exists an idempotent $e \in M_n(C^*)$ such that $M \cong (C \otimes W)e$. From the isomorphism that $e_{-C}(C \otimes W) \cong M^c(C, n)$ and *i)*, it follows that $e_{-C}(M) \cong eM^c(C, n)e$. ∎

Example 2.2 Write $e_{ij} = \varepsilon \otimes s_{i,j}$ in $M_n(C^*)$. The $(C, M^c(C, n))$-bicomodule $M^c(C, n)e_{11}$ may be identified with $C \otimes W$, where $W = k\{x_{11}, ..., x_{1n}\}$. Similarly, $e_{11}M^c(C, n)$ is identified with $C \otimes V$ for $V = k\{x_{11}, ..., x_{n1}\}$. Let e be an idempotent for $M^c(C, n)$, and consider $M = eM^c(C, n)e_{11}$ and $N = e_{11}M^c(C, n)e$ which are $(C, eM^c(C, n)e)$ and $(eM^c(C, n)e, C)$-bicomodules respectively via the maps:

$$\begin{aligned}&\rho^+_M(e(c \otimes x_{1i})) = \textstyle\sum_{(c)} c_{(1)} \otimes e(c_{(2)} \otimes x_{1i})\\ &\rho^-_M(e(c \otimes x_{1i})) = \textstyle\sum_{(c)} \sum_{l=1}^n e(c_{(1)} \otimes x_{1k}) \otimes e(c_{(2)} \otimes x_{li})e\\ &\rho^+_N((c \otimes x_{i1})e) = \textstyle\sum_{(c)} (c_{(1)} \otimes x_{i1})e \otimes c_{(2)}\\ &\rho^-_N((c \otimes x_{i1})e) = \textstyle\sum_{(c)} \sum_{l=1}^n e(c_{(1)} \otimes x_{il})e \otimes (c_{(2)} \otimes x_{l1})e\end{aligned}$$

We define

$$\begin{aligned}&f : C \to M\Box_{eM^c(C,n)e}N, \quad c \mapsto \textstyle\sum_{(c)} \sum_{i=1}^n e(c_{(1)} \otimes x_{1i}) \otimes (c_{(2)} \otimes x_{i1})e\\ &g : eM^c(C, n)e \to N\Box_C M, \quad e(c \otimes x_{ij})e \mapsto \textstyle\sum_{(c)} (c_{(1)} \otimes x_{i1})e \otimes e(c_{(2)} \otimes x_{1j})\end{aligned}$$

One may check that $(C, eM^c(C, n)e, M, N, f, g)$ is a Morita-Takeuchi context.

Proposition 2.3 *The foregoing Morita-Takeuchi context is strict if and only if $M_n(C^*)eM_n(C^*) = M_n(C^*)$.*

Proof: $\Leftarrow$) Let $c \in C$ such that

$$f(c) = \sum_{i=1}^{n} \sum_{(c)} e(c_{(1)} \otimes x_{1i}) \otimes (c_{(2)} \otimes x_{i1})e = 0.$$

Let $\alpha \in M_n(C^*)$ be arbitrary. By hypothesis, there are $\phi, \psi \in M_n(C^*)$ such that $\alpha = \phi e \psi$. Applying $\phi \otimes \psi$ to the foregoing equality yields:

$$\begin{aligned} 0 &= \textstyle\sum_{i=1}^{n} \sum_{(c)} \langle \phi, e(c_{(1)} \otimes x_{1i}) \rangle \langle \psi, (c_{(2)} \otimes x_{i1})e \rangle \\ &= \textstyle\sum_{i=1}^{n} \langle \phi e, c_{(1)} \otimes x_{1i} \rangle \langle e\psi, c_{(2)} \otimes x_{i1} \rangle \\ &= \langle \phi e \psi, c \otimes x_{11} \rangle \\ &= \langle \alpha, c \otimes x_{11} \rangle. \end{aligned}$$

Then $c \otimes x_{11} = 0$, and thus, $c = 0$. Hence f is injective.

