Flat comodules and perfect coalgebras *

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Abstract. Stenström introduced the notion of flat object in a locally finitely presented Grothendieck category $\mathcal{A}$. In this paper we investigate this notion in the particular case of the category $\mathcal{A} = C$-Comod of left C-comodules, where C is a coalgebra over a field K. Several characterizations of flat left C-comodules are given and coalgebras having enough flat left C-comodules are studied. It is shown how far these coalgebras are from being left semiperfect. As a consequence, we give new characterizations of a left semiperfect coalgebra in terms of flat comodules. Left perfect coalgebras are introduced and characterized in analogy with Bass's Theorem P. Coalgebras whose injective left C-comodules are flat are discussed and related to quasi-coFrobenius coalgebras.

1. Introduction

For a coalgebra C over a field K the category C-Comod of left C-comodules is an example of locally finitely presented Grothendieck category. For this type of categories a notion of flat object, due to Stenström [40], exists. An object F in such a category is flat if every epimorphism $M \to F$ is pure. Using this notion, a left C-comodule is called flat if it is a flat object in C-Comod. Since in C-Comod the classes of finitely presented objects and finite dimensional objects coincide, the definition of a flat comodule reads as follows: a left C-comodule F is flat if for every epimorphism $f : M \to F$ in C-Comod and every finite dimensional left C-comodule N the map $\mathrm{Hom}_C(N, f) : \mathrm{Hom}_C(N, M) \to \mathrm{Hom}_C(N, F)$ is surjective.

The aim of this paper is to investigate this notion of flat comodule. The special features of the category of comodules (finiteness conditions, duality, etc) endow the class of flat comodules with certain distinguished characteristics, not present in other kind of categories. This makes the class of flat comodules deserving attention. To give examples of such characteristics, in Theorem 3.8 is shown that any flat left C-comodule may be written as a union of flat subcomodules of countable dimension, whenever C-Comod has enough projective objects, i.e. C is left semiperfect. Such subcomodules may be indeed taken to be projective of finite dimension if, in addition, C is assumed to be hereditary. In our study several links of flat comodules with

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other notions (e.g. fin-projectivity, FP-injectivity) are revealed and some interesting problems come to the surface.

The flatness of a comodule can be viewed as a kind of "relative projectivity" (see [31]) since a left C-comodule F is flat if and only if every diagram in the category C-Comod

$$\begin{array}{ccccc} F_0 & \hookrightarrow & F & & \\ & & \downarrow{\scriptstyle f} & & \\ M & \xrightarrow{g} & N & \longrightarrow & 0 \end{array}$$

with a finite dimensional subcomodule F_0 of F and an epimorphism g, can be completed to a commutative diagram by a morphism $f_0 : F_0 \to M$. The notion of flatness is strongly related to the notion of projectivity. However C-Comod does not have in general enough projective objects, so a description of a flat comodule as a limit of projective comodules is not always possible although it is when C is left semiperfect. Under this latter assumption, we prove in Theorem 3.5 that a left C-comodule F is flat if and only if the dual C^*-module F^* is injective, or equivalently, F viewed as a rational right C^*-module is flat.

We focus our attention on those coalgebras having enough flat left comodules, i.e., those ones such that any left comodule may be expressed as a quotient of a flat comodule. We ask whether such coalgebras are left semiperfect. In an attempt to answer this question, which remains open, we find an interesting new class of comodules. A left C-comodule is *fin*-projective if it is projective with respect to sequences of finite dimensional left C-comodules. These comodules are characterized in two different ways in Proposition 4.5. We show there that a left C-comodule F is *fin*-projective if and only if the maximal rational submodule of F^* is injective in Comod-C. It is also proved that F is *fin*-projective if and only if the functor $h^F = \mathrm{Hom}_C(F, -) : C\text{-comod} \to \mathrm{Mod}\, K$ is an FP-injective object in the functor category $\mathcal{D}_l(C) = \mathrm{Add}(C\text{-comod}, \mathrm{Mod}\, K)$ of all covariant additive functors from the category C-comod of finite dimensional left C-comodules to the category of K-vector spaces. Precisely, the relation between the classes of flat and *fin*-projective left C-comodules show how far are these coalgebras of being semiperfect. We prove in Theorem 4.6 that a coalgebra C is left semiperfect if and only if C-Comod has enough flat objects and every flat left C-comodule is *fin*-projective. We also prove that a coalgebra C is left semiperfect if and only if C-Comod has enough flat objects and every flat left C-comodule is flat when viewed as a rational right C^*-module.

The existence of flat and projective covers for left C-comodules is also discussed. In this direction we introduce left perfect coalgebras as those such that every left C-comodule has a projective cover epimorphism. Every left perfect coalgebra is left semiperfect and any left comodule over a left semiperfect coalgebra has a flat cover. A characterization of left perfect coalgebras, analogous to Bass's Theorem P, is given in Theorem 5.6. In particular, over such coalgebras the classes of flat and projective left C-comodules coincide. We provide in Proposition 5.10 a way of constructing examples of left perfect coalgebras, showing that they are as abundant as left semiperfect ones. The path coalgebra KQ of a quiver Q is shown to be left perfect if and only if for every vertex $s \in Q_0$, the set $Q(s\!\to)$ of paths starting at s is finite and there is no infinite path $\ldots \to \bullet \to \ldots \to \bullet \to \bullet \to \bullet$ in Q, Corollary 5.9.

Quasi-coFrobenius coalgebras are analyzed in connection with flat comodules. A coalgebra C is left quasi-coFrobenius if the regular right comodule C_C is projective (hence C_C is flat). It is natural then to ask when C_C is flat. Coalgebras satisfying this property are called right IF-coalgebras. We prove in Proposition 6.3 that a coalgebra C is left quasi-coFrobenius if and only if C is a right IF-coalgebra and left semiperfect. For coalgebras being IF on both sides the classes of injective and flat comodules coincide. Finally, the notion of left weak global dimension of a coalgebra is treated and it is observed that when C is left semiperfect this dimension coincides with the global dimension of C.

The paper is organized as follows: In Section 2 we present the notion of flat object in a locally finitely presented Grothendieck category $\mathcal{A}$ and we list elementary properties of such objects. The following problem is posed: Assume that $\mathcal{A}$ has enough flat objects. Has $\mathcal{A}$ enough projective objects? In Section 3 we study the notion of flat object in the particular case $\mathcal{A} = C$-Comod. Main results of this section are: Theorems 3.4 and 3.5 which contains numerous characterizations of flat comodules; and the above mentioned Theorem 3.8 on the structure of flat left C-comodules when C is left semiperfect; Section 4 deals with the coalgebras C having enough flat left C-comodules and with the class of *fin*-projective comodules. Proposition 4.5 contains the aforementioned characterizations of *fin*-projective comodules. Theorem 4.6 gives the previously announced partial answer to Problem 2.9 for $\mathcal{A} = C$-Comod. Left perfect coalgebras are introduced and studied in Section 5. Several characterizations of left perfect coalgebras are shown in Theorem 5.6. A detailed analysis of the path coalgebra of the infinite quiver $\mathbb{A}_\infty^{(0)}$ is also carried out in this section. Section 6 deals with IF-coalgebras and quasi-coFrobenius coalgebras. Finally, in Section 7 the notion of left weak global dimension of a coalgebra is briefly treated.

The reader is referred to [8], [14] and [27] for the coalgebra and comodule terminology, to [2], [3], [37] and [38] for the representation theory terminology, and to [16], [17] and [30] for the category theory terminology.

2. Flat objects in a locally finitely presented Grothendieck category

The notion of flat comodule studied in this paper as well as its elementary properties may be framed in the more general setting of locally finitely presented Grothendieck categories.

In the sequel $\mathcal{A}$ will stand for a Grothendieck category. We recall that an object A of $\mathcal{A}$ is called **finitely generated** if for each directed family $\{A_i\}_{i\in I}$ of subobjects of A, with $A = \bigcup_{i\in I} A_i$, there is $j \in I$ such that $A = A_j$. An object A of $\mathcal{A}$ is **finitely presented** if it is finitely generated and every epimorphism $N \to A$ with N finitely generated has a finitely generated kernel. It is known that A is finitely presented if and only if the functor $\mathrm{Hom}_{\mathcal{A}}(A, -)$ preserves direct limits. The category $\mathcal{A}$ is said to be **locally finitely presented** if it has a family of finitely presented generators. An exact sequence $0 \longrightarrow A' \xrightarrow{f} A \xrightarrow{g} A'' \longrightarrow 0$ in $\mathcal{A}$ is called **pure** if for every finitely presented object N in $\mathcal{A}$ the map $\mathrm{Hom}_{\mathcal{A}}(N, g) : \mathrm{Hom}_{\mathcal{A}}(N, A) \to \mathrm{Hom}_{\mathcal{A}}(N, A'')$ is surjective, see [40], [32] and [33].

Following Stenström [40] and [42] we introduce the following definition.

DEFINITION 2.1. Let $\mathcal{A}$ be a locally finitely presented Grothendieck category.
(a) An object F of $\mathcal{A}$ is **flat** if every epimorphism $g : M \to F$ is pure.
(b) The category $\mathcal{A}$ **has enough flat objects** if every object L of $\mathcal{A}$ admits an epimorphism $F \to L$, with F flat.

We note that the definition of flat object is dual to the definition of FP-injective module, see [41, Proposition 2.6 (c)]. We record some facts that are easily deduced from the definition.

PROPOSITION 2.2. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category.*
(a) *Every projective object of $\mathcal{A}$ is flat.*
(b) *Every finitely presented flat object of $\mathcal{A}$ is projective.*
(c) *Assume that $0 \to A' \to A \to A'' \to 0$ is an exact sequence in $\mathcal{A}$.*
(i) *If the sequence is pure and A is flat, then A'' is flat.*
(ii) *If A' and A'' are flat, then A is flat.*

The following proposition shows that the class of flat objects is closed under coproducts and directed limits, see [40], [42] and [33, Section 2].

PROPOSITION 2.3. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category.*
(a) *The coproduct $\bigoplus_{i\in I} F_j$ of a family $\{F_i\}_{i\in I}$ of objects F_i of $\mathcal{A}$ is flat if and only if F_i is flat for every $i \in I$.*
(b) *The direct limit $\varinjlim_{i\in I} F_i$ of any directed family $\{F_i\}_{i\in I}$ of flat objects of $\mathcal{A}$ is flat.*
(c) *Assume that $\mathcal{A}$ is locally noetherian and $h : E \to F$ is a non-zero epimorphism in $\mathcal{A}$. If E is injective and F is flat then F is injective.*

Proof. (a) The finite case follows from (ii) in Proposition 2.2 (c). The infinite case is derived from the finite case by noting that the image of a finitely generated object lies in a finite coproduct of the objects F_i.

(b) The natural epimorphism $\bigoplus_{i\in I} F_i \to \varinjlim_{i\in I} F_i$ is pure, see [33, Corollary 2.5]. Then (b) follows by applying (a) and (i) in Proposition 2.2 (c).

(c) We prove that F is injective by applying the Baer injectivity criterion for Grothendieck categories, see [42, Proposition 5.2.9]. For this purpose, consider the diagram in $\mathcal{A}$,

$$\begin{array}{ccccc} & 0 \to & X & \xrightarrow{u} & Y \\ & & \downarrow{\scriptstyle\varphi} & & \\ E & \xrightarrow{h} & F & \to & 0 \end{array}$$

with a monomorphism $u : X \to Y$ between noetherian objects X and Y. Since $\mathcal{A}$ is locally noetherian, an object in $\mathcal{A}$ is noetherian if and only if it is finitely presented, or equivalently, it is finitely generated. Using the flatness of F, there is a morphism $\varphi' : X \to E$ such that $\varphi = \varphi' h$. Since E is injective, there is $\varphi'' : Y \to E$ such that $\varphi' = u\varphi''$. The morphism $\psi = h\varphi'' : Y \to F$ satisfies then $\varphi = \psi u$. By the Baer injectivity criterion, F is injective. □

COROLLARY 2.4. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Then $\mathcal{A}$ has enough flat objects if and only if for every finitely presented object N of $\mathcal{A}$ there exists an epimorphism $F \to N$ with F flat.*

Proof. The necessity is obvious. For the sufficiency, let L be an object of $\mathcal{A}$ and take a directed system $\{L_j\}_{j\in J}$ of finitely presented objects of $\mathcal{A}$ such that $L = \varinjlim_{j\in J} L_j$. By assumption, for each $j \in J$, there exists an epimorphism $f_j : F_j \to L_j$ with F_j flat. If $h : \bigoplus_{j\in J} L_j \longrightarrow L$ is the canonical limit epimorphism, then the composite morphism $\bigoplus_{j\in J} F_j \xrightarrow{\oplus_{j\in J} f_j} \bigoplus_{j\in J} L_j \xrightarrow{h} L$ is an epimorphism and the object $\bigoplus_{j\in J} F_j$ is flat by Proposition 2.3. □

The next result extends the well-known characterization of flat modules given by Govorov

[21] and Lazard [25].

