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Copulae, Self-Affine Functions, and Fractal Dimensions

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Abstract. We use the fact that the functions defined on the unit interval whose graphs support a copula are those that are Lebesgue-measure-preserving in order to characterize self-affine functions whose graphs are the support of a copula. This result allows computation of the Hausdorff, packing, and box-counting dimensions. The discussion is applied to classic examples such as the Peano and Hilbert curves, and the results are extended to discontinuous self-affine functions.

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1. Introduction

The notion of copula was introduced by Sklar [26] when he proved his celebrated theorem in 1959. His aim was to express the relationship between multivariate distribution functions and their univariate margins. Therefore, the primary importance of this notion lies in Probability Theory and Statistics. From that moment, the study of copulas and their applications have shown themselves to be tools of great interest in several fields, such as Markov operators, multivariate distributions, statistical models, doubly stochastic processes, dependence, and mass transportation theory. (For an introduction to copulas, see [8,17,25].)

Many authors in various fields have drawn attention to methods to generate fractal sets and to describe the concept of "size" for sets in the plane, computing different types of fractal dimensions (in particular, Hausdorff, packing, and box-counting dimensions). Fractal features are often exhibited by measures. This allows the investigation of the connection between fractals and measure-preserving transformations, and the use of methods from Probability Theory and Ergodic Theory.

In addition, some authors describe several ways in which fractal geometry interacts with the notion of copula. Specifically, recent studies have been carried out on examples where the copula has a fractal support, and on the relationship between copulas and measure-preserving transformations on the Borel sets of the unit interval in [3,6,7,14]. Moreover, sufficient conditions for the graph of a function to be the support of a copula are given in [14], and a necessary and sufficient condition is given in [3].

Finally, fractals that are invariant under simple families of transformations include self-similar and self-affine sets. In particular, Kamae [13], using a definition of self-affine function that generalizes that given by Kôno in [15], gives a characterization of them as functions generated by finite automata. Urbański [30] has given conditions to determine dimensions of the graphs of continuous self-affine functions.

In this paper, we establish closer relations between the notions of copulas and measure-preserving transformations, self-affine functions whose graphs are the support of a copula, and their applications to computing several fractal dimensions.

In Sect. 3, Theorem 5 gives necessary and sufficient conditions for the graphs of a family of self-affine functions to support a copula. To prove this result, we use the fact that a continuous function on the unit interval preserves the measure of Lebesgue if and only if its graph supports a copula (Proposition 1).

In Sect. 4, we use these results to compute Hausdorff, packing, and box-counting dimensions (Theorem 9). In particular, our methods can be performed on classic examples such as the coordinate functions of Peano and Hilbert curves.

Finally, in Sect. 5, we extend these results to the discontinuous case (Theorem 16).

2. Preliminaries

In this section, we recall some notions and definitions used below. First, we provide a definition of copula and some of its elementary properties (see [17]).

(1.1) Let $\mathbb{I} := [0,1]$ be the closed unit interval and let $\mathbb{I}^2$ be the unit square. A *two-dimensional copula* (or a *copula,* for brevity) is a function $C : \mathbb{I}^2 \longrightarrow \mathbb{I}$ with the following properties: (i) For every u, v in $\mathbb{I}$, $C(u,0) = 0 = C(0,v)$ and $C(u,1) = u$, $C(1,v) = v$, and ii) For every u_1, u_2, v_1, v_2 in $\mathbb{I}^2$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$,

$$C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \geq 0.$$

Alternatively, we can say that a copula is a bidimensional distribution whose restriction to $\mathbb{I}^2$ has its marginal distribution functions uniformly distributed. Therefore, each copula C induces a probability measure μ_C on $\mathbb{I}^2$ via the formula

$$\mu_C([a,b] \times [c,d]) = C(b,d) - C(b,c) - C(a,d) + C(a,c)$$

in a similar fashion to joint distribution functions. Through standard measure-theoretical techniques, μ_C can be extended from the semi-ring of rectangles in $\mathbb{I}^2$ to the σ-algebra $\mathcal{B}(\mathbb{I}^2)$ of Borel sets in the unit square. We denote by λ the standard Lebesgue measure on the σ-algebra $\mathcal{B}(\mathbb{I})$ of Borel sets in the unit interval. The *support of a copula* C is the complement of the union of all open subsets of $\mathbb{I}^2$ with μ_C-measure equal to zero.

