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HARMONIC ANALYSIS ON THE SIERPIŃSKI GASKET AND SINGULAR FUNCTIONS

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Abstract. We study the restriction on $[0,1]$ of harmonic functions on the Sierpiński gasket, proving they are singular functions whenever they are monotone. We show that their derivatives are zero or infinity on certain non-denumerable sets. Finally, we show they are among a wider class of functions that contains some already known and studied functions.

1. Introduction

The history of singular functions, i.e. monotone increasing continuous functions with zero derivatives on a set of measure one in the unit interval $\mathbb{I} := [0,1]$, can be traced back to the end of the 18th century (see [8,30]). But they won a new dimension with the publication of the celebrated Lebesgue's book [17] in 1904. In that year Minkowski [19], with the target to enumerate the quadratic irrational numbers, gave an example of a function which is known as Minkowski's question mark function. In [13] the relation between Minkowski's representation system for reals and simple continuous fractions is shown. Moreover, it is shown that it is a singular function. Different aspects of this have been studied. For example, Viader et al. [21,29] showed this function as the asymptotic distribution function of an enumeration of the rationals in the unit interval $\mathbb{I}$, and studied its derivative function.

Afterwards, another family of functions, namely $\{S_a\}$, was simultaneously introduced by Césaro in 1906 and Hellinger in 1907. They have been

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studied from a wide variety of viewpoints (for example, geometric, arithmetic, probabilistic), as can be seen in [1,6,15,22,27,28,32].

In 1956, De Rham [11] characterized these functions via a system of functional equations:

$$S_a\left(\frac{x}{2}\right) = aS_a(x), \quad S_a\left(\frac{1+x}{2}\right) = (1-a)S_a(x) + a. \tag{1}$$

Applications for plastic deformation and in fuzzy logic can be found in [4] and [2,18], respectively.

We can see the graphs of these functions in Fig. 1.

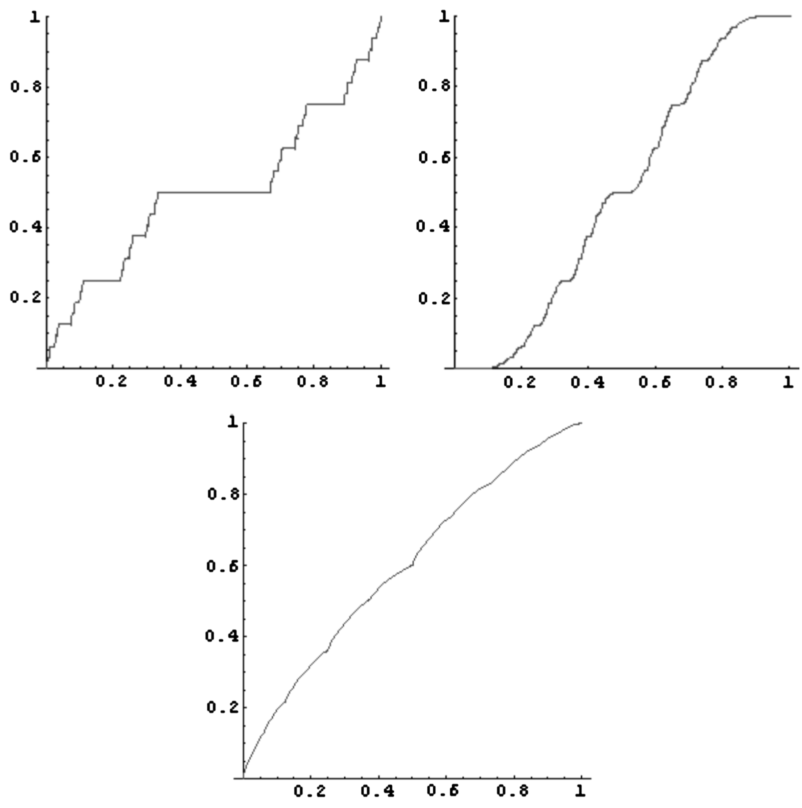

Fig. 1: Graphs of Cantor, Minkowski and $S_{.15}$ functions

We find singular functions to relate representation number systems, in a similar way to the Minkowski function in [5,20].

Under certain conditions, the Riesz products in the theory of trigonometric series are singular functions (see [34, p. 208]). Another field of mathematical knowledge where singular functions play an important role is in fractal sets. We find them as conjugating homeomorphisms (see [7, Ex. 4.6]) or Perron–Frobenius measures (see [14]).

Our target is to study another relation between singular functions and fractal sets; we refer to harmonic functions on the Sierpiński gasket S. Let us recall that the Sierpiński gasket S is a self-similar fractal object. It was first studied in 1915 in [31], but representations of it can be found in art from the Italian renaissance.