THEOREM 2.5. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category with enough projective objects and let F be an object of $\mathcal{A}$.*

(a) *The object F is flat if and only if F is a direct limit of a directed system of finitely generated projective objects of $\mathcal{A}$.*

(b) *If $\mathcal{A}$ is locally noetherian and* $\mathrm{gl.dim}\,\mathcal{A} \leq 1$*, then*

(i) *any noetherian subobject of a flat object F is projective,*

(ii) *any subobject of a flat object F is flat, and*

(iii) *F is flat if and only if F is a directed union of the form $F = \bigcup_{i \in I} F_i$, where each F_i is a finitely generated projective subobject of F.*

Proof. (a) The necessity follows from Proposition 2.3. For the sufficiency, assume that F is flat. First we note that $\mathcal{A}$ has a family of finitely generated projective generators $\{P_i\}_{i \in I}$. Then there is a pure epimorphism $g : \bigoplus_{j \in J} P'_j \to F$ such that every object P'_j is isomorphic to one of the objects in $\{P_i\}_{i \in I}$. Then the result follows by applying the arguments used in [21] and [25], see also [33, Section 4].

(b) Suppose that $\mathcal{A}$ is locally noetherian and $\mathrm{gl.dim}\,\mathcal{A} \leq 1$. An object X of $\mathcal{A}$ is noetherian if and only if X finitely generated, or equivalently, if X is finitely presented.

(i) Let X be a noetherian subobject of a flat object F. By hypothesis, there is a pure epimorphism $g : P \to F$, with P projective. Since F is flat, there is a morphism $u : X \to P$ such that gu is the inclusion morphism $X \hookrightarrow F$. Then u is a monomorphism, X is isomorphic to the subobject $\mathrm{Im}\, u$ of the projective object P and therefore X is projective, because $\mathrm{gl.dim}\,\mathcal{A} \leq 1$.

(ii) Assume that F is a flat object of $\mathcal{A}$ and let F' be a subobject of F. Since $\mathcal{A}$ is locally noetherian, F' is a directed union of finitely generated (= noetherian) subobjects F_j of F'. By (i), each F_j is projective and (ii) follows from (a).

(iii) The necessity follows from (a). For the sufficiency, we apply the arguments given in (ii) to the flat object $F' = F$. The proof is complete. $\square$

The above theorem gives the key to apply [4, Theorem 3.2] and deduce the existence of flat covers in any locally finitely presented Grothendieck category $\mathcal{A}$ with enough projective objects. Following Enochs [15], we define a morphism $\varphi : F \to A$ in a locally finitely presented Grothendieck category $\mathcal{A}$ to be a **flat precover of** A if F is flat and for every flat object F' of $\mathcal{A}$ the map $\mathrm{Hom}_{\mathcal{A}}(F', F) \to \mathrm{Hom}_{\mathcal{A}}(F', A)$ is surjective. If, in addition, given an endomorphism $f : F \to F$ of F, the equality $\varphi f = f$ implies that f is an automorphism, the morphism $\varphi : F \to A$ is defined to be a **flat cover** of the object A. Note that if $\mathcal{A}$ has enough projective objects then a flat cover of any object of $\mathcal{A}$ is an epimorphism.

THEOREM 2.6. *Assume that $\mathcal{A}$ is a locally finitely presented Grothendieck category with enough projective objects. Then every object of $\mathcal{A}$ has a flat cover.*

Proof. Let $\{P_i\}_{i \in I}$ be a family of finitely generated projective generators of the category $\mathcal{A}$. Let $\mathcal{F}$ be the class consisting of all flat objects of $\mathcal{A}$, and let $\mathcal{F}'$ be the class consisting of all objects in $\mathcal{F}$ of the form $\bigoplus_{j \in J} P_j^{n_j}$, with $J \subseteq I$ finite and n_j finite. Here $P_j^{n_j}$ denotes the coproduct of n_j copies of P_j. It is easy to see that $\mathcal{F}'$ is a set and we show as in [21] and [25] that any object of $\mathcal{F}$ is a direct limit of a directed system of projective objects of the set $\mathcal{F}'$.

Now the following result of El Bashir [4, Theorem 3.2] applies. □

Theorem 2.7. *Let $\mathcal{A}$ be a Grothendieck category and let $\mathcal{F}$ be a class of objects of $\mathcal{A}$ closed under coproducts and directed limits. If there is a subset $\mathcal{F}'$ of $\mathcal{F}$ such that every object in $\mathcal{F}$ is a directed limit of objects from $\mathcal{F}'$, then each object of $\mathcal{A}$ has an $\mathcal{F}$-cover.* □

Remark 2.8. Assume that $\mathcal{A}$ is a locally finitely presented Grothendieck category with enough projective objects. If $\mathcal{P}$ is the full subcategory of $\mathcal{A}$ consisting of pairwise nonisomorphic finitely generated projective objects in $\mathcal{A}$ then $\mathcal{A}$ is equivalent to the category $\mathrm{Add}(\mathcal{P}^{op}, \mathcal{A}b)$ of all contravariant functors from $\mathcal{P}$ to the category $\mathcal{A}b$ of abelian groups, see [16]. Then the flatness of an object in $\mathcal{A}$ can be rephrased in terms of the exactness of the tensor product in the functor category as explained by Stenström in [40].

Open problems 2.9. (a) *Give a characterization of locally finitely presented Grothendieck categories $\mathcal{A}$ that have enough flat objects.*

(b) *Give a characterization of locally finitely presented Grothendieck categories $\mathcal{A}$ such that every object of $\mathcal{A}$ has a flat cover.*

(c) *Do any of the conditions* (a) *and* (b) *imply that the category $\mathcal{A}$ has enough projective objects?*

In the next section we study in details these problems for the category $\mathcal{A} = C\text{-Comod}$ of left comodules over a coalgebra C. In this case a partial answer is given in Theorem 4.6. We finish this section by providing an example of locally finitely presented Grothendieck categories with a lack of flat objects.

Example 2.10. Given a quiver Q and a field K, we consider the Grothendieck K-category $\mathrm{Rep}_K(Q)$ of all K-linear representations of Q. We denote by $\mathcal{A} = \mathrm{Rep}_k^{\ell f}(Q)$ the full K-subcategory of $\mathrm{Rep}_K(Q)$ consisting of all locally finite dimensional representations, that is, directed unions of representations of finite dimension. This subcategory is locally finite. It is proved in [29, Theorem 2.2] that for the infinite locally Dynkin quiver

$$Q = \mathbb{A}_\infty^{(0)}: \qquad 0 \to 1 \to 2 \to 3 \to \ldots \to m \to m+1 \to \ldots,$$

the category $\mathrm{Rep}_K^{\ell f}(Q)$ is pure-semisimple, that is, every object in $\mathrm{Rep}_K^{\ell f}(Q)$ is a direct sum of finite dimensional subobjects. It is shown in [29, Corollary 3.9] that $\mathcal{A} = \mathrm{Rep}_K^{\ell f}(Q)$ has no non-zero projective objects. We claim that $\mathrm{Rep}_K^{\ell f}(Q)$ has neither non-zero flat objects. Let F be a flat object in $\mathrm{Rep}_K^{\ell f}(Q)$. Then $F = \bigoplus_{i \in I} F_i$ for a family $\{F_i\}_{i \in I}$ of finite dimensional (= finitely presented) subobjects of F. Each F_i is flat and, by Proposition 2.2 (b), F_i is projective. Hence $F = (0)$.

3. Flat comodules

Assume that K is a field and C is a K-coalgebra. We denote by C-Comod the category of left C-comodules, and by C-comod the full subcategory of C-Comod consisting of comodules of finite K-dimension. The category of right C-comodules is denoted by Comod-C. Given two C-comodules M and N, we denote by $\mathrm{Hom}_C(M, N)$ the K-vector space of all C-comodule homomorphisms from M to N.

We recall that the K-dual space $C^* = \mathrm{Hom}_K(C, K)$ to C is a K-algebra with respect to the convolution product and is viewed as a pseudocompact K-algebra (see [14], [36] and [38]). Any left C-comodule N with structure map $\delta_N : N \to C \otimes N$ can be viewed as a right rational (= discrete, see [14], [23], [36]) C^*-module via the action

$$n\varphi^* = \sum_{(n)} \varphi(n_{(-1)})n_{(0)},$$

where $\varphi \in C^*$, $n \in N$ and $\delta_N(n) = \sum_{(n)} n_{(-1)} \otimes n_{(0)} \in C \otimes N$.

We also recall that C-Comod is isomorphic to the category Rat-C^* (= Dis-C^*) of rational (= discrete) right C^*-modules. Given a right C^*-module M, we denote by $\mathrm{Rat}(M)$ its unique maximal rational submodule.

Note that the classes of finitely presented, finitely generated, and finite dimensional objects coincide in C-Comod, because the Grothendieck category $\mathcal{A} = C$-Comod is locally finite. Then the definition of a flat object in the category $\mathcal{A} = C$-Comod yields the following definition, see also [23, Section 6].

DEFINITION 3.1. (a) A left C-comodule F is defined to be **flat** if every epimorphism $f : M \to F$ in C-Comod is pure, that is, for every comodule N in C-comod, the linear map $\mathrm{Hom}_C(N, M) \to \mathrm{Hom}_C(N, F)$ induced by f is surjective.

(b) A coalgebra C **has enough flat left comodules** if every comodule L in C-Comod admits an epimorphism $F \to L$ with F flat.

We start with a characterization of pure exact sequences in C-Comod by means of the exactness of the cotensor product bifunctor $-\Box_C- \ : \text{Comod-}C \times C\text{-Comod} \longrightarrow \mathrm{Mod}\, K$, see [13].

PROPOSITION 3.2. *Let* $0 \to X \to Y \to Z \to 0$ *be a short exact sequence in* C-Comod. *The following statements are equivalent:*

(a) *The sequence* $0 \to X \to Y \to Z \to 0$ *is pure.*

(b) *The induced sequence* $0 \to M\Box_C X \to M\Box_C Y \to M\Box_C Z \to 0$ *is exact, for any comodule* M *in* Comod-C.

(c) *The induced sequence* $0 \to N\Box_C X \to N\Box_C Y \to N\Box_C Z \to 0$ *is exact, for any comodule* N *in* comod-C.

Proof. (a)$\Rightarrow$(b) Assume that the sequence $0 \to X \to Y \to Z \to 0$ is pure. Let M be an arbitrary comodule in Comod-C and let $\{M_i\}_{i\in I}$ be a directed family of finite dimensional C-subcomodules of M such that $M = \bigcup_{i\in I} M_i$. We recall that given a comodule N in comod-C, the K-dual space $N^* = \mathrm{Hom}_K(N, K)$ is a left C-comodule in a natural way and there is a natural isomorphism $N\Box_C U \cong \mathrm{Hom}_C(N^*, U)$ for any left C-comodule U, see [13, p. 32]. By applying this to $N = M_i$ we conclude that the induced sequence $0 \to M_i\Box_C X \to M_i\Box_C Y \to M_i\Box_C Z \to 0$ is exact, for all $i \in I$. Since the direct limit functor is exact and the cotensor functor commutes with direct limits, the sequence $0 \to M\Box_C X \to M\Box_C Y \to M\Box_C Z \to 0$ is exact.