We use Mandelbrot's original definition of *fractal set* (i.e., a set whose topologial dimension is less than its Hausdorff dimension $\dim_{\mathcal{H}}$). Dimensions of different types are particularly useful in describing the concept of "size" of sets in the plane, in particular, sets of zero Lebesgue measure. Several definitions are of widespread use, so we summarize the basic concepts used in this paper. For basic properties concerning dimensions (Hausdorff, box-counting, and packing), and other useful notions for expressing the fractal properties of sets, the reader is referred to [9,11].

(1.2) Let A be a subset of $\mathbb{R}^n$, and $0 \le s \le n$, $\delta > 0$. For every s, the (*outer*) *s-dimensional Hausdorff measure* of A is defined as

$$\mathcal{H}^s(A) = \lim_{\delta \to 0} \left\{ \inf \sum_i d(U_i)^s \right\},$$

the infimum being taken over all countable covers $\{U_i\}$ of A by sets in X with $0 < d(U_i) \le \delta$. The *Hausdorff dimension* of A, $\dim_{\mathcal{H}}(A)$, is the parameter s_0 such that $\mathcal{H}^s(A) = \infty$ for $s < s_0$ and $\mathcal{H}^s(A) = 0$ for $s > s_0$. We write $d(U) = \sup\{\|x-y\| : x, y \in U\}$ for the diameter of the set U.

Let A be a bounded subset of $\mathbb{R}^n$. For $\delta > 0$ we denote by $N_\delta(A)$ the minimum number of sets of diameter less than δ needed to cover A. The *lower* and *upper box-counting dimensions* are defined, respectively, as

$$\underline{\dim}_B(A) = \underline{\lim}_{\delta \to 0} \frac{\log N_\delta(A)}{-\log \delta} \quad \text{and} \quad \overline{\dim}_B(A) = \overline{\lim}_{\delta \to 0} \frac{\log N_\delta(A)}{-\log \delta}.$$

If these numbers are equal, we call the common value the *box-counting dimension*, abbreviated to *box dimension* (or *capacity*). We recall that the Hausdorff and box-counting dimensions may be defined using economical coverings by small balls. This provides another definition for dimension in terms of dense packings by disjoint balls of different small radius. More precisely, the *s-dimensional packing measure* is defined by

$$P^s(A) = \inf \left\{ \sum_i P_0^s(F_i) : A \subset \bigcup_{i=1}^{\infty} F_i \right\},$$

where

$$P_0^s(A) = \lim_{\delta \to 0} \left(\sup \left\{ \sum_i^s d(B_i)^s \right\} \right),$$

and the supremum is taken on collections $\{B_i\}$ of disjoint balls of radii at most δ with centre in A. The *packing dimension* is defined in the usual way

$$\dim_P(A) = \sup\{s : P^s(A) = \infty\} = \inf\{s : P^s(A) = 0\}.$$

The following relations are established: if $A \subset \mathbb{R}^n$, then

$$\dim_{\mathcal{H}}(A) \le \dim_P(A) \le \overline{\dim}_B(A).$$

Fredricks et al. [12], using an iterated function system, construct the first example of a family of copulas whose supports are fractals. In particular, they give sufficient conditions for the support of a self-similar copula to be a fractal whose Hausdorff dimension is between 1 and 2. The main result of these authors states that for every $s \in [1, 2]$ there exists a copula whose support has dimension equals to s. New results concerning 2-copulas of fractal support can be found in [3–5,27], and the generalization to dimension greater than 2 in [28].

(1.3) Given a measurable space (X, Ω, μ), a measurable function $F : X \to X$ is said to be *measure-preserving (*or *F preserves μ*) iff $\mu\left(F^{-1}(A)\right) = \mu(A)$, for all $A \in \Omega$.