As is known, for a given equilateral triangle in the plane, with vertices $p_0 = 0$, $p_1 = 1$ and $p_2 = 1/2 + i\sqrt{3}/2$, the set S is the unique compact in $\mathbb{C}$ that is invariant under the contraction functions system $\{M_n : \mathbb{C} \to \mathbb{C} : n = 0, 1, 2\}$, given by

$$\begin{cases} M_0(z) = z/2; \\ M_1(z) = 1 + (z-1)/2; \\ M_2(z) = 1/2 + i\sqrt{3}/2 + \left(z - 1/2 - i\sqrt{3}/2\right)/2; \end{cases}$$

and, therefore, $S = M_0(S) \cup M_1(S) \cup M_2(S)$. This self-similarity property and differential equations on smooth sets, have been used to describe the motion of waves and dispersion (see for example [23,24,33]). This property has been also used in the study of fractal interpolation (see [3,9]). Recall that in this case S is a fractal set with Hausdorff and box dimensions $\ln 3/\ln 2$. Its representation can be seen in Fig. 2.

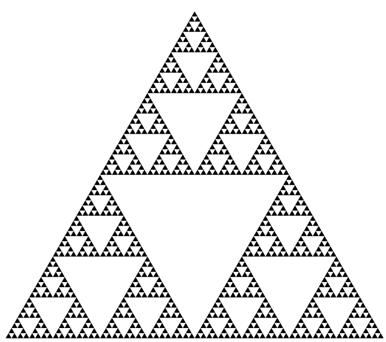

Fig. 2: Sierpiński gasket

The development of applications for sets of this type (modelling physical processes given in nature) has given rise to new analytical techniques over the last twenty years. The study of harmonic functions is a good example of the modelling of such physical processes. An introduction to this field can be found in [16].

For the purposes of this study, we restrict ourselves to considering that a function H is *harmonic on* S if for a given equilateral triangle in S with vertices p_0', p_1', p_2', such that

$$H(p_0') = \alpha_0', \quad H(p_1') = \alpha_1', \quad H(p_2') = \alpha_2',$$

then, for the middle points $p_0'' = \frac{p_1'+p_2'}{2}$, $p_1'' = \frac{p_0'+p_2'}{2}$, $p_2'' = \frac{p_0'+p_1'}{2}$, we have

$$H(p_0'') = \frac{\alpha_0' + 2\alpha_1' + 2\alpha_2'}{5}, \quad H(p_1'') = \frac{2\alpha_0' + \alpha_1' + 2\alpha_2'}{5},$$

$$H(p_2'') = \frac{2\alpha_0' + 2\alpha_1' + \alpha_2'}{5}.$$

Therefore, for initial conditions on the vertices, we can obtain corresponding values for the middle points on the sides. If we proceed by induction on the interior triangles, then, by continuity arguments, it follows that, for a given triplet of values, there exists one and only one harmonic function with given "boundary values".

NOTATION 1. Let α, β, γ be arbitrary real numbers. Let $H_{\alpha,\beta,\gamma}$ denote the harmonic function such that $H_{\alpha,\beta,\gamma}(0) = \alpha$, $H_{\alpha,\beta,\gamma}(1) = \gamma$, $H_{\alpha,\beta,\gamma}\big(1/2 + i\sqrt{3}/2\big) = \beta$. Its restriction to the unit interval $\mathbb{I}$ will be denoted by $h_{\alpha,\beta,\gamma}$.

As an example, we show the graphs of $h_{0,.1,1}$ and $h_{0,.9,1}$ in Fig. 3.

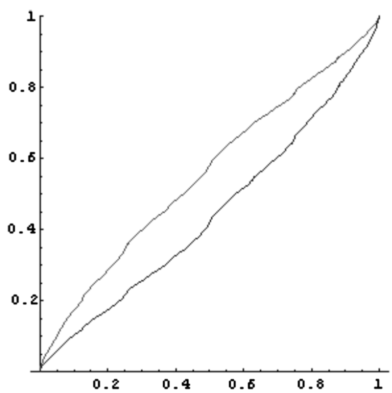

Fig. 3: Graphs of $h_{0,.1,1}$ *(down) and* $h_{0,.9,1}$ *(top)*

The first results obtained for the restrictions of H on the sides of the triangle of vertices p_0, p_1, p_2 were given by Dalrymple et al. [10, Theorem 2]. Specifically, they obtained the following result.