(b)$\Rightarrow$(c) Obvious.

(c)$\Rightarrow$(a) Assume that $0 \to X \to Y \to Z \to 0$ is an exact sequence in C-Comod and that L is a left C-comodule of finite K-dimension. By (c) applied to the right C-comodule $L = N^*$, the induced sequence $0 \to N^*\Box_C X \to N^*\Box_C Y \to N^*\Box_C Z \to 0$ is exact. Hence, using the above natural isomorphism, we conclude that the map $\mathrm{Hom}_C(N, Y) \to \mathrm{Hom}_C(N, Z)$ induced by the epimorphism $Y \to Z \to 0$ is surjective. This shows that the short exact sequence $0 \to X \to Y \to Z \to 0$ is pure and finishes the proof. $\Box$

We recall from [26] that a coalgebra C is said to be **left semiperfect** if every comodule L in C-comod admits a projective cover $P \to L$ in C-comod, or equivalently, if the right C-comodule C is a direct sum of finite dimensional comodules. We also recall from [11] that a coalgebra

C is said to be **left $\mathcal{F}$-noetherian** (resp. right $\mathcal{F}$-noetherian) if every closed and cofinite left ideal (resp. right ideal) of C^* is finitely generated. It follows from [11, Theorem 2.12] that C is right $\mathcal{F}$-noetherian if C is left semiperfect.

Throughout we need the following simple but useful observation.

LEMMA 3.3. *Assume that C is a right $\mathcal{F}$-noetherian coalgebra. If N is a comodule in C-comod then N, viewed as a right C^*-module, is finitely presented.*

Proof. Since C is right $\mathcal{F}$-noetherian, the endofunctor $\mathrm{Rat}(-) : \text{Mod-}C^* \longrightarrow \text{Mod-}C^*$ preserves direct limits, see Proposition 12 in [42, p. 263]. Let $\{M_i\}_{i\in I}$ be a directed system of right C^*-modules. Since $\dim_K N$ is finite then, under the identification $C\text{-Comod} \cong \text{Rat-}C^*$, we get the isomorphisms

$$\begin{aligned}\mathrm{Hom}_{C^*}(N, \varinjlim_{i\in I} M_i) &\cong \mathrm{Hom}_{C^*}(N, \mathrm{Rat}(\varinjlim_{i\in I} M_i)) \cong \mathrm{Hom}_C(N, \varinjlim_{i\in I} \mathrm{Rat}(M_i)) \\ &\cong \varinjlim_{i\in I} \mathrm{Hom}_C(N, \mathrm{Rat}(M_i)) \cong \varinjlim_{i\in I} \mathrm{Hom}_{C^*}(N, M_i).\end{aligned}$$

Consequently, N viewed as a right C^*-module is finitely presented. □

Now we give several characterizations of flat left C-comodules. In particular, we show that if the coalgebra C is left semiperfect then the category $C\text{-}\mathcal{F}\ell$ of flat left C-comodules is isomorphic to the intersection of the category $\text{Rat-}C^*$ with the full subcategory $\mathcal{F}\ell\text{-}C^*$ of $\text{Mod-}C^*$ consisting of all right flat C^*-modules.

THEOREM 3.4. *The following conditions about a left C-comodule F are equivalent:*

(a) *F is flat.*

(b) *For every finite dimensional subcomodule F_0 of F and every epimorphism $f : M \to F$ in C-Comod there exists a subcomodule F_0' of M such that $f_0(F_0') = F_0$ and the restriction $f_0 : F_0' \to F_0$ of f to F_0' is an isomorphism.*

(c) *Every diagram*

$$(*) \qquad \begin{array}{ccccc} F_0 & \hookrightarrow & F & & \\ & & \big\downarrow{\scriptstyle f} & & \\ M & \xrightarrow{g} & N & \longrightarrow & 0 \end{array}$$

in the category C-Comod, with a finite dimensional subcomodule F_0 of F and an epimorphism g, can be completed to a commutative diagram

$$(**) \qquad \begin{array}{ccccc} F_0 & \hookrightarrow & F & & \\ \big\downarrow{\scriptstyle f_0} & & \big\downarrow{\scriptstyle f} & & \\ M & \xrightarrow{g} & N & \longrightarrow & 0. \end{array}$$

(d) *For every epimorphism $f : M \to F$ in C-Comod and for every (finite dimensional) right C-comodule N, the linear map $N\square_C f : N\square_C M \to N\square_C F$ is surjective.*

Proof. (a)⇒(b) Assume that F is flat and that $f : M \to F$ is an epimorphism in C-Comod. If F_0 is a finite dimensional subcomodule of F then the inclusion $u : F_0 \hookrightarrow F$ extends to a C-comodule homomorphism $u' : F_0 \to M$ such that $u = fu'$. It follows that the subcomodule $F_0' = u'(F_0)$ satisfies the required conditions.

(b)⇒(c) Suppose that a diagram $(*)$ in C-Comod is given as in (c). By constructing the pull-back diagram for g and f we get the diagram

$$\begin{array}{ccccc} F_0 & \hookrightarrow & F & & \\ & & \Big\downarrow \mathrm{id}_F & & \\ M' & \xrightarrow{g'} & F & \longrightarrow & 0 \\ \Big\downarrow f' & & \Big\downarrow f & & \\ M & \xrightarrow{g} & N & \longrightarrow & 0 \end{array}$$

in C-Comod, where the square commutes and g' is surjective. By applying (b) to F_0 and to g' we get a C-comodule monomorphism $h : F_0 \to M'$ such that $g'h$ is the inclusion $F_0 \hookrightarrow F$. Hence the diagram $(**)$ with $f_0 = f'h$ is commutative and (c) follows.

(c)⇒(a) Assume that $g : M \to F$ is an epimorphism in C-Comod and L is a comodule in C-comod. If $h \in \mathrm{Hom}_C(L, F)$ then, by applying (c) to $N = F$, $f = \mathrm{id}_F$ and to the subcomodule $F_0 = h(L)$ of F, we get a C-comodule homomorphism $h' : F_0 \to M$ such that $gh'h'' = h$, where $h'' : L \to F_0$ is the epimorphism defined by h. Consequently, $\mathrm{Hom}_C(L, g) : \mathrm{Hom}_C(L, M) \to \mathrm{Hom}_C(L, F)$ is surjective.

(a)⇔(d) It follows from Proposition 3.2. □

THEOREM 3.5. *Assume that C is a left semiperfect coalgebra and let F be a left C-comodule. The following statements are equivalent:*

(a) *F is flat.*

(b) *The statement* (b) *in Theorem 3.4 with a fixed epimorphism $f : M \to F$, where M is projective.*

(c) *The rational right C^*-module F is flat.*

(d) *The left C^*-module $F^* = \mathrm{Hom}_K(F, K)$ is injective.*

(e) *F is a directed limit of finite dimensional projective C-comodules.*

Proof. The equivalence of (b) and Theorem 3.4 (b) easily follows because C-Comod has enough projective objects. So the equivalence of (a) and (b) follows from Theorem 3.4. The equivalence of (a) and (e) follows from Theorem 2.5 (a).

(e)⇒(c) Assume that F is the direct limit in C-Comod of a family of finite dimensional projective C-comodules $\{P_i\}_{i \in I}$. By [19, Lemma 2.1], under the identification C-Comod $\cong$ Rat-C^*, each P_i is projective as a right C^*-module and the direct limit in the category C-Comod is just the direct limit in Mod-C^*. Hence (c) follows.

(c)⇒(a) Assume that $g : M \to F$ is an epimorphism in C-Comod and make the identification C-Comod $\cong$ Rat-C^*. Let $f : L \to F$ be a C-comodule homomorphism, where L is finite dimensional. Since C is left semiperfect then, according to [11, Theorem 2.12], C is right $\mathcal{F}$-noetherian and, by Lemma 3.3, L viewed as a right C^*-module is finitely presented. Then there is a C^*-module homomorphism $h : L \to M$ such that $gh = f$. In view of the identification C-Comod $\cong$ Rat-C^*, (a) follows.

(c)⇔(d) This is the well-known characterization of flat modules in terms of the injectivity of their character modules (see [42, Proposition 10.4]). So the proof of the theorem is complete. □

For left semiperfect hereditary coalgebras the equivalence (a)⇔(e) in Theorem 3.5 takes a stronger form.

COROLLARY 3.6. *Suppose that C is a left semiperfect and hereditary coalgebra. Then:*
(a) *Any finite dimensional subcomodule of a flat left C-comodule F is projective.*
(b) *Any subcomodule of a flat left C-comodule F is flat.*
(c) *A left C-comodule F is flat if and only if F is a directed union of the form $F = \bigcup_{i\in I} F_i$, where each F_i is a finite dimensional projective C-subcomodule of F.*

Proof. Apply Theorem 3.5 and Theorem 2.5 (b). □

COROLLARY 3.7. *Assume that C is a left semiperfect coalgebra and take $0 \to M' \to M \to M'' \to 0$ an exact sequence in C-*Comod*. If M and M'' are flat, then M' is also flat.*

Proof. It follows from the equivalences (a)⇔(c) in Theorem 3.5 that a left C-comodule F is flat if and only if, for F viewed as a right C^*-module, we have $\mathrm{Tor}_m^{C^*}(F, -) = 0$, for all $m \geq 1$. Hence the corollary follows, by applying the $\mathrm{Tor}_m^{C^*}$ long exact sequence induced by the given exact sequence $0 \to M' \to M \to M'' \to 0$. □

The technique developed in [31] and [39] allows us to go deeper into the structure of flat comodules. We next show that a flat comodule may be written as a union of flat subcomodules of at most countable dimension.

THEOREM 3.8. *Suppose that C is a left semiperfect coalgebra and F is a flat left C-comodule. Then F is an $\aleph_0$-directed union of the form $F = \bigcup_{i\in I} F_i$, where each F_i is a flat subcomodule of F of dimension less or equal than $\aleph_0$.*

Proof. We prove the result by showing that any subcomodule F' of F of dimension less or equal than $\aleph_0$ is contained in a flat subcomodule F'' of F with $\dim_K F'' \leq \aleph_0$. First we view F' as a union $F' = \bigcup_{m=1}^{\infty} F'_m$ of an ascending sequence $F'_1 \subseteq F'_2 \subseteq \ldots \subseteq F'_m \subseteq \ldots$ of finite dimensional subcomodules. We construct a new ascending sequence $F''_1 \subseteq F''_2 \subseteq \ldots \subseteq F''_m \subseteq \ldots$ of finite dimensional subcomodules of F such that $F'_1 \subseteq F''_1$, $F'_2 \subseteq F''_2$, $\ldots$, $F'_m \subseteq F''_m, \ldots$ and $F'' = \bigcup_{m=1}^{\infty} F''_m$ is a flat subcomodule of F with $\dim_K F'' \leq \aleph_0$.

Since C is left semiperfect, according to [26], there is an epimorphism $f : P \to F$ in C-Comod with P projective of the form $P = \bigoplus_{j\in J} P_j$, and each P_j is of finite K-dimension. We proceed to construct the chain $F''_1 \subseteq F''_2 \subseteq \ldots \subseteq F''_m \subseteq \ldots$. We set $F''_1 = F'_1$. To construct the comodule F''_2 we note that since F is flat then, by Theorem 3.5 (b) applied to $M = P$ and $F_0 = F'_1$, there exists a C-comodule homomorphism $h_1 : F''_1 \to P$ such that fh_1 is the inclusion homomorphism $F'_1 = F''_1 \hookrightarrow F$. Let J_1 be a finite subset of J such that $\mathrm{Im}\, h_1$ is a subcomodule of $P'_1 = \bigoplus_{j\in J_1} P_j \subseteq P$. Then $F''_1 = F'_1 \subseteq f(P'_1)$ and there exists a finite dimensional subcomodule F''_2 of F such that $f(P'_1) + F'_2 \subseteq F''_2$.