If the σ-algebra Ω is generated by a family Ω_0 that is closed for finite intersections (i.e., a π-system), a sufficient condition for F to be measurable and measure-preserving (see [2, Sect. 24]) is that $F^{-1}(A) \in \Omega$ and $\mu\left(F^{-1}(A)\right) = \mu(A)$, for all $A \in \Omega_0$.

We are interested in the case $(X, \Omega, \mu) = (\mathbb{I}, \mathcal{B}(\mathbb{I}), \lambda)$.

Many authors have established a correspondence between copulas and measure-preserving transformations f, g on the unit interval via the formula

$$C_{f,g}(u, v) = \lambda\left(f^{-1}[0, u] \cap g^{-1}[0, v]\right)$$

(as we can see in [6,7,14,18,29]). In [3], the authors investigate the hardest implication in this correspondence; that is, for a given copula C, the goal is to find a pair of measure-preserving transformations (f, g) such that $C = C_{f,g}$. In particular, we study representation systems for a certain family of self-similar copulas in terms of measure-preserving transformations.

On the other hand, for the general problem of determining just what functions in $\mathbb{I}^2$ satisfy the property their graphic can concentrates the mass of a copula, in [7] it is proven that, for every copula obtained as a shuffle of Min (see [17, p. 67]), there exists a piece-wise linear function whose graph concentrates the probability mass.

In 1986 Kôno [15] introduced the notion of a self-affine function f of order $\alpha > 0$, whose paradigm is the component functions of the Peano curve.

"Self-affinity" properties have been studied by different authors, with definitions that generalize the Kôno notion using different methods (e.g., Kamae [13] or Peitgen et al. [19]). The main fact is that the graphs of self-affine functions are expected to show strong fractal features. In [16], the author obtains Hausdorff, box, and packing dimensions for graphs of self-affine functions under some conditions. We note that [1,20,23] have related results. In particular, with [16,30], for a given continuous and self-affine function $f : \mathbb{I} \to \mathbb{I}$, a necessary and sufficient condition for the probability distribution $\lambda \circ f^{-1}$ to be absolutely continuous with respect to Lebesgue measure is that the Hausdorff and box dimensions of the graph of f be equal to $2-\alpha$.

3. Copulas and Self-Affine Functions

We start with the general problem of determining just what functions in $\mathbb{I}$ have graphs that can serve as the support of a copula. We recall that [7] gives an answer to this problem using the notion of shuffle of an arbitrary copula. In [3], we proved the following general result:

Proposition 1. *Let $f : \mathbb{I} \to \mathbb{I}$ be a Borel measurable function. Then, there exists a copula C whose associated measure μ_C has its mass concentrated in the graph of f (denoted by Γ, $\mu_C(\Gamma) = 1$) if and only if the function f preserves the Lebesgue measure λ.*

For the sake of brevity, we say that f supports C.

Now, we introduce a family of self-affine functions on $\mathbb{I}$. It is adapted from those of Kamae [13] that generalizes the previous concept given by Kôno [15]. See also Peitgen et al. [19].

We use the following notations: for $k \in \mathbb{Z}^+$, let us denote by $[k]$ the set $\{0, 1, 2, \ldots, k-1\}$, and by $[k]^* = \{a_1 \cdots a_k : a_j \in [k], 1 \le j \le k\}$.

Definition 2. *A family of functions $x_0, x_1, \ldots, x_{N-1} : \mathbb{I} \longrightarrow \mathbb{I}$ is called self-affine of order $\alpha \in]0,1[$ and with base $m \in \mathbb{Z}^+ \backslash \{1\}$ (or simply, (m,α)-self-affine) iff the following conditions are satisfied:*

(a) $x_j(0), x_j(1) \in \{0,1\}$ *for all* $j \in [N]$.