Theorem 2 [10]. *Let us arrange the sides of the triangle of vertices* p_0, p_1, p_2, *according to the values* $\left|H(p_i) - H(p_j)\right|$. *By decreasing order*, l_1 *denotes the corresponding side to the longest number*, l_2 *denotes the side to the intermediate one*, *and we name* l_3 *the shortest. Then*, *the function* H *is monotone on* l_1 *and* l_2, *and there are two possibilities on* l_3: a) *to be monotone*, *or* b) *to have only one extreme.*

Advances on those restrictions have recently been given by Demir et al. [12]. They show that, for each triplet (α, β, γ), the function $h_{\alpha,\beta,\gamma}$ has a derivative at $1/3$ of value zero. Moreover, they prove that it has a generalised derivative at $1/2$ of value $\pm\infty$. With the aid of the self-similarity of S, and the specific way of building the harmonic functions, they find two denumerable sets where the function has zero derivatives at each point of the first set and infinity generalized derivative at each point of the second.

This paper is devoted to the study of harmonic functions defined on S but considering their restrictions on $\mathbb{I}$. In Section 2 we prove our main result, which is that the function $h_{\alpha,\beta,\gamma}$ does not admit non-zero derivatives. As a consequence, when it is monotone, then it is a singular function. We summarize this as follows.

Theorem 3 (main result). *The function* $h_{\alpha,\beta,\gamma}$ *is singular whenever it is monotone.*

In Section 3, we further establish two non-denumerable sets where the derivatives are, respectively, zero or infinity.

The last section is devoted to wider classes of functions containing the one already studied, which are determined under some functional relations. These classes have, as particular cases, the functions S_a.

2. Proof of the main result

We distinguish each one of the three different edges. From now on, unless we say another thing, we shall adapt the convention $\alpha < \beta < \gamma$. Let us note that the relation $f_{\alpha,\beta,\gamma}(x) = f_{\gamma,\beta,\alpha}(1-x)$, and other similar, allow us to reduce to this case.

2.1. The side l_1. Let us suppose that the unit interval $\mathbb{I}$ is the side l_1. In this case we use F and f to refer to $F_{\alpha,\beta,\gamma}$ as the harmonic function and the corresponding harmonic function restriction on the side $\mathbb{I}$ is denoted by $f_{\alpha,\beta,\gamma}$. The next result allows us the study to be reduced under the restrictions $f(0) = 0$ and $f(1) = 1$. Therefore, the functions only depend on the central parameter, and we will denote $F_{0,\beta,1}$ and $f_{0,\beta,1}$ by F_β and f_β, respectively.

PROPOSITION 4. *The following equality holds*:

$$f_{\alpha,\beta,\gamma} = \alpha + (\gamma - \alpha) f_{\frac{\beta-\alpha}{\gamma-\alpha}}.$$

Now, we establish functional relations among f_β and the other elements in the family. They will be the main tools used below.

PROPOSITION 5 (functional relations). *For $x \in \mathbb{I}$ and $\beta \in]0,1[$,*

$$f_\beta\left(\frac{x}{2}\right) = \frac{2+\beta}{5} f_{\frac{1+2\beta}{2+\beta}}(x), \quad f_\beta\left(\frac{1+x}{2}\right) = \frac{3-\beta}{5} f_{\frac{\beta}{3-\beta}}(x) + \frac{2+\beta}{5}. \tag{2}$$

PROOF. The function f_β is obtained from F_β. In the triangle $M_0(S)$ of vertices 0, $1/2$ and $1/4 + i\sqrt{3}/4$, the function takes the values $F_\beta(0) = 0$, $F_\beta(1/2) = \frac{2+\beta}{5}$ and $F_\beta\big(1/4 + i\sqrt{3}/4\big) = \frac{2\beta+1}{5}$, and the method of construction is the same as for S. Therefore, $f_\beta(x) = f_{0,\frac{1+2\beta}{5},\frac{2+\beta}{5}}(2x)$ for $0 \leqq x \leqq 1/2$. Because $f_{0,\frac{1+2\beta}{5},\frac{2+\beta}{5}} = \frac{2+\beta}{5} f_{\frac{1+2\beta}{2+\beta}}$, we obtain the first of the relations.

The same is true for the interval $[1/2, 1]$, but the relation is $f_{\frac{2+\beta}{5},\frac{2+2\beta}{5},1} = \frac{2+\beta}{5} + \frac{3-\beta}{5} f_{\frac{\beta}{3-\beta}}$, with $\frac{1+x}{2}$ for the righthand term in this second equation. $\square$

From Proposition 5 it follows that the graph of the map $(x, \beta) \to f_\beta(x)$ is the attractor of an IFS on $\mathbb{I}^3$ defined by the functional relations.