To construct the comodule F''_3 we apply again Theorem 3.5 (b) to the flat comodule F, to $M = P$ and $F_0 = F''_2$. We conclude that there exists a C-comodule homomorphism $h_2 : F''_2 \to P$ such that fh_2 is the inclusion homomorphism $F''_2 \hookrightarrow F$. Let J_2 be a finite subset of J containing J_1 such that $\mathrm{Im}\, h_2$ is a subcomodule of $P'_2 = \bigoplus_{j\in J_2} P_j \subseteq P$. Then $P'_1 \subseteq P'_2$ is a direct summand embedding, $F''_1 \subseteq F''_2 \subseteq f(P'_2)$ and there exists a finite dimensional subcomodule F''_3 of F such that $f(P'_2) + F''_2 \subseteq F''_3$. Continuing this procedure we construct an infinite chain

$$J_1 \subseteq J_2 \subseteq J_3 \subseteq \ldots \subseteq J_m \subseteq J_{m+1} \subseteq \ldots$$

of finite subsets of J, a chain

$$P'_1 \subseteq P'_2 \subseteq P'_3 \subseteq \ldots \subseteq P'_m \subseteq P'_{m+1} \subseteq \ldots$$

of finite dimensional subcomodules of P, where $P'_m = \bigoplus_{j \in J_m} P_j$, a chain

$$F''_1 \subseteq F''_2 \subseteq \ldots \subseteq F''_m \subseteq \ldots$$

of finite dimensional subcomodules of F, and C-comodule homomorphisms

$$h_1 : F''_1 \to P, \quad h_2 : F''_2 \to P, \quad \ldots \quad h_m : F''_m \to P, \quad \ldots$$

such that $F'_m \subseteq F''_m$, fh_m is the inclusion homomorphism $F''_m \hookrightarrow F$ and $f(P'_m) + F''_m \subseteq F''_{m+1}$, for each $m \geq 1$.

We set $J' = \bigcup_{m=1}^{\infty} J_m$, $P' = \bigoplus_{j \in J'} P_j = \bigcup_{m=1}^{\infty} P'_m$ and $F'' = \bigcup_{m=1}^{\infty} F''_m$. It is clear that

- $\dim_K P' \leq \aleph_0$ and $\dim_K F'' \leq \aleph_0$,
- P' is projective, F'' is a subcomodule of F containing F', and
- the epimorphism $f : P \to F$ restricts to an epimorphism $f' : P' \to F''$ such that the diagram

$$\begin{array}{ccccc} P' & \xrightarrow{f'} & F'' & \longrightarrow & 0 \\ \Big\downarrow{\scriptstyle u'} & & \Big\downarrow{\scriptstyle u''} & & \\ P & \xrightarrow{f} & F & \longrightarrow & 0 \end{array}$$

is commutative, where u' and u'' are the canonical embeddings. Moreover, by applying the properties of the maps $h_1, h_2, h_3, \ldots$ constructed above, we can easily show that, given a finite dimensional subcomodule F''_0 of F'', there exists a C-comodule homomorphism $h : F''_0 \to P'$ such that $f'h$ is the embedding $F''_0 \hookrightarrow F''$. It then follows from the equivalences (a)$\Leftrightarrow$(b) of Theorem 3.5 that the comodule F'' is flat. This finishes the proof, because F'' contains the comodule F'. □

We finish this section by applying Theorem 2.6 to C-Comod in order to get:

Theorem 3.9. If C is a left semiperfect coalgebra, then every left C-comodule has a flat cover.

4. Coalgebras with enough flat comodules

The aim of this section is to study coalgebras with enough flat left comodules, which means that every comodule L in C-Comod admits an epimorphism $F \to L$, with F flat. We would like to get necessary and sufficient conditions for a coalgebra C to ensure that C has enough flat left comodules. In this context, the following question arises.

Question 4.1. *Assume that C is a coalgebra and F is flat in C-*Comod*. Is the left C^*-module* Rat(F^*)*, viewed as a right C-comodule, injective in the category* Comod-C*?*

It follows from the arguments in the proof of Theorem 3.5 that if C is a left semiperfect coalgebra then the answer is affirmative. We show in this section that an affirmative answer to Question 4.1 would lead to the fact that the coalgebra C is left semiperfect if C has enough flat left comodules.

We start with an example of comodule category with a lack of non-zero flat comodules and projective comodules.

Example 4.2. Here we interpret Example 2.11 from a comodule point of view, according to [29, Theorem 2.2]. Recall that for a quiver Q the path coalgebra KQ is the K-vector space spanned by the paths in Q with comultiplication and counit given by

$$\Delta(\alpha) = \sum_{\beta\gamma=\alpha} \beta \otimes \gamma \quad \text{and} \quad \varepsilon(\alpha) = \begin{cases} 0, & if\ |\alpha| > 0, \\ 1, & if\ |\alpha| = 0, \end{cases}$$

where $\beta\gamma$ is the concatenation of paths and $|\alpha|$ the length of α. The concatenation of paths will be done in a reverse way. It is shown in [8, Section 5.3] and [29, Theorem 2.2] that KQ-Comod $\cong \mathrm{Rep}_K^{\ell f \ell n}(Q)$. Hence, if C is the path coalgebra associated to the quiver

$$Q = \mathbb{A}_\infty^{(0)} : \quad 0 \to 1 \to 2 \to 3 \to \ldots \to m \to \ldots$$

then, by Example 2.11 and [29, Corollary 3.9], the category KQ-Comod $\cong \mathrm{Rep}_K^{\ell f}(Q)$ has neither non-zero flat object nor non-zero projective object. On the other hand, Comod-KQ is equivalent to $\mathrm{Rep}_K^{\ell f}(Q^\circ)$, where Q° is the quiver $0 \leftarrow 1 \leftarrow 2 \leftarrow \ldots \leftarrow m \leftarrow \ldots$ dual to $Q = \mathbb{A}_\infty^{(0)}$. It follows that KQ is right semiperfect and every right KQ-comodule has a flat cover.

In an attempt to give an answer to Question 4.1 a new class of comodules appears in a natural way. As we will see later this class may be recognized as the class of FP-injective objects in a certain functor category.

Definition 4.3. A left C-comodule F is called fin-**projective** if for any epimorphism $g : X \to Y$ in C-comod, the linear map $\mathrm{Hom}_C(F, X) \to \mathrm{Hom}_C(F, Y)$ induced by g is surjective.

In other words, F is fin-projective if and only if every diagram in C-Comod

(4.4)
$$\begin{array}{ccccc} & & F & & \\ & & \big\downarrow{\scriptstyle f} & & \\ X & \xrightarrow{g} & Y & \to & 0 \end{array}$$

with an epimorphism $g : X \to Y$ in C-comod, can be completed to a commutative diagram

(4.4a)
$$\begin{array}{ccccc} & & F & & \\ & \swarrow & \big\downarrow{\scriptstyle f} & & \\ X & \xrightarrow{g} & Y & \to & 0. \end{array}$$

Proposition 4.5. *Let F be a left C-comodule. The following assertions are equivalent:*

(a) *The comodule F is fin-projective.*

(b) *The right C-comodule* $\mathrm{Rat}(F^*)$ *is injective.*

(c) *The functor $h^F = \mathrm{Hom}_C(F, -) : C\text{-comod} \longrightarrow \mathrm{Mod}\, K$ is an FP-injective object of the functor category $\mathcal{D}_l(C) = \mathrm{Add}(C\text{-comod}, \mathrm{Mod}\, K)$ of all covariant additive functors $T : C\text{-comod} \longrightarrow \mathrm{Mod}\, K$, that is, $\mathrm{Ext}^1_{\mathcal{D}_l(C)}(T, h^F) = 0$ for any finitely presented object T of $\mathcal{D}_l(C)$, see* [41].

Proof. Before we begin the proof we fix some notation. Given a left C-comodule L, we denote by $\lambda_L : L \to L^{**}$ the canonical embedding. In case L is finite dimensional, λ_L is an isomorphism. The inclusion map $j : \mathrm{Rat}(L^*) \to L^*$ gives rise to a dual map $j^* : L^{**} \to \mathrm{Rat}(L^*)^*$. The image of the composite map $\alpha_L = j^*\lambda_L : L \to \mathrm{Rat}(L^*)^*$ is contained in $\mathrm{Rat}(\mathrm{Rat}(L^*)^*)$.

(b)$\Rightarrow$(a) Suppose a diagram as (4.4) in C-Comod is given, with an epimorphism $g : X \to Y$ in C-comod. It leads to the dual diagram

$$\begin{array}{ccccc} & & \mathrm{Rat}(F^*) & & \\ & & \big\uparrow{\scriptstyle f^*} & & \\ X^* & \xleftarrow{g^*} & Y^* & \longleftarrow & 0 \end{array}$$

in Comod-C with a monomorphism $g^* : Y^* \to X^*$. Since the comodule $\mathrm{Rat}(F^*)$ is injective, there is a C-comodule homomorphism $\bar{h} : X^* \to \mathrm{Rat}(F^*)$ such that $\bar{h}g^* = f^*$. Let $h = \bar{h}^*\alpha_F : F \to Y^{**}$. One may check that $g^{**}h = \lambda_Y f$ and, hence, we get $f = \lambda_Y^{-1} g^{**}\bar{h}^*\alpha_F = g\lambda_Y^{-1}\bar{h}^*\alpha_F$. Then the map $\lambda_Y^{-1}\bar{h}^*\alpha_F : F \to X$ is the desired C-comodule homomorphism making the diagram (4.4a) commutative. This shows that F is fin-projective.

(a)$\Rightarrow$ (b) Assume that F is fin-projective. We prove that $\mathrm{Rat}(F^*)$ is injective by showing that any diagram

$$(*) \qquad \begin{tikzcd} 0 \arrow[r] & X \arrow[r, "g"] \arrow[d, "f"] & Y \\ & \mathrm{Rat}(F^*) & \end{tikzcd}$$

in Comod-C, with a monomorphism $g : X \to Y$ in comod-C, can be completed to a commutative diagram by a C-comodule homomorphism $h : Y \to \mathrm{Rat}(F^*)$, see [26, Lemma 11]. The diagram give rise to the dual diagram

$$\begin{tikzcd} 0 & X^* \arrow[l] & Y^* \arrow[l, "g^*"'] \\ & F \arrow[u, "f^*\alpha_F"'] & \end{tikzcd}$$

in C-Comod, with an epimorphism $g^* : Y^* \to X^*$ in C-comod. By our hypothesis, there is a C-comodule homomorphism $\bar{h} : F \to Y^*$ such that $g^*\bar{h} = f^*\alpha_F$. Consider $\bar{h}^* : Y^{**} \to \mathrm{Rat}(F^*)$ and let $h = \bar{h}^*\lambda_Y : Y \to \mathrm{Rat}(F^*)$. It is easy to check that $hg = f$, that is, $h : Y \to \mathrm{Rat}(F^*)$ completes $(*)$ to a commutative diagram.

(a)$\Leftrightarrow$(c) Note that (a) means that the functor $h^F = \mathrm{Hom}_C(F, -) : C\text{-comod} \longrightarrow \mathrm{Mod}\, K$ is exact. To prove (a)$\Rightarrow$(c), we assume that h^F is exact and we show that h^F is FP-injective, compare with [34]. Since any finitely presented object T of $\mathcal{D}_l(C)$ has the form $T = h^X/I$, where $h^X = \mathrm{Hom}_C(X, -)$, X is in C-comod and I is finitely generated subfunctor of h^X (see [2, Chapter IV]), then it is sufficient to show that any morphism $\varphi : I \to h^F$ extends to a morphism $\varphi' : h^X \to h^F$.