(b) *There is an application $\theta : [N] \to [N]^*$ of constant length m (i.e., $\theta(j)$ has the same number m of terms for all $j \in [N]$) such that, for all $(j,h) \in [N] \times [m]$ and for $t \in \mathbb{I}$, we have*

$$x_j\left(\frac{h+t}{m}\right) - x_j\left(\frac{h}{m}\right) = \frac{x_{\theta_h(j)}(t) - x_{\theta_h(j)}(0)}{m^\alpha},$$

where $\theta_h(j)$ is the element in $[N]$ in the hth position in $\theta(j)$.

We say that each one of the functions x_j is self-affine.

Observe that any self-affine function is continuous, because for all $z, z' \in \mathbb{I}$, $|z - z'| < 2m^{-n}$ implies that $|x_j(z) - x_j(z')| < m^{-n\alpha}$.

A typical example of a self-affine function is each coordinate function in the Peano curve (see for instance [15,19]). Let us see the details below.

Example 3. (Coordinate functions for the Peano curve). Let us define the operator $k(\beta) = 2 - \beta$, with $\beta \in \{0,1,2\}$. If $t = \sum_{n=1}^\infty \frac{t_n}{3^n}$, then

$$\begin{cases} x(t) = \frac{t_1}{3} + \frac{k^{t_2}(t_1)}{3^2} + \frac{k^{t_2+t_4}(t_5)}{3^3} + \cdots \\ y(t) = \frac{k^{t_1}(t_2)}{3} + \frac{k^{t_1+t_3}(t_4)}{3^2} + \frac{k^{t_1+t_3+t_5}(t_6)}{3^3} + \cdots \end{cases}$$

The x coordinate for the Peano curve is self-affine with values:

(a) $N = 2; \quad m = 9; \quad \alpha = 1/2$

(b) $x_0(t) = x(t), \quad x_1(t) = 1 - x(t)$

(c) $\theta(0) = 010010010; \quad \theta(1) = 101101101.$

In order to characterize the self-affine functions whose graphs can support a copula, we use the set $\mathbb{I}^2 \times [N]$. We can define the next metric on it:

$$d((x,y,j),(x',y',j')) = \begin{cases} \rho, & \text{if } j \ne j' \\ \sqrt{(x-x')^2 + (y-y')^2}, & \text{if } j = j' \end{cases}$$

with $\rho > \sqrt{2}$. Let $\kappa(\mathbb{I}^2 \times [N])$ be the space of compact sets in $\mathbb{I}^2 \times [N]$, endowed with the Hausdorff metric given by d. Let us consider the attractor F given by the Contraction Mapping Theorem (see for example [11, Chap. 9]) for the contraction τ given in the following form. Let us introduce the functions:

$$\tau_{jh} : \mathbb{I}^2 \times \{\theta_h(j)\} \longrightarrow$$
$$\left[\frac{h}{m}, \frac{h+1}{m}\right] \times \left[x_j\left(\frac{h}{m}\right) - \frac{x_{\theta_h(j)}(0)}{m^\alpha}, x_j\left(\frac{h}{m}\right) + \frac{-x_{\theta_h(j)}(0)+1}{m^\alpha}\right] \times \{j\}$$

given by

$$(x, y) \times \{\theta_h(j)\} \to \left(\frac{h+x}{m}, x_j\left(\frac{h}{m}\right) + \frac{y - x_{\theta_h(j)}(0)}{m^\alpha}\right) \times \{j\};$$

and let us define:

$$\begin{aligned} \tau : \kappa(\mathbb{I}^2 \times [N]) &\longrightarrow \kappa(\mathbb{I}^2 \times [N]) \\ D &\to \underset{jh}{\cup} \tau_{jh}\left(D \cap \left(\mathbb{I}^2 \times \{\theta_h(j)\}\right)\right) \end{aligned} \tag{1.4}$$

Now, we can characterize the self-affine functions in Definition 2:

Proposition 4. *The function x_j is self-affine if and only if its graph is the intersection of the square $\mathbb{I}^2 \times \{j\}$ and the attractor F, in the space $\kappa(\mathbb{I}^2 \times [N])$, given by* (1.4).

Proof. Since τ_{jh} is a contraction, it follows easily that τ is a contraction as well.