The interest of these functional relations lies in the fact that one can calculate the values for f_β on reals with dyadic finite representation and, therefore, deduce an approximation by polygonals.

NOTATION 6. (i) $h_0(\beta) = \frac{2+\beta}{5}$; $h_1(\beta) = \frac{3-\beta}{5}$, $r_0(\beta) = \frac{1+2\beta}{2+\beta}$; $r_1(\beta) = \frac{\beta}{3-\beta}$.

(ii) $\beta_i(x) = r_{d(T^i(x))}\big(\beta_{i-1}(x)\big)$ with $\beta_0 = \beta$, where d and T are functions given by

$$d(x) := \begin{cases} 0, & 0 \leqq x \leqq \dfrac{1}{2} \\ 1, & \dfrac{1}{2} < x \leqq 1 \end{cases} \qquad T(x) := \begin{cases} 2x, & 0 \leqq x \leqq \dfrac{1}{2} \\ 2x-1, & \dfrac{1}{2} < x \leqq 1 \end{cases}$$

THEOREM 7. *The function f_β does not admit non-zero derivatives.*

PROOF. Let $x \in]0,1[$. For $k \in \mathbb{Z}^+$, we will look for the natural m satisfying

$$\frac{m}{2^k} \leqq x < \frac{m+1}{2^k}.$$

Let us study the quotient

$$\frac{f_\beta\left(\frac{m+1}{2^k}\right) - f_\beta\left(\frac{m}{2^k}\right)}{\frac{1}{2^k}}.$$

With the aid of the polygonals previously built, this quotient is equal to

$$2^k \prod_{i=1}^{k} \Big(\big(1 - d\big(T^i(x)\big)\big) h_0\big(\beta_i(x)\big) + d\big(T^i(x)\big)\, h_1\big(\beta_i(x)\big)\Big).$$

If the sequence of these products converges to a non-zero real number, it is necessary that

$$2\big(1 - d\big(T^i(x)\big)\big) h_0\big(\beta_i(x)\big) + 2d\big(T^i(x)\big)\, h_1\big(\beta_i(x)\big) \longrightarrow 1$$

if $i \to +\infty$. By definition of h_j, the number

$$2\big(1 - d\big(T^i(x)\big)\big) h_0\big(\beta_i(x)\big) + 2d\big(T^i(x)\big)\, h_1\big(\beta_i(x)\big)$$

is near 1 if and only if $\beta_i(x)$ is near $1/2$. But, if this is the case, then $\beta_{i+1}(x)$ is not near $1/2$, and, as a consequence, there exists $\varepsilon > 0$ such that, for all i, at least one of the values

$$2\big(1 - d\big(T^i(x)\big)\big) h_0\big(\beta_i(x)\big) + 2d\big(T^i(x)\big)\, h_1\big(\beta_i(x)\big),$$

or

$$2\big(1 - d\big(T^{i+1}(x)\big)\big) h_0\big(\beta_{i+1}(x)\big) + 2d\big(T^{i+1}(x)\big)\, h_1\big(\beta_{i+1}(x)\big),$$

does not belong to $]1-\varepsilon, 1+\varepsilon[$. In consequence, the above quotient does not converge to any real number different from zero. $\square$

The following results are known and can be found in [6], [25, p. 11] and [26, p. 149].

LEMMA 8. *Let r be a bounded variation function. Then, it is differentiable for almost all x (with respect to the Lebesgue measure).*

LEMMA 9. *Let us suppose that a function f has a (finite) derivative at a point x. Then,*

$$\lim_{u \to x^-,\ v \to x^+} \frac{f(u) - f(v)}{u - v} = f'(x).$$

COROLLARY 10. *The function f_β is singular.*

PROOF. On the one hand, Lemma 9 implies that the derivative, if it exists, has to be zero. On the other hand, this result, together with the fact that monotone functions have derivatives on a set of full measure, proves that the function f_β is differentiable on a set of measure 1, and its derivative vanishes. $\square$

2.2. The side l_2. If the unit interval $\mathbb{I}$ is the side l_2, then we will use letters G, g and g^*. We distinguish two cases.

Case a: the harmonic function is of type $G_{\beta,\alpha,\gamma}$ with $\gamma-\beta \geqq \beta-\alpha$. If $\beta \leqq 1/2$, then the restriction of $G_{\beta,0,1}$ is denoted as $g_{\beta,0,1}$ by g_β.

We can restrict our study to these functions as it is exposed by the next result.

PROPOSITION 11. *The following equality holds*:

$$g_{\beta,\alpha,\gamma} = \alpha + (\gamma-\alpha) g_{\frac{\beta-\alpha}{\gamma-\alpha}}.$$

Now we establish functional relations among g_β and the other elements in the family.

