# Regularity for quasilinear elliptic systems with critical growth. Critical groups computations and applications.

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## Quasilinear elliptic system

$$(P)\quad \begin{cases} -\operatorname{div}\left((\gamma+|\nabla u|^{p-2})\nabla u\right)=\lambda|u|^{p-2}u+\dfrac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta}, & x\in\Omega\\ -\operatorname{div}\left((\gamma+|\nabla v|^{r-2})\nabla v\right)=\mu|v|^{r-2}v+\dfrac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v, & x\in\Omega\\ u=v=0, \quad x\in\partial\Omega \end{cases}$$

## Variational setting

Let $\Omega\subset\mathbb{R}^N$ be a smooth bounded domain, $2\le p,r$, $\gamma\ge 0$, $1\le\alpha,\beta$ and $N\ge\max\{p^2,r^2\}$. Solutions of $(P)$ correspond to critical points of the functional $J: X\equiv W_0^{1,p}(\Omega)\times W_0^{1,r}(\Omega)\to\mathbb{R}$ given by

$$J(z)=J(u,v)=\frac{1}{p}\int_\Omega\left(\gamma+|\nabla u|^2\right)^{\frac{p}{2}}+\frac{1}{r}\int_\Omega\left(\gamma+|\nabla v|^2\right)^{\frac{r}{2}}-\int_\Omega F(u,v),$$

where $F(u,v)=\dfrac{\lambda}{p}|u|^p+\dfrac{\mu}{r}|v|^r+\dfrac{2}{\alpha+\beta}|u|^\alpha|v|^\beta$. Moreover, $J\in C^2(X)$ and given $z_i=(u_i,v_i)\in X$ we have:

$$\langle J'(z_0),z_1\rangle=\int_\Omega(\gamma+|\nabla u_0|^2)^{\frac{p-2}{2}}\nabla u_0\nabla u_1+\int_\Omega(\gamma+|\nabla v_0|^2)^{\frac{r-2}{2}}\nabla v_0\nabla v_1-\int_\Omega\left(\mathrm{D}_uF(u_0,v_0)u_1+\mathrm{D}_vF(u_0,v_0)v_1\right),$$

$$\begin{aligned}\langle J''(z_0)z_1,z_2\rangle=&\int_\Omega\left((\gamma+|\nabla u_0|^2)^{(p-2)/2}(\nabla u_1|\nabla u_2)+(p-2)(\gamma+|\nabla u_0|^2)^{(p-4)/2}(\nabla u_0|\nabla u_1)(\nabla u_0|\nabla u_2)\right)\\&+\int_\Omega\left((\gamma+|\nabla v_0|^2)^{(r-2)/2}(\nabla v_1|\nabla v_2)+(r-2)(\gamma+|\nabla v_0|^2)^{(r-4)/2}(\nabla v_0|\nabla v_1)(\nabla v_0|\nabla v_2)\right)\\&-\int_\Omega\left(D^2_{uu}F(u_0,v_0)u_1u_2+D^2_{vv}F(u_0,v_0)v_1v_2+D^2_{uv}F(u_0,v_0)u_1v_2+D^2_{vu}F(u_0,v_0)u_2v_1\right).\end{aligned}$$

## Morse theory in the Banach framework

Let $z_0$ be a critical point of $J$:

1. $z_0$ is degenerate in the usual sense and we **can not use classical Morse theory**
2. $J''(z_0)$ is not a Fredholm operator and we **can not use the perturbation results in [6]**
3. $J''(z_0)$ injective is the **nondegeneracy** for critical points in the Banach setting (see [3])
4. **Regularity** of solutions allows the use of Morse theory and the knowledge of the local behavior of $J$ near its critical points by means of the associated critical groups.

## Critical growth. Main difficulties

Antecedents exists for nonlinear terms $F$ having subcritical growth or even, in the scalar case, for $F$ having critical growth (regularity stablished in [5]), in this case we have the additional *"lack of compactness"*.

$F$ has critical growth if $\dfrac{\alpha}{p^*}+\dfrac{\beta}{r^*}=1$, and the main problems to solve are

1. to prove some kind of local compactness
2. to prove regularity for solutions of $(P)$.

## Local compactness

There exists $R>0$ such that, for any fixed $\gamma\ge 0$ and any $z\in X$, the functional $J$ is weakly lower semicontinuous and satisfies $(P.S.)$ condition on $\overline{B}_R(z)$.

The proof relies on

- A Clarkson type inequality obtained in [2]

$$(\gamma+|y|^2)^{q/2}\ge(\gamma+|x|^2)^{q/2}+q(\gamma+|x|^2)^{(q-2)/2}(x|y-x)+\frac{|y-x|^q}{2^{q-1}-1}$$

  for any $\gamma\ge 0$, $q\ge 2$ and $x,y\in\mathbb{R}^N$.
- The concentration-compactness principle in [7].

## Regularity

- Partial results are known for $p=r=2$ ([5] extended).
- The case $p\ne r$ requires a more detailed analysis.