We prove now that g is injective. In fact, we are going to prove that g is always injective, without further hypothesis. Let $x = e(\sum_{i,j} c_{ij} \otimes x_{ij})e \in eM^c(C,n)e$ verifying

$$g(x) = \sum (c_{ij(1)} \otimes x_{i1})e \otimes e(c_{ij(2)} \otimes x_{1j}) = 0.$$

Let $c^* \in C^*$ and $u, v \in \{1, ..., n\}$ be arbitrary. Then,

$$\begin{aligned} \langle e(c^* \otimes s_{uv})e, x \rangle &= \langle e(c^* \otimes s_{uv})e, \textstyle\sum_{i,j} c_{ij} \otimes x_{ij} \rangle \\ &= \langle e(c^* \otimes s_{u1})e_{1v}e, \textstyle\sum_{i,j} c_{ij} \otimes x_{ij} \rangle \\ &= \langle e(c^* \otimes s_{u1})e_{11}e_{11}e_{1v}e, \textstyle\sum_{i,j} c_{ij} \otimes x_{ij} \rangle \\ &= \textstyle\sum \langle e(c^* \otimes s_{u1})e_{11}, c_{ij(1)} \otimes x_{il} \rangle \langle e_{11}e_{1v}e, c_{ij(2)} \otimes x_{lj} \rangle \\ &= \textstyle\sum \langle c^* \otimes s_{u1}, e_{11}(c_{ij(1)} \otimes x_{il})e \rangle \langle e_{1v}, e(c_{ij(2)} \otimes x_{lj})e_{11} \rangle \\ &= \textstyle\sum \langle c^* \otimes s_{u1}, (c_{ij(1)} \otimes x_{i1})e \rangle \langle e_{1v}, e(c_{ij(2)} \otimes x_{1j}) \rangle \\ &= 0. \end{aligned}$$

Hence $x = 0$, and thus g is injective.

$\Rightarrow$) Suppose that the Morita-Takeuchi context is strict. By [1, Theorem 2.5], M is an injective cogenerator for $\mathcal{M}^C$. Since M is finitely cogenerated, we get that M is an ingenerator for $\mathcal{M}^C$. Similarly, N is an ingenerator as $eM^c(C,n)e$-comodule. Hence C is strongly equivalent to $eM^c(C,n)e$. By [9, Theorem 5], C^* is Morita equivalent to $eM_n(C^*)e$ via the functors

$$M^* \otimes_{C^*} - : {}_{C^*}\mathcal{M} \to {}_{eM_n(C^*)e}\mathcal{M}, \qquad N^* \otimes_{eM_n(C^*)e} - : {}_{eM_n(C^*)e}\mathcal{M} \to {}_{C^*}\mathcal{M}.$$

We may identify M^* (resp. N^*) with $eM_n(C^*)e_{11}$ (resp. $e_{11}M_n(C^*)e$), and C^* with $e_{11}M_n(C^*)e_{11}$. Since both are generators, by [18, 5.3, page 81],

$$e_{11}M_n(C^*)e_{11} = C^* = (e_{11}M_n(C^*)e)(eM_n(C^*)e_{11}).$$

$M_n(C^*)eM_n(C^*)$ is a two-sided ideal of $M_n(C^*)$ containing to e_{11}. Then $M_n(C^*)e$ $M_n(C^*) = M_n(C^*)$. ∎

Theorem 2.4 *Two coalgebras C and D are strongly equivalent if and only if there is $n \in \mathbb{N}$ and an idempotent $e \in M_n(C^*)$ such that $D \cong eM^c(C,n)e$ and $M_n(C^*)eM_n(C^*) = M_n(C^*)$.*

Proof: $\Leftarrow$) By Proposition 2.3, the context of Example 2.2 is strict and M, N are ingenerators. Then C is strongly equivalent to D.