PROPOSITION 12 (functional relations). *In the case where $\beta \leqq 1/2$, the unit interval $\mathbb{I}$ is l_2, and then*

$$\tag{3} \begin{cases} g_\beta\left(\frac{x}{2}\right) = \begin{cases} \beta + \dfrac{2-3\beta}{5} f_{\frac{1-3\beta}{2-3\beta}}(x), & \text{if } \beta < 1/3, \\ \dfrac{2\beta+1}{5} + \dfrac{1}{5} g_{3\beta-1}(x), & \text{if } 1/3 < \beta, \end{cases} \\ g_\beta\left(\dfrac{1+x}{2}\right) = \dfrac{3-\beta}{5} g_{\frac{\beta}{3-\beta}}(x) + \dfrac{2+\beta}{5} \end{cases}$$

where the functions f_a are those introduced in the preceding subsection.

Case b: the harmonic function is in the form $G_{\alpha,\gamma,\beta}$, with $\gamma-\beta < \beta-\alpha$. If $\beta > 1/2$, then we denote $g^*_{0,1,\beta}$ by g^*_β.

PROPOSITION 13. *With the above notation*,

$$g^*_{\alpha,\gamma,\beta} = \alpha + (\gamma-\alpha) g^*_{\frac{\beta-\alpha}{\gamma-\alpha}}.$$

Now, we establish functional relations among g^*_β and the other elements in the family. They are the main tools we use below.

PROPOSITION 14 (functional relations). *If $\beta > 1/2$, the unit interval $\mathbb{I}$ corresponds to l_2, and, in this case,*

$$(4)\qquad \begin{cases} g^*_\beta\left(\dfrac{x}{2}\right) = \dfrac{\beta+2}{5} g^*_{\frac{1+2\beta}{\beta+2}}(x), \\ g^*_\beta\left(\dfrac{1+x}{2}\right) = \begin{cases} \dfrac{1+2\beta}{5} + \dfrac{3\beta-1}{5} f_{\frac{1}{3\beta-1}}(x) & \text{if } \beta > 2/3, \\ \dfrac{2\beta+1}{5} + \dfrac{1}{5} g^*_{3\beta-1}(x) & \text{if } 2/3 > \beta. \end{cases} \end{cases}$$

With the help of these relations, and applying the same arguments as in the previous case, we can prove the following result.

THEOREM 15. *The functions g_β and g^*_β are singular, and they do not admit non-zero derivatives.*

2.3. The side l_3. Let us suppose that $\gamma - \beta > \beta - \alpha$. Then, the behavior of the harmonic function $H_{\alpha,\gamma,\beta}$ on the side l_3 is similar to the behavior of $H_{0,1,\beta}$ under the condition $\beta < \frac{1}{2}$. The function $H_{0,1,\beta}$ takes the values

$$\begin{cases} H_{0,1,\beta}(0) = 0; \quad H_{0,1,\beta}\big(1/2 + i\sqrt{3}/2\big) = 1; \quad H_{0,1,\beta}(1) = \beta, \\ H_{0,1,\beta}\big(1/4 + i\sqrt{3}/4\big) = \dfrac{2+\beta}{5}; \quad H_{0,1,\beta}\big(3/4 + i\sqrt{3}/4\big) = \dfrac{2+2\beta}{5}; \\ H_{0,1,\beta}(1/2) = \dfrac{1+2\beta}{5}. \end{cases}$$

This implies that if $x \in [0, 1/2]$, then $H_{0,1,\beta}(x) = H_{0,\frac{2+\beta}{5},\frac{2\beta+1}{5}}(2x)$; that is, the behavior of the function $H_{0,1,\beta}$ in $[0, 1/2]$ is similar to the behavior of a harmonic function in the side l_2.

On the other hand, if $x \in [1/2, 1]$, then $H_{0,1,\beta}(x) = H_{\frac{2\beta+1}{5},\frac{2+2\beta}{5},\beta}(2x-1)$. Now, the behavior of the function $H_{0,1,\beta}$ in $[1/2, 1]$ is similar to the behavior of a harmonic function in the side l_3. If we repeat the process, then we obtain a denumerable division of the unit interval $\mathbb{I}$ with subintervals of lengths $1/2, 1/2^2, \ldots$ where the behavior of $H_{0,1,\beta}$ is similar to the behavior of a harmonic function in the side l_2.

Therefore, the harmonic function $H_{\alpha,\gamma,\beta}$, on the side l_3, inherits the property of not having non-zero derivatives. Let us recall that in the side l_3, the function can miss monotonicity, having only one maximum in this case.