Since I is finitely generated, there exists an epimorphism $\psi : h^Y \to I$, with Y in C-comod. The composed morphism $h^Y \xrightarrow{\psi} I \hookrightarrow h^X$ has the form $h^f = \mathrm{Hom}_C(f, -)$, where $f : X \to Y$ is a C-comodule homomorphism. Let $\pi : Y \longrightarrow \overline{Y} = Y/\mathrm{Im}\, f$ be the quotient epimorphism. Then the sequence $0 \to h^{\overline{Y}} \xrightarrow{h^\pi} h^Y \xrightarrow{h^f} h^X$ is exact and $I = \mathrm{Im}\, h^f = \mathrm{Ker}\, h^\pi$. Hence we derive the exact sequence $0 \longrightarrow \mathrm{Hom}_{\mathcal{D}_l(C)}(I, h^F) \xrightarrow{\widetilde{\psi}} \mathrm{Hom}_{\mathcal{D}_l(C)}(h^Y, h^F) \xrightarrow{\widetilde{h^\pi}} \mathrm{Hom}_{\mathcal{D}_l(C)}(h^{\overline{Y}}, h^F)$, where $\widetilde{\psi}$ is the map induced by the epimorphism ψ. On the other hand, by the exactness of h^F and the Yoneda Lemma, the exact sequence $X \xrightarrow{f} Y \xrightarrow{\pi} \overline{Y} \longrightarrow 0$ induces the commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{D}_l(C)}(h^X, h^F) \arrow[r, "\widetilde{h^f}"] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathcal{D}_l(C)}(h^Y, h^F) \arrow[r, "\widetilde{h^\pi}"] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathcal{D}_l(C)}(h^{\overline{Y}}, h^F) \arrow[r] \arrow[d, "\cong"] & 0 \\ h^F(X) \arrow[r, "h^F(f)"] & h^F(Y) \arrow[r, "h^F(\pi)"] & h^F(\overline{Y}) \arrow[r] & 0 \end{tikzcd}$$

with exact rows. Hence, if $\varphi \in \mathrm{Hom}_{\mathcal{D}_l(C)}(I, h^F)$ then $\widetilde{h^\pi}(\widetilde{\psi}(\varphi)) = 0$ and, by the exactness of the top row in the diagram above, there is $\varphi' \in \mathrm{Hom}_{\mathcal{D}_l(C)}(h^X, h^F)$ such that $\widetilde{h^f}(\varphi') = \widetilde{\psi}(\varphi)$. It follows that φ' extends φ, and the implication (a)$\Rightarrow$(c) is proved.

To prove the inverse implication (c) $\Rightarrow$(a), assume that h^F is FP-injective in $\mathcal{D}_l(C)$ and apply the functor $h^{(-)} : C\text{-Comod} \longrightarrow \mathcal{D}_l(C)$ to the diagram (4.4). We get the dual diagram in $\mathcal{D}_l(C)$ with a monomorphism $h^g : h^Y \to h^X$ and, by the FP-injectivity of h^F, there is a morphism $\xi : h^X \to h^F$ such that $h^f = \xi h^g$. Since ξ has the form $\xi = h^t$, for some $t : F \to X$, then $h^f = \xi h^g = h^t h^g = h^{gt}$ and we get $f = gt$, see [30, Section 30 1.3] and [33, Theorem 2.8]. Consequently, t completes (4.4a) to a commutative diagram. The proof is complete. □

We are now in a position to prove the main result of this section. It shows how far are the coalgebras with enough flat left comodules from the left semiperfect coalgebras. In particular, we give a partial answer to Problem 2.9 in the case $\mathcal{A} = C$-Comod. This result also gives a new characterization of left semiperfect coalgebras.

THEOREM 4.6. *Let C be a coalgebra. The following assertions are equivalent.*

(a) *The category C-*Comod *has enough flat left objects and every flat left C-comodule is fin-projective.*

(b) *The category C-*Comod *has enough flat left objects and* $\mathrm{Rat}(F^*)$ *is injective in* Comod-C *for every flat left C-comodule F.*

(c) *The category C-*Comod *has enough flat left objects and every flat left C-comodule is flat when viewed as a right C^*-module.*

(d) *The coalgebra C is left semiperfect.*

Proof. (a)⇔ (b) It follows from Proposition 4.5.

(d)⇒(b) and (d)⇒(c) Assume that C is left semiperfect. Then the category C-Comod has enough projective (hence flat) objects. Furthermore, by Theorem 3.5, every flat left C-comodule F is flat when viewed as a right C^*-module and the left C-comodule $\mathrm{Rat}(F^*)$ is injective.

(b)⇒(d) Given a vector space V, the K-dual space V^* is equipped with the finite topology. A subspace Y of V^* is dense in V^* if and only if $Y^{\perp(V)} = \{0\}$. It follows from [26, Proposition 9, Theorem 10] that a coalgebra C is left semiperfect if and only if $\mathrm{Rat}(E(S)^*)$ is dense in $E(S)^*$, for each simple right C-comodule S. The following arguments are inspired by the proof of the implication (a)⇒(b) in [26, Theorem 10].

Let S be a simple right C-comodule. We show that $\mathrm{Rat}(E(S)^*)$ is dense in $E(S)^*$. Since S^* is a simple left C-comodule then, by our hypothesis, there is an epimorphism $f : F \to S^*$ in C-Comod with F flat. The dual map $f^* : S \to F^*$ is a monomorphism of left C^*-modules. Let $i : S \to E(S)$ be the injective hull of S. Since, by hypothesis, $\mathrm{Rat}(F^*)$ is injective in Comod-C then there exists a homomorphism $h : E(S) \to \mathrm{Rat}(F^*)$ of right C-comodules such that $ih = f^*$. Since i is essential and f^* is injective, h is injective. Hence $\mathrm{Rat}(F^*) \cong E(S) \oplus N$, for some right C-comodule N. Then $\mathrm{Rat}(\mathrm{Rat}(F^*)^*) \cong \mathrm{Rat}(E(S)^*) \oplus \mathrm{Rat}(N^*)$. Now consider the epimorphism $j^* : F^{**} \to \mathrm{Rat}(F^*)^*$ induced by the inclusion $j : \mathrm{Rat}(F^*) \to F^*$. Since $\mathrm{Rat}(F^{**})$ is dense in F^{**} (it contains F), then $\mathrm{Rat}(\mathrm{Rat}(F^*)^*)$ is dense in $\mathrm{Rat}(F^*)^*$. Consequently, $\mathrm{Rat}(E(S)^*)$ is dense in $E(S)^*$ and we are done.

(c)⇒(b) Suppose that F is flat in C-Comod. Then, by hypothesis, F is flat as a right C^*-module. Hence F^* is injective as a left C^*-module and, consequently, the left C-comodule $\mathrm{Rat}(F^*)$ is injective in Comod-C. This completes the proof. □

REMARK 4.7. (a) Observe that in the proof of the theorem the first hypothesis in statements (a), (b) and (c) may be replaced by the following one: *Every simple left C-comodule has a non-zero flat cover.*

(b) Using Lemma 5.3 of Section 5, one can prove that the statements of Theorem 4.6 are equivalent to this other one: *The category C-*Comod *has flat covers and the flat cover of any* *object in* C-comod *lies in* C-comod.

We do not know whether in general $\mathrm{Rat}(F^*)$ is injective in Comod-C, for every flat left C-comodule F. An answer depends on the following problem.

PROBLEM 4.8. Consider the diagram (4.4) in C-Comod with F flat and an epimorphism $g : X \to Y$ in C-comod. Let

$$\begin{array}{ccccc} P & \xrightarrow{g'} & F & \longrightarrow & 0 \\ \Big\downarrow{\scriptstyle f'} & & \Big\downarrow{\scriptstyle f} & & \\ X & \xrightarrow{g} & Y & \longrightarrow & 0. \end{array}$$

be the pull-back of the maps f and g.

Consider the set $\mathcal{F}$ consisting of pairs of the form (N, h_N) where N is a subcomodule of F and $h_N : N \to X$ is a C-comodule homomorphism such that $gh_N = f|_N$. Let N be a finite dimensional subcomodule of F and $i_N : N \to F$ be the inclusion map. Since F is flat, there is a C-comodule homomorphism $h : N \to P$ such that $g'h = i_N$. Hence $g(hf') = f|_N$. This shows that the set $\mathcal{F}$ is not empty. We view $\mathcal{F}$ as a partially ordered set by defining $(N, h_N) \le (N', h_{N'})$ if $N \subseteq N'$ and $h_{N'}|_N = h_N$.

It is easy to check that $\mathcal{F}$ is an inductive set and then, by Zorn's Lemma, there exists a maximal element (M, h_M) in $\mathcal{F}$.

We claim that M is an essential C-subcomodule of F. For, if M is not essential then $M \cap (x) = \{0\}$, for some $x \in F$, where (x) denotes the C-subcomodule generated by x. Since F is flat, there is a map $h_{(x)} : (x) \to X$ such that $gh_{(x)} = f|_{(x)}$. Then the map $h = h_M \oplus h_{(x)} : M \oplus (x) \to X$ satisfies $gh = f|_{M \oplus (x)}$ and extends to h_M. This contradicts the maximality of (M, h_M).

We do not know whether $M = F$. In affirmative case, it would follow from Proposition 4.5 that $\mathrm{Rat}(F^*)$ is injective in C-Comod, for every flat left C-comodule F providing a positive answer to Question 4.1. Then Theorem 4.6 would also answer in the affirmative Problem 2.9 for C arbitrary.

5. Perfect coalgebras

This section is devoted to the study of coalgebras with enough flat covers and with enough projective covers in the following sense, compare with Bass [5].

DEFINITION 5.1. A coalgebra C is defined to be **left perfect** if every comodule L in C-Comod admits a projective cover epimorphism $P \to L$.

REMARK 5.2. (a) The condition that every comodule in C-Comod admits a projective cover epimorphism is equivalent to the existence of a non-zero projective cover of any comodule. Indeed, in this case any simple comodule has a non-zero projective cover, that is surjective. It then follows, as in [26, Theorem 10], that every finite dimensional comodule has a projective cover epimorphism and, hence, every comodule has a projective cover epimorphism.

(b) In Definition 5.1 we require that a projective cover $P \to L$ of any comodule L is an epimorphism, because otherwise we would admit zero projective covers $0 \to L$. An example of this type is provided by the coalgebra given in Example 4.2 since it has not non-zero projective right C-comodules.

(c) Any left perfect coalgebra is left semiperfect.

The following technical fact will be needed later on.

Lemma 5.3. *Let M be a left C-comodule and $p : P \to M$ a projective cover of M. If $f : F \to M$ is a flat cover of M, then $M \cong P$. In particular, if M is flat, then $M \cong P$.*

Proof. It is well-known that $p : P \to M$ is a projective cover in the sense of Bass [5] if and only if $p : P \to M$ is a precover satisfying the minimality condition. Then there exist C-comodule homomorphisms $h : F \to P$ and $h' : P \to F$ such that $p = fh'$ and $f = ph$. It follows that $p = phh'$ and $f = fh'h$. Hence $1_P = hh'$ and $1_F = h'h$, that is, $M \cong P$. □

Before going on, we analyze in details the following hereditary left semiperfect coalgebra that is not left perfect. This coalgebra plays a distinguished role in the theory of pure-semisimple coalgebras and serial coalgebras, see [29], [24] and [12].