$\Rightarrow$) Let ${}_DM_C$ and ${}_CN_D$ be the comodules giving the equivalence. By hyphotesis, M_C, N_D are finitely cogenerated injective. [1, Theorem 3.5] combined with Proposition 2.1 yields $D \cong e_{-C}(M) \cong eM^c(C,n)e$ for some $n \in \mathbb{N}$ and an idempotent $e \in M_n(C^*)$. We may identify M (resp. N) with $e_{11}M^c(C,n)e$ (resp. $eM^c(C,n)e_{11}$). An argument as in the last part of the proof of Proposition 2.3 gives that $M_n(C^*)eM_n(C^*) = M_n(C^*)$. ∎

Remark 2.5 If C and D have finite dimensional coradical, then any equivalence is a strong equivalence, see [9, page 322]. Therefore, Theorem 2.4 applies.

3 Basic coalgebras

The basic coalgebra of a coalgebra C is defined in [6] as the coendomorphisms coalgebra of a minimal injective cogenerator for $\mathcal{M}^C$. In this section, we offer an easier description in terms of idempotents.

Definition 3.1 *i) A family $\{M_i\}_{i\in I}$ in $\mathcal{M}^C$ is called a basic set if it is a full set of indecomposable injective objects in $\mathcal{M}^C$.*

ii) A family $\{e_i\}_{i\in I}$ consisting of orthogonal primitive idempotents of C^* is said to be basic if the set $\{Ce_i\}_{i\in I}$ is a basic set in $\mathcal{M}^C$.

iii) An idempotent e for C is said to be basic if there exists a basic set of idempotents $\{e_i\}_{i\in I}$ such that $e = \sum_{i\in I} e_i$.

Proposition 3.2 *Every coalgebra has a basic idempotent.*

Proof: Let $\{I_\alpha\}_{\alpha\in\Gamma}$ be a basic set in $\mathcal{M}^C$. Since $\oplus_{\alpha\in\Gamma}I_\alpha$ is a direct summand of C, by Proposition 1.12 and Remark 1.14, there exists a family $\{e_i\}_{i\in I}$ of primitive orthogonal idempotents such that $I_\alpha \cong Ce_\alpha$ for all $\alpha \in \Gamma$. The element $e = \sum_{\alpha\in\Gamma} e_\alpha$ is a basic idempotent. ∎

Proposition 3.3 *For a coalgebra C the following assertions are equivalent:*

i) $\{e_i\}_{i\in I}$ *is a basic set of idempotents for C.*

ii) $\{\bar{e}_i\}_{i\in I}$ *is a basic set of idempotents for C_0.*

Proof: *i)* $\Rightarrow$ *ii)* By hypothesis, $\{Ce_i\}_{i\in I}$ is a basic set of right C-comodules. But an indecomposable injective comodule is the injective hull of a simple comodule. Hence the family $\{soc(Ce_i)\}_{i\in I}$ is a full set of simple comodules. By Lemma 1.9 ii), $soc(Ce_i) \cong C_0\bar{e}_i$ for any $i \in I$. By Proposition 1.17, every $\bar{e}_i$ is primitive. Hence $\{\bar{e}_i\}_{i\in I}$ is a basic set of idempotents for C_0.

ii) $\Rightarrow$ *i)* If $\{C_0\bar{e}_i\}$ is a full set of simple comodules in $\mathcal{M}^C$, then $\{E(C_0\bar{e}_i)\}$ is a basic set in $\mathcal{M}^C$. But $E(C_0\bar{e}_i) \cong Ce_i$ by Lemma 1.9 i). Every e_i is primitive by Proposition 1.17. ∎

Definition 3.4 *A coalgebra D is said to be a basic coalgebra for C if there exists a basic idempotent e for C such that $D \cong eCe$ as coalgebras.*

Proposition 3.5 *Any coalgebra has a basic coalgebra which is unique up to isomorphisms.*

Proof: The existence is deduced from Proposition 3.2. Suppose now that C is a coalgebra and there are basic idempotents e, e'. Let $e = \sum_{i\in I} e_i, e' = \sum_{i\in I} e'_i$ where $\{e_i\}_{i\in I}, \{e'_i\}_{i\in I}$ are basic sets of idempotents. Then, $Ce = \oplus_{i\in I}Ce_i$ $\cong \oplus_{i\in I}Ce'_i = Ce'$. Using Corollary 1.4, $eCe \cong e_{-C}(Ce) \cong e_{-C}(Ce') \cong e'Ce'$. ∎