2.4. The degenerated case. After the study of the three cases before, we now consider functions where the relation $\alpha < \beta < \gamma$ fails. The case $\alpha = \beta = \gamma$ trivializes. Therefore, we consider two cases, each with two subcases.

a) $\alpha = \beta < \gamma$.

a.1) The study of $H_{\alpha\beta\gamma}$ is equivalent to the study of H_{001}. We denote by h_{001} its restriction to the unit interval $\mathbb{I}$. This function decomposes into two functions that are similar to $f_{0,1/5,2/5}$ and $g_{2/5,1/5,1}$, such that the behavior of h_{001} in $\mathbb{I}$ coincides with the respective behaviors of $f_{0,1/5,2/5}$ in $[0,1/2]$ and of $g_{2/5,1/5,1}$ in $[1/2,1]$.

a.2) The case $H_{\alpha\gamma\beta}$ is the same as H_{010}. Again, the decomposition in the two subintervals $[0,1/2]$ and $[1/2,1]$ gives, respectively, an equivalent function to $f_{0,2/5,1/5}$ and to $f_{1/5,2/5,1}$.

b) $\alpha < \beta = \gamma$.

b.1) The study of $H_{\beta\alpha\gamma}$ is similar to that of H_{101}. In $[0,1/2]$, the behaviour of h_{101} is similar to $g_{1,3/5,4/5}$, and in $[1/2,1]$ as $g_{4/5,3/5,1}$.

b.2) The case of $H_{\alpha\beta\gamma}$ is analogous to that of H_{011}. The function h_{011} has the same behaviour in $[1/2,1]$ that $f_{3/5,4/5,1}$. In $[0,1/2]$, its behaviour is like a replica of itself. Precisely, $h_{011}(x) = h_{011}(2x)$. Therefore, if we iterate this idea in $[1/2^{n+1}, 1/2^n]$, the behaviour of h_{011} likes $f_{3/5,4/5,1}$.

In summary, $h_{\alpha\beta\gamma}$ is a singular function in each case.

3. Properties in the side l_1 case

In light of the above results, it is of interest to find a set of points where derivatives exist and are zero. Moreover, we will look for another set where the function has infinity generalised derivative at each point. We now restrict ourselves to the edge l_1.

We should first remark on the behavior of the sequence

$$\big(1 - d\big(T^i(x)\big)\big)h_0\big(\beta_i(x)\big) + d\big(T^i(x)\big)\, h_1\big(\beta_i(x)\big).$$

Each term depends upon β and all $d\big(T^j(x)\big)$ with $j \leqq i$. Independently of β, we obtain the following bounds: if

$$d\big(T^i(x)\big) = d\big(T^{i-1}(x)\big) = \cdots = d\big(T^{i-k}(x)\big) = 1$$

or

$$d\big(T^i(x)\big) = d\big(T^{i-1}(x)\big) = \cdots = d\big(T^{i-k}(x)\big) = 0,$$

then

$$1/2 < \theta(k) < \big(1 - d\big(T^i(x)\big)\big)h_0\big(\beta_i(x)\big) + d\big(T^i(x)\big)\, h_1\big(\beta_i(x)\big) < 3/5,$$

with $\theta(k) \to 3/5$ for $k \to +\infty$.

Proposition 16. *The set of points whose dyadic expansions are given by chains of* 0*'s and* 1*'s of length k or $k+1$ (this k is found and fixed during the proof) is non-denumerable with f_β having infinity generalised derivative at each point.*

Proof. With k large enough, if

$$d\big(T^i(x)\big) = d\big(T^{i-1}(x)\big) = \cdots = d\big(T^{i-k}(x)\big) = \begin{cases} 0 \\ 1 \end{cases}$$

then,

$$2^{k+1} \prod_{j=0}^{k} \Big(\big(1 - d\big(T^{i+j}(x)\big)\big) h_0\big(\beta_{i+j}(x)\big) + d\big(T^{i+j}(x)\big) h_1\big(\beta_{i+j}(x)\big) \Big) > \rho > 1.$$

Let x be in the set of the statement. There exists C such that if $h > 0$, then we can choose numbers $x_h = \frac{r}{2^t}$, $x'_h = \frac{r+1}{2^t}$ satisfying that $x \leqq x_h \leqq x'_h \leqq x + h$, with $x'_h - x_h \geqq Ch$ and

$$\frac{f(x+h) - f(x)}{h} \geqq \frac{f(x'_h) - f(x_h)}{h} \geqq C \frac{f(x'_h) - f(x_h)}{x'_h - x_h} \geqq C\rho^n;$$

where n depends on x and h tends to infinity when $h \to 0$.