In general, we have equalities

$$\tau(F) \cap \left(\mathbb{I}^2 \times \{j\}\right) = \underset{h}{\cup} \tau_{jh}\left(F \cap \left(\mathbb{I}^2 \times \{\theta_h(j)\}\right)\right),$$

and

$$\tau(F) \cap \left[\frac{h}{m}, \frac{h+1}{m}\right] \times \mathbb{I} \times \{j\} = \tau_{jh}\left(F \cap \left(\mathbb{I}^2 \times \{\theta_h(j)\}\right)\right).$$

However, the last equality is equivalent to

$$x_j\left(\frac{h+t}{m}\right) - x_j\left(\frac{h}{m}\right) = \frac{x_{\theta_h(j)}(t) - x_{\theta_h(j)}(0)}{m^\alpha}.$$

$\square$

Now, we establish the main result in this section.

Theorem 5. *An element of the family of (m, α)-self-affine functions $\{x_j\}_{j \in [N]}$ can support a copula if and only if $m^{1-\alpha} \in \mathbb{Z}^+$ and, for each $j \in [N]$ and $r \in [m^\alpha]$:*

$$\text{Card}\left\{h : 0 \le h \le m-1, x_j\left(\frac{h}{m}\right) - \frac{x_j(0)}{m^\alpha} = \frac{r}{m^\alpha}\right\} = m^{1-\alpha}.$$

Proof. If the functions are self-affine and can support the copula, then their range is the unit interval $\mathbb{I}$. Now, $x_j(0), x_j(1) \in \{0, 1\}$ for all j, implies that the images of the intervals $\left[\frac{h}{m}, \frac{h+1}{m}\right]$ are, once again, intervals in the form $\left[\frac{r}{m^\alpha}, \frac{r+1}{m^\alpha}\right]$ with $r \in [m^\alpha]$. Moreover, such intervals are either coincident, disjoint, or have only one point in common. These facts imply that $m^\alpha \in \mathbb{Z}^+$.

If x_j supports a copula, then it is Lebesgue-measure-preserving (by Proposition 1). Therefore:

$$\frac{1}{m^{\alpha}}=\lambda\left(\left[\frac{r}{m^{\alpha}},\frac{r+1}{m^{\alpha}}\right]\right)=\lambda\left(x_j^{-1}\left(\left[\frac{r}{m^{\alpha}},\frac{r+1}{m^{\alpha}}\right]\right)\right),$$

is the union of intervals of length $1/m$, and as a consequence, we have $m^{1-\alpha}\in\ \mathbb{Z}^{+}$.

The above reasoning provides a bonus result:

$$\text{Card}\left\{h:0\leq h\leq m-1,x_j\left(\frac{h}{m}\right)-\frac{x_{\theta_h(j)}(0)}{m^{\alpha}}=\frac{r}{m^{\alpha}}\right\}=m^{1-\alpha}.$$

For the reverse implication, we use the consequences of the extension theorem. It gives a sufficient condition for a function to be Lebesgue-measure-preserving (see (1.3) above). In fact, since the intervals in $]0,1]$ form a π-system that generates the σ-algebra $\mathcal{B}$ of Borel sets, if we show that $\frac{1}{m^{n\alpha}}=\lambda\left(x_j^{-1}\left(\left[\frac{r}{m^{\alpha n}},\frac{r+1}{m^{\alpha n}}\right]\right)\right)$, then the assertion follows.

Therefore, we have seen that the result is true in the case $n=1$. The rest can be done by induction on n. □

4. Application to Computing Fractal Dimensions

For a family of self-affine functions whose graphs support a copula, this is the property we use to computing fractal dimensions (see (1.2) above for definitions we use here):

Lemma 6. *Let us consider a self-affine function f of order $\alpha\in\ ]0,1[$, then the upper box-counting dimension of Γ is not greater than $2-\alpha$.*

Proof. Let us consider a cover for the graphs with squares of the mesh of side $1/m^n$. Using τ in (1.4), then $\tau^m(\mathbb{I}^2\times[N])$ is a cover for the graphs. On each set $\mathbb{I}^2\times\{j\}$, the graph of x_j, namely $\Gamma(x_j)$, is covered by m^n rectangles of mesh $\frac{1}{m^n}\times\frac{1}{m^{\alpha n}}$. Therefore, we can divide them for obtaining a recover with $m^n m^{(1-\alpha)n}$ squares of sides of length $1/m^n$.