Assume that $(u,v)\in X$ is a solution for $(P)$ then $u,v\in C^{1,\eta}(\overline{\Omega})$, for some $0<\eta<1$.

1. test function: $\displaystyle\int_0^u\left|h'_{k,\theta}(s)\right|^p ds,\ \int_0^v\left|h'_{k,\theta}(s)\right|^r ds$ (Fig. 1)
2. pass to the limit, as $k\to+\infty$, using that

$$\left|h_{k,\theta}(|s|^{\frac{r^*}{p^*}})\right|^{p^*}\le C\left|h_{k^{\frac{p^*}{r^*}},\theta}(|s|)\right|^{r^*}$$

3. $u\in L^{\theta p^*}(\Omega)$ and $v\in L^{\theta r^*}(\Omega)$ for every $\theta>1$
4. the proof is finished in the standard way.

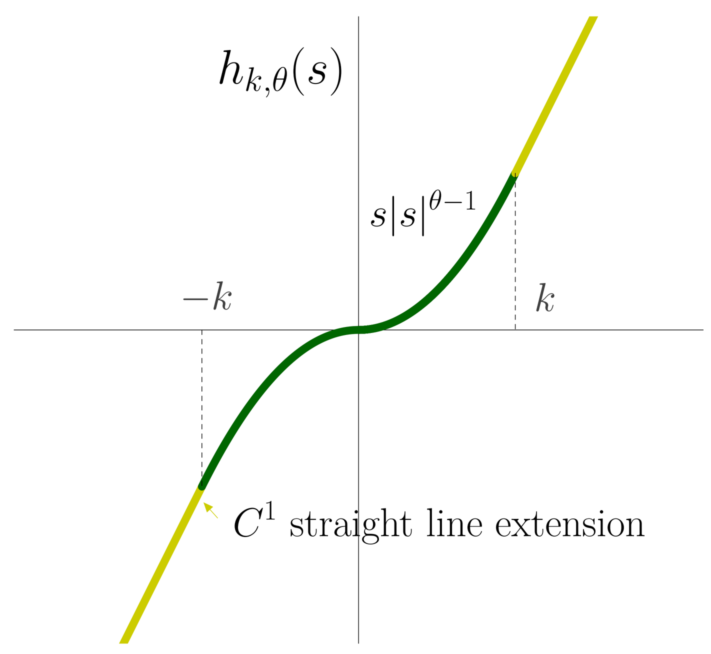


Figure 1: Truncation function $h_{k,\theta}$

## Finite dimensional reduction

- Regularity and the expression for $J''$ (green part) induce a Hilbert space (depending on $z_0$), in which $X$ is continuously embedded
- this allows the splitting $X=V\oplus W$ where $m^*(J,z_0)=\dim V<+\infty$
- for some $\rho_0,r_0>0$, are well defined and continuous
  - $\Psi: V\cap\overline{B}_\rho(0)\to W\cap B_{r_0}(0)$ given by $\Psi(v)=\overline{w}$ the unique minimum point of the function $w\in W\cap\overline{B}_{r_0}(0)\mapsto J(z_0+v+w)$
  - $\Phi(v)=J(z_0+v+\Psi(v))$
- arguing as in [2] we can prove that $C_j(J,z_0)\cong C_j(\Phi,0)$.

## Critical group computations

Let $z_0$ be a non degenerate critical point of $J$ then

$$C_j(J,z_0)\cong\mathbb{K},\quad \text{if } j=m(J,z_0),$$
$$C_j(J,z_0)=\{0\},\quad \text{if } j\ne m(J,z_0).$$

## Multiplicity result

In the case $p=r$, $\gamma=0$ and $\lambda,\mu$ are small enough, it is showed in [4] the existence of $cat_\Omega(\Omega)$ distinct positive solutions of $(P)$. We prove that:

Either $(P)$ has at least $\mathcal{P}_1(\Omega)$ distinct solutions or given $0<\gamma_n\to 0$ there exist $f_n$ and $g_n$ in $C^1(\overline{\Omega})$, such that $\|f_n\|_{C^1(\overline{\Omega})},\|g_n\|_{C^1(\overline{\Omega})}\to 0$ and problem

$$\begin{cases} -\operatorname{div}\left((|\nabla u|^2+\gamma_n)^{(p-2)/2}\nabla u\right)=\lambda|u|^{p-2}u+\dfrac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^\beta+f_n,\\ -\operatorname{div}\left((|\nabla v|^2+\gamma_n)^{(p-2)/2}\nabla v\right)=\mu|v|^{p-2}v+\dfrac{2\beta}{\alpha+\beta}|u|^\alpha|v|^{\beta-2}v+g_n,\\ u=v=0,\quad x\in\partial\Omega, \end{cases}$$

has at least $\mathcal{P}_1(\Omega)$ distinct positive solutions, for $n$ large enough.

## Conclusion

The application of the Morse theory yields better results than the application of Ljusternik–Schnirelman theory for topologically rich domains.

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