Example 5.4. Let $C = KQ^\circ$ be the path coalgebra of the quiver dual to $Q = \mathbb{A}_\infty^{(0)}$

$$Q^\circ : \quad 0 \longleftarrow 1 \longleftarrow 2 \longleftarrow 3 \longleftarrow \ldots \longleftarrow m \longleftarrow \ldots$$

For each $u, i \in \mathbb{N}$, we denote by $p_{u+i,u}$ the path starting in the vertex $u+i$ of Q° and ending in the vertex u. The comultiplication and counit of C are defined by the formulae $\Delta(p_{u+j,u}) = \sum_{s=0}^{j} p_{u+s,u} \otimes p_{u+j,u+s}$, and $\varepsilon(p_{u+j,u}) = \delta_{j,0}$, where $\delta_{j,0}$ is the Kronecker symbol. The set $\mathcal{B} = \{p_{u+j,u};\ u, j \in \mathbb{N}\}$ is a K-basis of C. For each $v \in \mathbb{N}$, $S(v) = kp_{v,v}$ is a simple left subcomodule of C and the family $\{S(v)\}_{v\in\mathbb{N}}$ is a complete set of representatives of simple left C-comodules. The injective hull of $S(v)$ is the left coideal $E(v) = \oplus_{j\geq v} Kp_{j,v}$ of C. The projective cover $P(v)$ of $S(v)$ is the subcomodule $P(v) = \oplus_{j\leq v} Kp_{j,0}$ of $E(0)$. For each $v \geq 0$, we consider the exact sequence

$$(*) \qquad 0 \longrightarrow P(v) \longrightarrow E(0) \xrightarrow{f_v} E(v+1) \longrightarrow 0$$

in C-Comod, where f_v is the surjective homomorphism of C-comodules defined by the formula

$$f_v(p_{j,0}) = \begin{cases} p_{j,v+1} & if\ j \geq v+1, \\ 0 & if\ j \leq v. \end{cases}$$

The coalgebra C is right pure-semisimple. Then C is left semiperfect and, according to Example 4.2, C has no non-zero flat right comodules. So C is not right perfect. By Theorem 3.9 every left C-comodule has a flat cover. We next prove the following statements:

(1) For each $v \geq 0$, $\mathrm{End}_C E(v) \cong K$ and $\mathrm{End}_C P(v) \cong K$.

(2) The family $\{P(v)\}_{v\in\mathbb{N}}$ is a complete set of representatives of the indecomposable projective left C-comodules.

(3) For each $v \geq 0$, the chain $P(0) \subset P(1) \subset \ldots \subset P(v)$ is a unique composition series of the projective comodule $P(v)$, $P(v)/P(v-1) \cong S(v)$, and the canonical epimorphism $P(v) \to S(v)$ is a projective cover of $S(v)$.

(4) The comodule $E(0)$ is uniserial, flat, non-projective, and has no projective cover. Each proper non-zero subcomodule of $E(0)$ is one of the comodules in the chain $P(0) \subset P(1) \subset \ldots \subset P(m) \subset \ldots$ and $E(0) = \bigcup_{v=0}^{\infty} P(v)$.

(5) For each $v \geq 1$, the injective comodule $E(v)$ is not flat and the homomorphism f_v in $(*)$ is a flat cover of $E(v)$.

(6) The coalgebra C is neither left nor right perfect.

In the proof we apply the equivalence of categories C-Comod $\cong \mathrm{Rep}_K^{\ell f}(Q^\circ) = \mathrm{Rep}_K(Q^\circ)$ established in [8] and [38]. By applying this equivalence one can show that the comodules $S(v)$, $E(v)$, and $P(v)$ can be viewed as the representations of Q°

$$S(v): \qquad 0 \longleftarrow 0 \longleftarrow 0 \longleftarrow 0 \longleftarrow 0 \longleftarrow \dots \longleftarrow 0 \longleftarrow \underset{v}{K} \longleftarrow 0 \longleftarrow \dots$$

$$P(v): \quad K \xleftarrow{1_K} K \xleftarrow{1_K} K \xleftarrow{1_K} K \xleftarrow{1_K} \dots \xleftarrow{1_K} K \xleftarrow{1_K} \underset{v}{K} \longleftarrow 0 \longleftarrow 0 \longleftarrow \dots$$

$$E(v): \quad 0 \longleftarrow 0 \longleftarrow 0 \longleftarrow 0 \longleftarrow \dots \longleftarrow 0 \longleftarrow 0 \longleftarrow \underset{v}{K} \xleftarrow{1_K} K \xleftarrow{1_K} K \xleftarrow{1_K} \dots$$

Hence, by applying the quiver representation technique (see [2, Section III.2], [3, Section III.1], [24, Proposition 2.7] and [35, Section 2]), the statements (1)-(4) easily follow. To see that $E(0)$ is flat, we apply Corollary 3.6. Furthermore, the comodule $E(0)$ is not projective because of (2), and $E(0)$ has no projective cover by Lemma 5.3.

To prove (5), assume to the contrary, that $v \geq 1$ and $E(v)$ is flat. Since the coalgebra is hereditary then, by Corollary 3.6, $E(v)$ is a directed union of projective subcomodules of $E(v)$ of finite dimension. It follows that there is a monomorphism $P(j) \to E(v)$, for some $j \geq 0$. This is a contradiction because by looking at the representations $P(j)$ and $E(v)$ presented above we get the following:

(7) There is a monomorphism $P(j) \to E(v)$ if and only if $v = 0$.

(8) There is no non-zero C-comodule homomorphism $h : P(j) \to P(r)$ if and only if $j \leq r$. If $h \neq 0$, then h is injective.

Now we prove that the homomorphism $f_v : E(0) \to E(v)$ in $(*)$ is a flat cover of $E(v)$, for $v \geq 1$. Let F be a flat left C-comodule. We show that the linear map $\mathrm{Hom}_C(F, f_v) : \mathrm{Hom}_C(F, E(0)) \longrightarrow \mathrm{Hom}_C(F, E(v+1))$ is surjective. Since C is hereditary then, by Corollary 3.6, F has a directed union form $F = \bigcup_{\beta \in T} P_\beta$, where each P_β is a projective subcomodule of F of finite dimension. It follows that P_β is a finite direct sum of copies of the comodules $P(0), P(1), P(2), \dots, P(v), P(v+1), \dots$, and the decomposition is unique up to isomorphism. For each $\beta \in T$ we fix such a decomposition of P_β and consider the decomposition $P_\beta = P'_\beta \oplus \overline{P_\beta}$, where P'_β and $\overline{P_\beta}$ is the direct sum of the summand of P_β isomorphic to any of the comodules $P(v+1), P(v+2), \dots$ and $P(0), P(1), P(2), \dots P(v)$, respectively. In view of (8) , it is easy to see that, for $\gamma > \beta$, the embedding $u_{\gamma,\beta} : P_\beta \to P_\gamma$ restricts to an embedding $u'_{\gamma,\beta} : P'_\beta \to P'_\gamma$ and induces the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P'_\beta & \longrightarrow & P_\beta & \xrightarrow{\pi_\beta} & \overline{P_\beta} & \longrightarrow & 0 \\ & & \big\downarrow {\scriptstyle u'_{\gamma,\beta}} & & \big\downarrow {\scriptstyle u_{\gamma,\beta}} & & \big\downarrow {\scriptstyle \overline{u}_{\gamma,\beta}} & & \\ 0 & \longrightarrow & P'_\gamma & \longrightarrow & P_\gamma & \xrightarrow{\pi_\gamma} & \overline{P_\gamma} & \longrightarrow & 0, \end{array}$$

where $u'_{\gamma,\beta}$ and $\overline{u}_{\gamma,\beta}$ are monomorphisms. By passing to the direct limits we get the short exact sequence

$$(**) \qquad 0 \longrightarrow F' \xrightarrow{u'} F \xrightarrow{\pi} \overline{F} \longrightarrow 0$$

in C-Comod, where $F' = \bigcup_{\beta \in T} P'_\beta$ and $\overline{F} = \varinjlim_{\beta \in T} \overline{P_\beta}$ are flat. The exact sequences $(*)$ and $(**)$ yield the commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
0 \longrightarrow & \mathrm{Hom}_C(F', P(v)) & \longrightarrow & \mathrm{Hom}_C(F', E(0)) & \xrightarrow{\theta'} & \mathrm{Hom}_C(F', E(v+1)) & \\
 & \uparrow & & \uparrow & & \uparrow \varphi & \\
0 \longrightarrow & \mathrm{Hom}_C(F, P(v)) & \longrightarrow & \mathrm{Hom}_C(F, E(0)) & \xrightarrow{\theta} & \mathrm{Hom}_C(F, E(v+1)) & \\
 & \uparrow \psi & & \uparrow & & \uparrow & \\
0 \longrightarrow & \mathrm{Hom}_C(\overline{F}, P(v)) & \longrightarrow & \mathrm{Hom}_C(\overline{F}, E(0)) & \longrightarrow & \mathrm{Hom}_C(\overline{F}, E(v+1)) & \\
 & \uparrow & & \uparrow & & \uparrow & \\
 & 0 & & 0 & & 0 &
\end{array}$$

with exact rows and columns, where $\theta' = \mathrm{Hom}_C(F', f_v)$ and $\theta = \mathrm{Hom}_C(F, f_v)$. Note that

$$\begin{aligned}
\mathrm{Hom}_C(F', P(v)) &= \mathrm{Hom}_C(\bigcup_{\beta \in T} P'_\beta, P(v)) \cong \varprojlim_{\beta \in T} \mathrm{Hom}_C(P'_\beta, P(v)) = 0,\\
\mathrm{Hom}_C(\overline{F}, E(v+1)) &= \mathrm{Hom}_C(\varinjlim_{\beta \in T} \overline{P_\beta}, E(v+1)) \cong \varprojlim_{\beta \in T} \mathrm{Hom}_C(\overline{P_\beta}, E(v+1)) = 0,
\end{aligned}$$

by (7), (8), and the definition of P'_β and $\overline{P_\beta}$. Thus, the maps ψ and φ are isomorphisms. Moreover, since P'_β is projective and $\mathrm{Hom}_C(P'_\beta, P(v)) = 0$, in view of (8), then the map $\theta'_\beta = \mathrm{Hom}_C(P'_\beta, f_v) : \mathrm{Hom}_C(P'_\beta, E(0)) \longrightarrow \mathrm{Hom}_C(P'_\beta, E(v+1))$ is bijective. It follows that θ' is bijective as an inverse limit of the bijections θ'_β. Consequently, the map $\theta = \mathrm{Hom}_C(F, f_v) : \mathrm{Hom}_C(F, E(0)) \longrightarrow \mathrm{Hom}_C(F, E(v+1))$ is surjective, and we are done.

It remains to show that if $g : E(0) \to E(0)$ is such that $f_v g = f_v$, then g is bijective. Since $f_v g = f_v$ yields $g \neq 0$ then, in view of (1), $g \in \mathrm{End}_C E(0) \cong K$ is invertible. The statement (6) is a consequence of remaining ones because C has no non-zero flat right comodules and the left C-comodule $E(0)$ has no projective cover by (4).

Now we give a characterization of perfect coalgebras analogous to the famous Theorem P of Bass [5] (compare with [32]). To this end we need some conventions. Given a coalgebra C, let J denote the Jacobson radical of C^*. It is known that $J = C_0^{\perp(C^*)}$, where $C_0 \subseteq C$ is the coradical of C, see [27, Proposition 5.2.9]. For any right C-comodule M, the quotient space M/MJ is viewed as a left C_0-comodule ([26, Lemma 14]) and hence it is a semisimple left C-comodule.

Theorem 5.6. *Let C be a coalgebra. The following assertions are equivalent.*

(a) *C is left perfect.*

(b) *C is left semiperfect and every flat left C-comodule is projective.*

(c) *C is left semiperfect and every left comodule has a maximal subcomodule.*

(d) *C is left semiperfect and $MJ \neq M$ for any non-zero left C-comodule M.*

(e) *C is left semiperfect and MJ is superfluous in M for any non-zero left C-comodule M.*

(f) *C is left semiperfect and given a sequence $P_1 \xrightarrow{f_1} P_2 \xrightarrow{f_2} \dots \longrightarrow P_m \xrightarrow{f_m} \dots$ in* C-comod *of non-zero non-isomorphisms $f_1, \dots, f_m, \dots$ between indecomposable projective C-comodules $P_1, P_2, \dots, P_m, \dots$, there exists $m \geq 2$ such that $f_m \dots f_2 f_1 = 0$.*

(g) *C has a decomposition $C = \bigoplus_{i \in I} E(i)$ in* Comod-C*, where $E(i)$ is indecomposable of finite dimension, and for any infinite sequence $E(i_1) \xleftarrow{g_1} E(i_2) \xleftarrow{g_2} \dots \longleftarrow E(i_m) \xleftarrow{g_m} \dots$ in* comod-C

of non-zero non-isomorphisms $g_1, \ldots, g_m, \ldots$ there exists $m \geq 2$ such that $g_1 g_2 \ldots g_m = 0$.