With this covering, we can deduce that:

$$\overline{\dim}_B\Gamma(x_j)\leq\lim_{n\to\infty}-\frac{\ln m^n m^{(1-\alpha)n}}{\ln 1/m^n}=2-\alpha.$$

□

As usual, to find lower bounds for dimensions, we use the mass distribution principle (see [10]). We recall that a *mass distribution* μ on a set $A\subset\mathbb{R}^2$ is a measure such that $\mu\left(\mathbb{R}^2\backslash A\right)=0$. Precisely, we use it in the following form.

Lemma 7. *Let μ be a mass distribution on $A\subset\mathbb{R}^2$. If there exist constants $c>0$ and $\delta>0$ such that, for all m-adic square $Q\subset\mathbb{R}^2$ with $l(Q)\leq\delta$ (where $l(Q)$ denotes the length of a side of Q),*

$$\mu(Q)\leq c\,(l(Q))^s,$$

then

$$s \leq \dim_{\mathcal{H}}(A).$$

Proposition 8. *Let us consider a family of self-affine functions with order of self-affinity α. If their graphs can support a copula, then their Hausdorff dimension is not less than $2-\alpha$.*

Proof. We apply Lemma 7 to the m-adic squares. Because the graphs support the copula C, the squares that remain outside of the rectangles used in Lemma 6 have null mass.

Besides, we consider a m-adic square S with side of length $1/m^n$ and positive mass, which is included in a rectangle R (such as those used in Lemma 6). But $\mu_C(R) = 1/m^n$, and R contains at least one (deformed) copy of some graph $\Gamma(x_j)$. Then, the vertical distribution for the mass is uniform; and therefore, the μ_C-measure of the square is $\frac{1}{m^n m^{(1-\alpha)n}}$. As a consequence, we can write $\mu_C(S) = [l(S)]^{2-\alpha}$, and we deduce the statement. □

Now, with (1.2), we summarize with this result:

Theorem 9. *Let us consider a self-affine function with affinity order α. If its graph Γ supports a copula, then the packing, Hausdorff, and box-counting dimensions for Γ are exactly $2-\alpha$.*

In [24, 2.8], the author proposes a generalization of the Hilbert curve to three dimensions; that is, a curve in the unit cube. The coordinates $x(t), y(t), z(t)$ are self-affine of order $1/3$, and they follow this scheme:

$$\begin{cases} x_0(t) = x(t), x_1(t) = y(t), x_2(t) = z(t), \\ x_3(t) = 1 - x(t), x_4(t) = 1 - y(t), x_5(t) = 1 - z(t). \\ \theta(0) = 02025050, \theta(1) = 21133115, \theta(2) = 10241234, \\ \theta(3) = 35352323, \theta(4) = 54400442, \theta(5) = 43514501. \end{cases}$$

Approximations to the graphs of x and y are given in Fig. 1 (right and left, resp.), and the corresponding to the graph of z is given in Fig. 2.

They are functions supporting copulas, which is a consequence of Proposition 5 when we observe the matrix