Proposition 3.6 *Any coalgebra is Morita-Takeuchi equivalent to its basic coalgebra. Moreover, two coalgebras are Morita-Takeuchi equivalent if and only if their basic coalgebras are isomorphic.*

Proof: Let e be a basic idempotent for C and $\{e_i\}_{i\in I}$ be a basic set of idempotents such that $e=\sum_{i\in I} e_i$. Since $Ce=\oplus_{i\in I}Ce_i$ and $\{Ce_i\}_{i\in I}$ is basic set in $\mathcal{M}^C$, Ce is a cogenerator. By Corollary 1.8, C is Morita-Takeuchi equivalent to eCe.

We prove the second claim. Let $F:\mathcal{M}^C\to\mathcal{M}^D$ be an equivalence, and $M\in\mathcal{M}^C$ be quasi-finite. The functors $h_{-C}(M,-):\mathcal{M}^C\to\mathcal{M}_k$ and $h_{-D}(F(M),F(-)):\mathcal{M}^C\to\mathcal{M}_k$ are naturally equivalent. From this, it is deduced that $e_{-C}(M)\cong e_{-D}(F(M))$.

The family $\{F(Ce_i)\}_{i\in I}$ is basic set in $\mathcal{M}^D$. Let $\{d_i\}_{i\in I}$ be a basic set of idempotents such that $F(Ce_i)\cong Dd_i$ for all $i\in I$, and consider the basic idempotent $d=\sum_{i\in I} d_i$. Then, $eCe\cong e_{-C}(Ce)\cong e_{-D}(F(Ce))\cong e_{-D}(Dd)\cong dDd$. The other part is clear. ∎

Corollary 3.7 *If C is Morita-Takeuchi equivalent to D, then C_0 is Morita-Takeuchi equivalent to D_0.*

Proof: Let e,d be basic idempotents for C and D respectively and consider its basic coalgebras eCe, dDd. From Proposition 3.3, $\bar{d},\bar{e}$ are basic idempotents for D_0 and C_0 respectively. Then $\bar{d}D\bar{d},\bar{e}C\bar{e}$ are basic coalgebras for D_0 and C_0 respectively. If C is Morita-Takeuchi equivalent to D, by Proposition 3.6 $dDd\cong eCe$. Then $\bar{d}D\bar{d}=(dDd)_0\cong(eCe)_0=\bar{e}C\bar{e}$. Proposition 3.6 yields that C_0 is Morita-Takeuchi equivalent to D_0. ∎

Definition 3.8 *A coalgebra is said to be basic if the counit is a basic idempotent for it.*

Proposition 3.9 *The basic coalgebra of any coalgebra is basic.*

Proof: Let $e=\sum_{i\in I} e_i$ be a basic idempotent for C where $\{e_i\}_{i\in I}$ is a basic set of idempotents. Then the functor $h_{-C}(Ce,-):\mathcal{M}^C\to\mathcal{M}^{eCe}$ is an equivalence. Since the family $\{Ce_i\}$ is a basic set in $\mathcal{M}^C$, the family $\{h_{-C}(Ce,Ce_i)\}_{i\in I}$ is a basic set in $\mathcal{M}^{eCe}$. But, by Corollary 1.6 ii), $h_{-C}(Ce,Ce_i)\cong eCe_i=(eCe)e_i$. Note that e_i is an idempotent in eCe since $ee_ie=e_i$. For $c\in C$,

$$\begin{aligned}\langle e_i^2, ece\rangle &= \textstyle\sum_{(c)}\langle e_i, ec_{(1)}e\rangle\langle e_i, ec_{(2)}e\rangle = \sum_{(c)}\langle ee_ie, c_{(1)}\rangle\langle ee_ie, c_{(2)}\rangle \\ &= \textstyle\sum_{(c)}\langle e_i, c_{(1)}\rangle\langle e_i, c_{(2)}\rangle = \langle e_i, c\rangle = \langle ee_ie, c\rangle = \langle e_i, ece\rangle. \quad ∎\end{aligned}$$

We recall from [6] that a coalgebra is said to be basic if the dual of any simple subcoalgebra is a division algebra. We see that our definition of basic coalgebra agrees with that.