The process is similar for negative h.

To show it is a non-denumerable set, it is enough to demonstrate that the subset of its points starting with 0 is under bijection with the sequences $\{a_n\}_{n\in\mathbb{Z}^+}$ with $a_{2j} \in \{s, t\}$ and $a_{2j+1} \in \{u, v\}$ (where $\{s, t\} \cap \{u, v\} = \emptyset$). □

Remark 17. It is unnecessary for all the chains to be in this form: it is enough for them to be so at the end; as we can observe in the proof, the beginning chains have no influence.

It is not difficult to transfer this reasoning to the case $d\big(T^i(x)\big) = 0$ for all $i \geqq m$. This is the result in [12, Lemma 6] for the l_1 case.

Proposition 18. *The set of points whose dyadic expansion ends with a chain of* 10*'s of length greater than* s (*determined during the proof*), *followed by a chain of* 100*'s, is a non-denumerable set. The function* f_β *has a zero derivative at all these points.*

Proof. Let x be in such a set. There exists C such that for each $h > 0$, we can choose numbers $x_h = \frac{k}{2^t}, x'_h = \frac{k+1}{2^t}$ satisfying $x_h \leqq x \leqq x + h \leqq x'_h$, with $x'_h - x_h \leqq Ch$ and

$$\frac{f(x) - f(x+h)}{h} = O\left(\frac{f(x_h) - f(x'_h)}{x'_h - x_h} \right).$$

On these points,

$$\frac{f_\beta\left(\frac{k+1}{2^t}\right) - f_\beta\left(\frac{k}{2^t}\right)}{\frac{1}{2^t}} = 2^t \prod_{i=1}^{t} \Big[\big(1 - d\big(T_i(x_h)\big)\big) h_0\big(\beta_i(x_h)\big)$$

$$+ d\big(T_i(x_h)\big)\, h_1\big(\beta_i(x_h)\big)\Big].$$

The way the set has been defined requires an examination of the behavior of

$$\prod \Big(\big(1 - d\big(T^i(x)\big)\big) 2h_0\big(\beta_i(x)\big) + d\big(T^i(x)\big)\, 2h_1\big(\beta_i(x)\big)\Big)$$

for chains of pairs of 10's. Because 1's and 0's alternate, the same occurs for $2h_1$ and $2h_0$ inside the product. The same is also true for r_0 and r_1 in the definition of β_i. If the length of the chain grows, then $\beta_i \to \frac{5-\sqrt{13}}{2}$ for the values associated with 0 and converges to $\frac{-3+\sqrt{13}}{2}$ for 1's. For these values, we have

$$2h_1\left(\frac{5-\sqrt{13}}{2}\right) = 2h_0\left(\frac{-3+\sqrt{13}}{2}\right) = \frac{1+\sqrt{13}}{5}.$$

For the second couple in the form 10, both $2r_0$ and $2r_1$ are less than $\mu < 1$, and for the last term in the final triple, 100, we can consider $2\frac{3}{5}$ as a bound. Therefore, we can consider s such that $\mu^{s-2} 2\frac{3}{5} = \delta < 1$. Then,

$$\frac{f(x) - f(x+h)}{h} = O\left(\frac{f(x_h) - f(x_h')}{x_h' - x_h}\right)$$

$$= 2^t \prod_{i=1}^{t} \Big[\big(1 - d\big(T_i(x_h)\big)\big) h_0\big(\beta_i(x_h)\big) + d\big(T_i(x_h)\big)\, h_1\big(\beta_i(x_h)\big)\Big] = O(\delta^n),$$

where n depends on x and h, and $n \to \infty$ if $h \to 0$, and the quotient tends to infinity when $h \to 0$.

The same process works for negative h. The set is non-denumerable because of its bijection with $\{s, s+1, s+2, \dots\}^{\mathbb{N}}$. □

REMARK 19. It is easy to see that these ideas can be used for the case $x = 1/3$ for all $i \geqq k$. This is the result shown by [12, Theorem 7] in the case of l_1.

It is easy to check that the result is still true doing the substitution of 0's by 1's and viceversa. (The last comment is necessary because the restriction of the harmonic function to the side l_3 may be not monotone.)

Proposition 20. *The above proposition is still true for the sides l_2 and l_3.*

Proof. The case for the side l_2 can be reduced to the above (for l_1). According to the functional relations in (3), if $d\big(T^i(x)\big) = 0$ and $d\big(T^{i-1}(x)\big) = 1$, we can find an interval containing x where the function is a replica of one in the class of f_β. Therefore, any property in the last two propositions is true. The conditions $d\big(T^i(x)\big) = 0$ and $d\big(T^{i-1}(x)\big) = 1$ are satisfied at every point but the origin. Therefore, the statement is true.