Proof. This result follows from [33, Theorem 5.4] where a characterization of perfect functor categories is provided. For the sake of completeness we give a direct proof with arguments similar to those used in proving Theorem P of Bass, as done in [5, Theorem 28.4]. Nevertheless, for the equivalence (a) $\Leftrightarrow$ (f), [33, Theorem 5.4] is invoked.

(a)$\Rightarrow$(b) Clearly C is left semiperfect. If F is a flat left C-comodule and $P \twoheadrightarrow F$ is a projective cover of F in C-Comod then, by Lemma 5.3, $P \cong F$ and hence F is projective.

(b)$\Rightarrow$(a) If C is left semiperfect, every left comodule has a flat cover by Theorem 3.9. Since flat left comodules are projective by hypothesis, (a) follows.

(a)$\Rightarrow$(c) Obviously, C is left semiperfect. Let M be an arbitrary left C-comodule and $g : P \twoheadrightarrow M$ its projective cover in C-Comod. Then $N = \operatorname{Ker} g$ is superfluous in P and $P/N \cong M$. By [19, Lemma 2.1], P is projective as a right C^*-module. By [1, Proposition 17.14], P has a maximal C^*-submodule L. Then L is a maximal subcomodule of P. Now $N \subseteq L$, because N is superfluous in P. Hence L/N is a maximal subcomodule of $P/N \cong M$.

(c)$\Rightarrow$(d) Assume that M is a non-zero left C-comodule and let L be a maximal subcomodule of M. Since M/L is simple, $(M/L)J = 0$. Then $MJ \subseteq L$, and so $MJ \neq M$.

(d)$\Rightarrow$(e) Take a non-zero left C-comodule M and let N be a proper subcomodule of it. By hypothesis, $(M/N)J \neq M/N$. This yields that $MJ + N \neq M$.

(e)$\Rightarrow$(a) Let M be a left C-comodule and let $\pi : M \to M/MJ$ be the canonical projection. Let $\{S_i\}_{i\in I}$ be a family of simple left C-comodules such that $M/MJ \cong \bigoplus_{i\in I} S_i$. As C is left semiperfect, each S_i has a projective cover $g_i : P_i \twoheadrightarrow S_i$ with kernel P_iJ. Set $P = \bigoplus_{i\in I} P_i$ and $g = \oplus_{i\in I} g_i$. There is $f : P \to M$ such that $\pi f = g$. By our hypothesis, PJ and MJ are superfluous in P and M, respectively. Then

$$\operatorname{Ker} g = \operatorname{Ker} \bigoplus_{i\in I} g_i = \bigoplus_{i\in I} \operatorname{Ker} g_i = \bigoplus_{i\in I} P_iJ = (\bigoplus_{i\in I} P_i)J = PJ.$$

It follows that $\operatorname{Ker} g = PJ$ is superfluous in P. Since g is surjective, $f(P) + MJ = M$, and then $f(P) = M$. Furthermore, $\operatorname{Ker} f$ is superfluous in P because $\operatorname{Ker} f \subseteq PJ$. Hence $f : P \twoheadrightarrow M$ is a projective cover of M.

(a)$\Leftrightarrow$(f) If C is left semiperfect then the category C-Comod has a set $\{P_\beta\}_{\beta\in T}$ of finite dimensional indecomposable projective generators. Then there is an equivalence of categories $C\text{-Comod} \cong \operatorname{Add}(\mathcal{P}^{op}, \operatorname{Mod} K)$, where $\mathcal{P}$ is the full subcategory of C-comod with the objects P_β, $\beta \in T$, and $\operatorname{Add}(\mathcal{P}^{op}, \operatorname{Mod} K)$ is the category of all contravariant functors from $\mathcal{P}$ to the category $\operatorname{Mod} K$ of vector spaces, see [16]. Since each of the conditions (a) and (f) implies that C is left semiperfect then the equivalence (a)$\Leftrightarrow$(f) follows from the characterization of perfect functor categories given in [32] and [33, Theorem 5.4]. Note that the Jacobson radical of the category $\mathcal{C} = \mathcal{P}^{op}$ is left T-nilpotent if and only if given a sequence $P_1 \xrightarrow{f_1} P_2 \xrightarrow{f_2} \ldots \longrightarrow P_m \xrightarrow{f_m} \ldots$ in C-comod of non-zero non-isomorphisms $f_1, \ldots, f_m, \ldots$ between indecomposable projective C-comodules $P_1, P_2, \ldots P_m \ldots$, there exists $m \geq 2$ such that $f_m \ldots f_2 f_1 = 0$.

(f)$\Leftrightarrow$(g) By [26, Theorem 10], the coalgebra C is left semiperfect if and only if $C = \bigoplus_{i\in I} E(i)$ in Comod-C, where each $E(i)$ is indecomposable of finite dimension. Then the equivalence (f)$\Leftrightarrow$(g) follows from the duality $(C\text{-comod})^{op} \cong \text{comod-}C$. This completes the proof. $\square$

We next establish a criterion for a quiver $Q = (Q_0, Q_1)$ to have the path coalgebra KQ left semiperfect and left perfect, respectively (compare with [24, Theorem 4.7]). The characterization of left semiperfect path coalgebras appears in [7, Corollary 6.3(a)] and [38, Proposition

8.1(b)]. To formulate it we introduce some notation. Given a vertex $s \in Q_0$, we denote by $Q(s\to)$ and $Q(\to s)$ the set of all paths in Q starting from s and ending at s, respectively.

COROLLARY 5.7. *Let $Q = (Q_0, Q_1)$ be a quiver and KQ the path coalgebra of Q.*

(a) *The coalgebra KQ is left semiperfect if and only if, for every vertex $s \in Q_0$, the set $Q(s\to)$ is finite.*

(b) *The coalgebra KQ is left perfect if and only if, for every vertex $s \in Q_0$, the set $Q(s\to)$ is finite and there is no infinite path $\ldots \to \bullet \to \ldots \to \bullet \to \bullet \to \bullet$ in Q.*

Proof. (a) Let $C = KQ$ be the path coalgebra of Q. It is known that the right C-comodule KQ has the decomposition $KQ = \bigoplus_{s\in Q_0} e_s KQ$ in Comod-C, where e_s is the stationary path at s and $e_s KQ = KQ(s\to)$ is the K-vector space generated by $Q(s\to)$. By [26, Theorem 10], $C = KQ$ is left semiperfect if and only if the injective hull $E(Ke_s)$ of every simple right C-comodule Ke_s is finite dimensional. But $E(Ke_s) = KQ(s\to)$, so the statement follows.

(b) Let $C = KQ$. Since, up to isomorphism, every indecomposable injective right C-comodule is of the form $e_s KQ$, for some $s \in Q_0$, then by Theorem 5.7 the coalgebra KQ is left perfect if and only if KQ is left semiperfect and for any infinite sequence

$$e_{s_1}KQ \xleftarrow{f_1} e_{s_2}KQ \xleftarrow{f_2} \ldots \longleftarrow e_{s_m}KQ \xleftarrow{f_m} \ldots$$

in comod-C of non-zero non-isomorphisms $f_1, \ldots, f_m, \ldots$ there exists $m \geq 2$ such that $f_1 f_2 \ldots f_m = 0$. Since C is hereditary, quotients of injectives are injectives and, hence, each of the maps f_j is surjective. It follows that there is no such an infinite sequence. By (a), C is semiperfect if and only if the set $Q(s\to)$ is finite, for each $s \in Q_0$. Since $\mathrm{Hom}_C(e_s KQ, e_t KQ) \cong e_s KQ e_t \cong KQ(s,t)$, where $Q(s,t)$ is the set of all paths from s to t (see [38]), then the infinite sequence condition above holds if and only if Q has no infinite path $\ldots \to \bullet \to \ldots \to \bullet \to \bullet \to \bullet$. Consequently, (b) follows and the proof is complete. □

PROPOSITION 5.8. (a) *Every subcoalgebra of a left perfect coalgebra is left perfect.*

(b) *Let $\{C_i\}_{i\in I}$ be a family of coalgebras. Then $\bigoplus_{i\in I} C_i$ is left perfect if and only if each C_i is left perfect.*

(c) *Let C and D be two Morita-Takeuchi equivalent coalgebras. Then C is left perfect if and only if D is left perfect.*

Proof. (a) Let D be a subcoalgebra of a left perfect coalgebra C. Since C is left semiperfect, D is left semiperfect. On the other hand, every left D-comodule M is a left C-comodule. By Theorem 5.6, M has a maximal C-subcomodule. But C-subcomodules of M are indeed D-subcomodules, so M has a maximal subcomodule.

(b) Let $C = \bigoplus_{i\in I} C_i$ and assume that each C_i is left perfect. Then C is left semiperfect, because each C_i is so. Any left C-comodule M is of the form $M = \bigoplus_{i\in i} M_i$ where each M_i is a left C_i-comodule. In particular, if M is flat, then each M_i is a flat C_i-comodule. By our hypothesis and Theorem 5.6, M_i is a projective C_i-comodule and, hence, M is a projective C-comodule. This proves that C is left perfect. The converse follows from (a).

(c) Note that projective cover epimorphisms are preserved under category equivalences. □

COROLLARY 5.9. *Assume that K is an algebraically closed field, C is a hereditary coalgebra and $Q = {}_CQ$ is the quiver opposite to the left* Ext*-quiver of C.*

(a) *The coalgebra C is left semiperfect if and only if, for every vertex $s \in {}_CQ_0$, the set ${}_CQ(s\to)$ is finite.*

(b) *The coalgebra C is left perfect if and only if, for every vertex $s \in {}_CQ_0$, the set ${}_CQ(s\!\to)$ is finite and there is no infinite path $\ldots \to \bullet \to \ldots \to \bullet \to \bullet \to \bullet$ in Q.*

Proof. We recall from [18, Section 7] (see also [9], [24], [38]) that the vertices of the left Ext-quiver of C are the pairwise non-isomorphic representatives S_j of simple left C-comodules, and there is an arrow $S_i \to S_j$ in the Ext-quiver of C if and only if $\mathrm{Ext}^1_C(S_j, S_i) \neq 0$. In view of Proposition 5.9, the left semiperfectness and left perfectness are Morita-Takeuchi invariant. We recall that the Ext-quiver is also Morita-Takeuchi invariant and the hereditary coalgebra C is Morita-Takeuchi equivalent to a hereditary basic coalgebra C' (see [6], [9], [38]). Since the field K is assumed to be algebraically closed, $C' \cong KQ$, see [6]. Hence, the corollary is a consequence of Corollary 5.7. $\square$

We provide a way to construct left perfect coalgebras from left semiperfect ones. Let $\{C_n\}_{n\in\mathbb{N}}$ be the coradical filtration of the coalgebra C. This is indeed the Loewy series of C, viewed as a left (or right) C-comodule. The Loewy series of any left C-comodule M may be constructed from the coradical filtration. Let $\{\mathrm{soc}^n(M)\}_{n\in\mathbb{N}}$ be the Loewy series of M and $\delta : M \to C \otimes M$ its structure map. Then $\mathrm{soc}^{n+1}(M) = \delta^{-1}(C_n \otimes M)$.

PROPOSITION 5.10. *Let C be a left semiperfect coalgebra.*
(a) *The coalgebra C_n is left perfect for all $n \in \mathbb{N}$.*
(b) *If $C = C_n$ for some $n \geq 0$, then C is left perfect.*

Proof. (a) Since C is left semiperfect, C_n is also left semiperfect. Any left C_n-comodule M satisfies $M = \mathrm{soc}^{n+1}(M)$ and hence M has finite Loewy series. From the definition of the Loewy series, $M/\mathrm{soc}^n(M)$ is a semisimple left C-comodule. Then M has a maximal subcomodule and (a) follows. Since (b) is a consequence of (a), the proposition is proved. $\square$

PROPOSITION 5.11. *A coalgebra C is left and right perfect if and only if C is left and right semiperfect.*

Proof. This is just rephrasing [26, Corollary 18] which states that C is left and right semiperfect if and only if every right C-comodule has a projective cover and every left C-comodule has a projective cover. $\square$

We finish this section by giving an example of right perfect coalgebra that is not left perfect.