Theorem 3.10 *A coalgebra C is basic if and only if the dual of any simple subcoalgebra of C is a division algebra.*

Proof: Suppose that $\varepsilon = \sum_{i\in I} e_i$ where $\{e_i\}_{i\in I}$ is a basic set of idempotents for C. Then $C = \oplus_{i\in I} Ce_i$. From Proposition 3.3, $\{\bar{e}_i\}_{i\in I}$ is a basic set of idempotents for C_0. Since $\bar{\varepsilon} = \sum_{i\in I} \bar{e}_i$, $C_0 = \oplus_{i\in I} C_0\bar{e}_i$. Each $C_0\bar{e}_i$ is a simple comodule, and its isotypic component consists only of itself because $\{\bar{e}_i\}_{i\in I}$ is a basic set. Hence $C_0\bar{e}_i$ is a simple subcoalgebra for every $i \in I$. Any simple subcoalgebra is also of this form because $C_0 = \oplus_{i\in I} C_0\bar{e}_i$. Then any simple subcoalgebra is simple (indecomposable) as comodule. Corollary 1.21 yields that it is isomorphic to a division algebra.

Conversely, suppose now that the dual of any simple subcoalgebra is a division algebra. Then every simple subcoalgebra is simple as comodule by Corollary 1.21, and they represent all the simples of $\mathcal{M}^C$. We may find a basic set $\{e'_i\}_{i\in I}$ of idempotents for C_0 such that $C_0 = \oplus_{i\in I} C_0 e'_i$. By Lemma 1.22 i), $\bar{\varepsilon} = \sum_{i\in I} e'_i$. From Proposition 3.3, we may find a basic set $\{e_i\}_{i\in I}$ of idempotents for C such that $\varepsilon = \sum_{i\in I} e_i$ and $\bar{e}_i = e'_i$ for all $i \in I$. Therefore ε is a basic idempotent. ∎

It is well-known that any coalgebra decomposes as direct sum of indecomposable subcoalgebras. In the following result we see that the associated basic coalgebra may be recovered from the basic coalgebra of each indecomposable one. Let $C = \oplus_{i\in I} C_i$ and $D = \oplus_{i\in I} D_i$ be coalgebra decompositions of two coalgebras C and D. If C_i is Morita-Takeuchi equivalent to D_i for every $i \in I$, then C is Morita-Takeuchi equivalent to D (see [20]).

Remark 3.11 The preceding result may be understaken in a categorical frame. An equivalence between the categories of right comodules over C and D induces an equivalence between the full subcategories of semisimple objects of these categories. But these categories may be identified to the categories of right comodules over the coradicals C_0 and D_0, see [19, Proposition 2.5.3, Example 3.1.2].

Corollary 3.12 *Let C be a coalgebra and B its basic coalgebra. Let $\{C_i\}_{i\in I}$ be a family of indecomposable subcoalgebras of C such that $C = \oplus_{i\in I}C_i$. For every $i \in I$, let B_i be the basic coalgebra of C_i. Then $B \cong \oplus_{i\in I}B_i$.*

Proof: Let $D = \oplus_{i\in I}B_i$. It is deduced that D is basic from Theorem 3.10. By Proposition 3.6, each C_i is Morita-Takeuchi equivalent to B_i. Therefore C is Morita-Takeuchi equivalent to D. Proposition 3.6 yields that $B \cong D$. ∎

Corollary 3.13 *Let C be a coalgebra and B its basic coalgebra. Suppose that $C_0 \cong \oplus_{i\in I}M^c(D_i, n_i)$ where the $D_i's$ are division algebras and the $n_i's \in I\!N$. Then $B_0 \cong \oplus_{i\in I}D_i$.*

Proof: As in the proof of Corollary 3.7, B_0 is the basic coalgebra for C_0. In light of the foregoing corollary, $D = \oplus_{i\in I}D_i$ is a basic colgebra for C_0. Then $B_0 \cong \oplus_{i\in I}D_i$. ∎

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