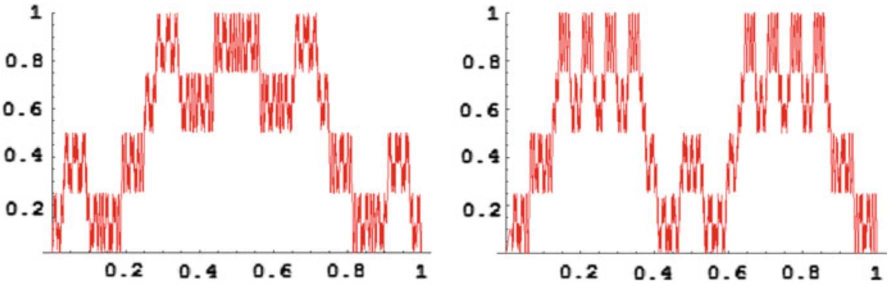

Figure 1. x and y coordinates

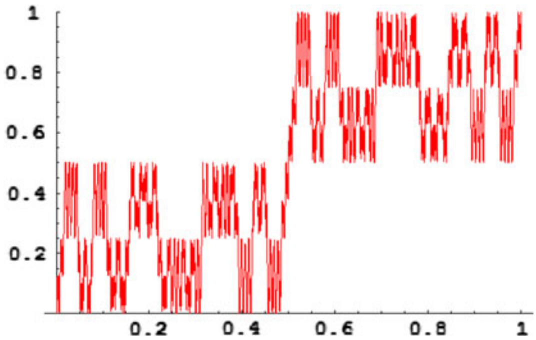

Figure 2. z coordinate

$$\left(x_j\left(\frac{h}{8}\right)-\frac{x_{\theta_h(j)}(0)}{2}\right)=\begin{pmatrix} 0 & 0 & 1/2 & 1/2 & 1/2 & 1/2 & 0 & 0\\ 0 & 1/2 & 1/2 & 0 & 0 & 1/2 & 1/2 & 0\\ 0 & 0 & 0 & 0 & 1/2 & 1/2 & 1/2 & 1/2\\ 1/2 & 1/2 & 0 & 0 & 0 & 0 & 1/2 & 1/2\\ 1/2 & 0 & 0 & 1/2 & 1/2 & 0 & 0 & 1/2\\ 1/2 & 1/2 & 1/2 & 1/2 & 0 & 0 & 0 & 0 \end{pmatrix}$$

because

$$\begin{aligned}&\mathrm{Card}\left\{h:0\le h\le 7, x_j\left(\frac{h}{8}\right)-\frac{x_{\theta_h(j)}(0)}{2}=0\right\}\\ &\quad=\mathrm{Card}\left\{h:0\le h\le 7, x_j\left(\frac{h}{8}\right)-\frac{x_{\theta_h(j)}(0)}{2}=1/2\right\}=4=8^{1-1/3},\end{aligned}$$

independently of j. Therefore, we can deduce that the corresponding graphs for these functions have fractal dimensions equal to $5/3$.

Remark 10. This technique can be applied to functions studied in [15] and [20]. For example, it is possible to obtain $3/2$ as the dimension of the coordinate functions in the curves of either Peano or Hilbert.

5. The Discontinuous Case

We recall that the conditions imposed in Definition 2 to the family of self-affine functions imply continuity for the functions in the family. We can introduce another definition that allows the study of those cases consisting of discontinuous functions. This target can be reached when we set the condition (b′) instead of (b) in Definition 2:

(b′) There is an application $\theta:[N]\to[N]^*$ of constant length m (i.e., $\theta(j)$ has the same number m of elements for all $j\in[N]$) such that, for all $(j,h)\in[N]\times[m]$ and for $t\in[0,1[$, we have

$$x_j\left(\frac{h+t}{m}\right)-x_j\left(\frac{h}{m}\right)=\frac{x_{\theta_h(j)}(t)}{m^\alpha},$$

where $\theta_h(j)$ is the element in $[k]$ in the hth position in $\theta(j)$.

These functions are called (m, α)-*quasi-self-affine functions*.

In this new context, we can obtain similar results to those we obtained in the continuous case:

Proposition 11. *The (m, α)-quasi-self-affine functions x_j are continuous on points with infinite m-adic representation; in particular, they are continuous on the irrationals. Moreover, at those points with finite representation, the coordinates have both one-side limits. Furthermore, one of them, at least, coincides with the value of the function at that point.*

Theorem 12. *A family $\{x_j\}_{j\in[N]}$ of (m, α)-quasi-self-affine functions can support a copula if and only if $m^{1-\alpha} \in \mathbb{Z}^+$, and for each $j \in [N]$ and $r \in [m^\alpha]$, it is satisfied that*

$$\operatorname{Card}\left\{h : 0 \le h \le m-1, x_j\left(\frac{h}{m}\right) = \frac{r}{m^\alpha}\right\} = m^{1-\alpha}.$$