EXAMPLE 5.12. Let $C = KQ$ be the path coalgebra of the infinite quiver

$$Q: \quad \begin{array}{ccccc} 1 & 2 & 3 \;\ldots & m & \ldots \\ \searrow & \downarrow & \swarrow \cdots & & \\ & 0 & & & \end{array}$$

The coalgebra C is right semiperfect but not left semiperfect and, consequently, C is not left perfect. Since all paths in Q are of length at most 1 then $C = C_1$ and Proposition 5.11 yields that C is right perfect.

6. Left IF-coalgebras and left quasi-coFrobenius coalgebras

In connection with left quasi-coFrobenius coalgebras studied in [19] and [20] in this section we introduce left IF-coalgebras in the following sense, compare with Colby [10].

DEFINITION 6.1. A coalgebra C is defined to be a **left IF-coalgebra** if the left C-comodule C is flat.

Left IF-coalgebras are characterized as follows.

LEMMA 6.2. *A coalgebra C is a left IF-coalgebra if and only if every injective left C-comodule is flat.*

Proof. Let $\{S_i\}_{i\in I}$ be a complete set of pairwise non-isomorphic representatives of simple left C-comodules. Then $\{E(S_i)\}_{i\in i}$ is a complete set of pairwise non-isomorphic representatives of injective indecomposable left C-comodules and there is a left C-comodule decomposition $C = \bigoplus_{i\in I} E(S_i)^{n_i}$, where each n_i a positive integer, see [22, 1.5g].

Assume that ${}_CC$ is flat. Then each $E(S_i)$ is flat and, since the category C-Comod is locally finite, every injective left C-comodule is a direct sum of copies of the comodules $E(S_i)$. Hence every injective left C-comodule is flat. The converse implication is obvious. □

We recall that C is a **left quasi-coFrobenius coalgebra** if C viewed as a right C-comodule is projective. A relation between left IF-coalgebras and left quasi-coFrobenius coalgebras is next established.

PROPOSITION 6.3. *A coalgebra C is left quasi-coFrobenius if and only if C is a right IF-coalgebra and left semiperfect.*

Proof. If C is left quasi-coFrobenius, then C is left semiperfect and the right C-comodule C_C is projective, see [19, Theorem 1.3, Corollary 1.4]. It follows that C is a right IF-coalgebra. Conversely, assume that C is a right IF-coalgebra and left semiperfect. With notation as above, $C = \bigoplus_{i\in I} E(S_i)^{n_i}$ as a right C-comodule and each $E(S_i)$ is finite dimensional. Since C_C is flat, each finite dimensional comodule $E(S_i)$ is flat and hence $E(S_i)$ is projective for each $i \in I$. Consequently, the right comodule C is projective. □

As a consequence of our previous results we get the following characterization of coalgebras that are both left and right quasi-coFrobenius.

COROLLARY 6.4. *Let C be a coalgebra. The following conditions are equivalent:*
(a) *C is both left and right quasi-coFrobenius.*
(b) *C is both left and right IF-coalgebra and is both left and right perfect.*
(c) *C is both left and right perfect and the dual K-algebra C^* is both left and right self-injective.*

Proof. By Proposition 5.11, a coalgebra C is both left and right perfect if and only if C is both left and right semiperfect. Then the equivalence (a)⇔(b) follows from Proposition 6.3.

(a)⇒(c) By [19, Corollaries 1.4 and 1.8], a coalgebra C that is both left and right quasi-coFrobenius is both left and right semiperfect and C^* is both left and right self-injective.

(c)⇒(b) If C is both left and right perfect, C is both left and right semiperfect. Then, by Theorem 3.5 (a)⇔(d), applied to the comodule C, C is both left and right IF-coalgebra, if C^* is both left and right self-injective. This finishes the proof. □

We note that the equivalence (a)⇔(c) is proved in [20] by a different technique. We present an example of a right IF-coalgebra that is not a left IF-coalgebra.

EXAMPLE 6.5. Let $C = C_1$ be the first term of the coradical filtration of the path coalgebra of the infinite quiver $\mathbb{A}_{\infty}^{(0)}$, see Example 2.10. By Corollary 5.9, C is right and left semiperfect (paths of length greater than two are not allowed). By [19, Example 1.6], C is left quasi-coFrobenius but not right quasi-coFrobenius. Hence, in virtue of Proposition 6.3, C is a right IF-coalgebra, but C is not a left IF-coalgebra.

We next show that over a coalgebra C that is both left and right quasi-coFrobenius the classes of injective left (resp. right) C-comodules, projective left (resp.) C-comodules, and flat left (resp. right) C-comodules coincide. We derive it from the following useful fact.

PROPOSITION 6.6. *Assume that C is a coalgebra such that the left C-comodule ${}_CC$ is flat and a generator of* C-Comod. *Then a left C-comodule E is injective if and only if E is flat.*

Proof. By Proposition 6.2, every injective left C-comodule is flat. Conversely, assume that F is a flat left C-comodule. Since ${}_CC$ is a generator of the category C-Comod, F may be written as a quotient of a direct sum of copies of the injective comodule ${}_CC$. Then F is injective, by the following lemma, which is Proposition 2.3 (c) applied to C-Comod. □

LEMMA 6.7. *Assume that C is a coalgebra and $h : E \to F$ is a non-zero epimorphism in* C-Comod. *If the comodule E is injective and F is flat then F is injective.*

COROLLARY 6.8. *Assume that C is a coalgebra that is both left and right quasi-coFrobenius.*
(a) *A left (resp. right) C-comodule is flat if and only if it is injective.*
(b) *A left (resp. right) C-comodule is projective if and only if it is injective.*

Proof. (a) Since C is left and right quasi-coFrobenius, then by [19, Theorem 2.6], C_C is a generator of Comod-C and ${}_CC$ is a generator of C-Comod. Furthermore, from Proposition 6.3, the comodules C_C and C_C are flat. Then (a) follows from Proposition 6.6.

(b) Let P be a projective left C-comodule and view P as a quotient of a direct sum of copies of the injective generator ${}_CC$. By Lemma 6.7, P is injective. Since the converse follows from [19, Theorem 1.3], the proof is complete. □

PROPOSITION 6.9. *Assume that C is a coalgebra. The following conditions are equivalent:*
(a) *C is left semiperfect, a left IF-coalgebra, and* $\mathrm{gl.dim}\, C$ *is finite.*
(b) *The coalgebra C is cosemisimple.*
(c) *C is left semiperfect, hereditary and a left IF-coalgebra.*

Proof. The implications (b)⇒(c)⇒(a) are obvious.

(a)⇒(b) First we prove that any X in C-comod is projective. Since $\mathrm{gl.dim}\, C$ is finite, there is an injective resolution of X of the form $0 \longrightarrow X \longrightarrow E_0 \xrightarrow{e_0} E_1 \xrightarrow{e_1} \dots \xrightarrow{e_n} E_n \longrightarrow 0$. By hypothesis, $E_0, \dots, E_n$ are flat. From Corollary 3.7 we get that $\mathrm{Ker}\, e_n, \mathrm{Ker}\, e_{n-1}, \dots, \mathrm{Ker}\, e_0 = X$ are also flat. Then X is a finite dimensional flat C-comodule and, hence, X is projective. It follows that any simple (hence, any semisimple) left comodule is projective.

Let M be a right C-comodule and let $\{\mathrm{soc}^n(M)\}_{n\in\mathbb{N}}$ be its Loewy series. Then $M = \bigcup_{n=0}^{\infty} \mathrm{soc}^n(M)$ and it follows from the definition of the Loewy series that $\mathrm{soc}^{n+1}(M)/\mathrm{soc}^n(M)$ is a semisimple left C-comodule. Hence, $\mathrm{soc}^{n+1}(M)/\mathrm{soc}^n(M)$ is projective. It follows that $M \cong \bigoplus_{n=0}^{\infty} \mathrm{soc}^{n+1}(M)/\mathrm{soc}^n(M)$ and, consequently, M is projective. This finishes the proof. □

7. Weak global dimension of coalgebras

When a coalgebra C has enough flat left comodules, it is natural to ask about the left weak global dimension $\mathrm{l.w.gl.dim}\, C$ of C, which is defined in the obvious way. In this short final section we observe that for C left semiperfect the left weak global dimension and the global dimension $\mathrm{gl.dim}\, C$ of C coincide.

Definition 7.1. Let C be a coalgebra and let M be a left C-comodule.

(a) The **flat dimension** $\mathrm{fld}(M)$ of M is defined to be the minimal natural number $n \geq 0$ such that there is an exact sequence

$$0 \to F_n \to F_{n-1} \to \ldots \to F_0 \to M \to 0$$

in C-Comod, called a **flat resolution** of M, where $F_0, F_1, \ldots, F_n$ are flat. If M has no finite flat resolution and has an infinite one, we set $\mathrm{fld}(M) = \infty$. The flat dimension $\mathrm{fld}(M)$ of M is not defined if there is no epimorphism $F_0 \to M$, where F_0 is flat.

(b) Assume that C has enough flat left comodules. The **left weak global dimension** of C is defined to be $\mathrm{l.w.gl.dim}\, C = \sup\{\mathrm{fld}(M);\ M \in C\text{-Comod}\}$. The right weak global dimension $\mathrm{r.w.gl.dim}\, C$ of C is defined analogously.

Example 7.2. Let C be the coalgebra of Example 5.4. Then we have:

(a) $\mathrm{l.w.gl.dim}\, C = \mathrm{gl.dim}\, C = 1$,

(b) $\mathrm{fld}(E(v)) = 1$, the sequence $(*)$ is a flat resolution of $E(v)$, and $(*)$ is not a projective resolution, for each $v \geq 1$,

(c) a projective resolution of $E(v)$ has the form $0 \to \bigoplus_{j=v-1}^{\infty} P(j) \to \bigoplus_{j=v}^{\infty} P(j) \to E(v) \to 0$,

(d) $\mathrm{fld}(E(0)) = 0$ and $\mathrm{pd}(E(0)) = 1$.

Arguing as in Proposition 6.9, one shows:

Proposition 7.3. *The coalgebra C is cosemisimple if and only if* $\mathrm{l.w.gl.dim}\, C = 0$, *that is, every left C-comodule is flat.*

Theorem 7.4. *If C is left semiperfect, then* $\mathrm{l.w.gl.dim}\, C = \mathrm{gl.dim}\, C$.

Proof. Assume that C is left semiperfect. Then any left C-comodule M has a projective resolution, the projective dimension $\mathrm{pd}(M)$ of M is defined and $\mathrm{pd}(M) \geq \mathrm{fld}(M)$, since every projective comodule is flat. Hence $\mathrm{gl.dim}\, C \geq \mathrm{l.w.gl.dim}\, C$, because in this case the global dimension of C may be computed either using projective resolutions or injective resolutions.

To prove the equality, we can suppose that $\mathrm{l.w.gl.dim}\, C$ is finite. We recall that $\mathrm{gl.dim}\, C = \sup\{\mathrm{id}(N);\ N \in \text{comod-}C\}$, see [28]. Let M be a right C-comodule of finite dimension and let $0 \to F_n \to F_{n-1} \to \ldots \to F_0 \to M^* \to 0$ be a flat resolution of M^* in C-Comod. Since C is left semiperfect, the functor Rat is exact [19, Proposition 2.2] and induces the exact sequence $0 \to M \to \mathrm{Rat}(F_0^*) \to \ldots \to \mathrm{Rat}(F_r^*) \to 0$ in Comod-C. By Theorem 4.6, the right C-comodule $\mathrm{Rat}(F_i^*)$ is injective, for all i. Then $\mathrm{id}(M) \leq \mathrm{fld}(M^*)$ and so $\mathrm{gl.dim}\, C \leq \mathrm{l.wgl.dim}\, C$. □

Corollary 7.5. *If C is left and right semiperfect, then* $\mathrm{l.w.gl.dim}\, C = \mathrm{r.w.gl.dim}\, C = \mathrm{gl.dim}\, C$.

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