Theorem 13. *The Hausdorff, packing, and box-counting dimensions of the graphs of self-affine functions supporting copulas are $2-\alpha$.*

Example 14. (Coordinates for Cantor function). Let us consider the coordinates of the map from $\mathbb{I}$ to $\mathbb{I}^2$ given by this rule: If $t = \sum \frac{t_n}{2^n}$, then:

$$\begin{cases} x(t) = \sum \frac{t_{2n-1}}{2^n} \\ y(t) = \sum \frac{t_{2n}}{2^n} \end{cases}$$

For the rest of the points (a denumerable set), we can choose one expansion or the other for t and define its values following these rules.

The above proposition can be used to obtain Hausdorff, packing, and box-counting dimensions for the graphs of these functions. Therefore, they are 3/2 (in any case). This example provides a particular case of one given in [21] because the graph of x and the set

$$\overline{R}_x = \left\{\left(\sum \frac{a_n}{4^n}, \sum \frac{b_n}{4^n}\right) : (a_n, b_n) \in \{(0,0), (1,0), (2,1), (3,1)\}\right\}$$

differ on a denumerable set. The same is true for the graph of y and the set

$$\overline{R}_y = \left\{\left(\sum \frac{a_n}{4^n}, \sum \frac{b_n}{4^n}\right) : (a_n, b_n) \in \{(0,0), (1,1), (2,0), (3,1)\}\right\}.$$

Let us note that the results of [21] cannot be achieved through the methods we propose here. However, on the other hand, in the case $N > 1$, we can obtain dimensions that are forbidden for those results in [21].

The hypothesis of self-affinity has been used throughout the paper. However, the main result remains true even if the functions are not necessarily self-affine. We summarize this result in the theorem below. But we still need a definition.

Definition 15. *Given the squares $R = [r, t] \times [r', t']$ and*

$$R_{ij} = \left[\frac{t-r}{a} i + r, \frac{t-r}{a}(i+1) + r\right] \times \left[\frac{t'-r'}{b} j + r', \frac{t'-r'}{b}(j+1) + r'\right],$$

with $a, b, a/b \in \mathbb{Z}^+$, *we say that* R' *is a* C_{ab}*-subset of* R *if* $R' = \cup_{(i,j)\in A} R_{ij}$, *where* $A \subset [a] \times [b]$, *satisfying:*

$$\mathrm{Card}\,\{h : 0 \le h \le a-1, (h,j) \in A\} = \frac{a}{b},$$

independently of j.

Theorem 16. *Let* m *and* m^α *be integers greater than* 1, *such that the second divides the first. Set* $R_1 = \mathbb{I}^2$. *Let* R_2 *be a* C_{ab}*-subset of* R_1, *and* R_3 *a subset in* (*the rectangle*) R_2 *satisfying that its intersection with one of the rectangles in* R_2 *is a* C_{ab}*-subset of this rectangle. In general, the rectangle* R_{n+1} *is obtained from* R_n *in the same way as* R_3 *was from* R_2. *If* $S = \cap_{i\in\mathbb{Z}^+} R_i$, *then:*

i. *The set* S *can be the support for some copula.*
ii. *The set* S *differs from the graph of a function on a denumerable set of points.*
iii. *The function in ii. is continuous at points that do not admit finite* m*-adic expansion* (*in particular, on the irrationals*), *and it has one-side limits at points with finite representation; one of them, at least, coincides with the value of the function at the point.*
iv. *The set* S (*and, therefore, the graph of the function in ii. and iii.*) *has a dimension equal to* $2 - \alpha$.

Remark 17. If we consider the case $\alpha = 1$, then we can take the function in statement ii. above as a bijection. The copulas associated to these functions are generalized Shuffles of Min (see [22]) in the sense of Durante et al. in [7]. These functions are examples of functions they use in that paper to define the Shuffle of an arbitrary copula.

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