Where the functional relations are those given by (4), a similar consideration gives the same result.

With respect to l_3, we have already called attention to the decomposition through replicas of functions in the form of f_β or g_β; therefore, the result follows. □

Theorem 21. *If $m_x(\beta) := f_\beta(x)$, then m_x is a continuous function.*

Proof. If $x = \frac{n}{2^k}, m_x(\beta)$ is determined, via functional relations, by k steps. Its expression is rational in β, with coefficients only depending upon $\frac{n}{2^k}$. If β' tends to β, then $m_{\frac{n}{2^k}}(\beta')$ tends to $m_{\frac{n}{2^k}}(\beta)$, and for this x, the function m_x is continuous.

For arbitrary x, the inequalities

$$\big|m_x(\beta) - m_x(\beta')\big| = \big|f_\beta(x) - f_{\beta'}(x)\big| \leqq \left|f_\beta(x) - f_\beta\left(\frac{n}{2^k}\right)\right|$$

$$+\left|f_\beta\left(\frac{n}{2^k}\right) - f_{\beta'}\left(\frac{n}{2^k}\right)\right| + \left|f_{\beta'}\left(\frac{n}{2^k}\right) - f_{\beta'}(x)\right|$$

give the continuity of m_x. □

Theorem 22. *The function m_x is monotone increasing.*

Proof. For $x = \frac{n}{2^k}$, monotonicity is a consequence of the respective of r_0, r_1, and h_0. By continuity arguments, we can deduce the result for the rest of the cases. □

Theorem 23. *The function f_β fills a region into the unit square whose perimeter is given by graphs of singular functions.*

Proof. The function m_x is continuous and monotone increasing and, therefore, $m_x\big(\,]0,1[\,\big)$ is an interval. Let us denote

$$\widehat{S}(x) := \lim_{\beta\to 1} m_x(\beta) = \lim_{\beta\to 1} f_\beta(x).$$

Taking limits in the functional relations (if $\beta \to 1$),

$$f_\beta\left(\frac{x}{2}\right) = \frac{2+\beta}{5} f_{\frac{1+2\beta}{2+\beta}}(x), \quad f_\beta\left(\frac{1+x}{2}\right) = \frac{3-\beta}{5} f_{\frac{\beta}{3-2\beta}}(x) + \frac{2+\beta}{5}$$

we have a new system for $\widehat{S}$:

$$\widehat{S}\left(\frac{x}{2}\right) = \frac{3}{5}\widehat{S}(x), \quad \widehat{S}\left(\frac{1+x}{2}\right) = \frac{2}{5}\widehat{S}(x) + \frac{3}{5}.$$

These functional equations are a particular case of those in (1). Therefore, $\widehat{S}$ coincides with $S_{3/5}$ whose graph can be seen in Fig. 1.

In the same way, writing $I(x) := \lim_{\beta\to 0} m_x(\beta) = \lim_{\beta\to 0} f_\beta(x)$, the equations that follow are:

$$I\left(\frac{x}{2}\right) = \frac{2}{5} f_{\frac{1}{2}}(x), \quad I\left(\frac{1+x}{2}\right) = \frac{3}{5} I(x) + \frac{2}{5}.$$

These functional equations allow us to see that I is a function built by replicas with different scales of $f_{\frac{1}{2}}$ on intervals of the form $\left[\frac{2^{k-1}-1}{2^{k-1}}, \frac{2^k-1}{2^k}\right]$. $\square$

The described region can be seen in grey color in Fig. 4.

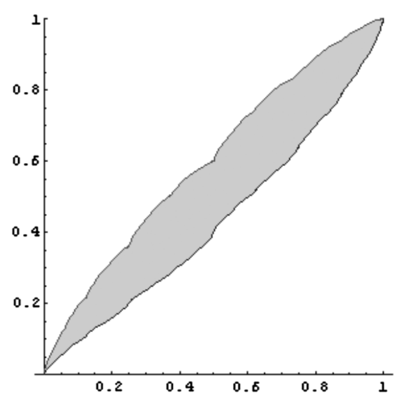

Fig. 4: The region in Theorem 23

4. Generalization

In the above proofs, we have often made use of the concrete definitions and properties of the functions h_0, h_1, r_0, and r_1. It is possible to generalize this family of functions to another with the same properties.

LEMMA 24. *Let continuous functions* $h_0, r_0, r_1 : \mathbb{I} \to \mathbb{I}$ *be given, satisfying*:

a) $h_0(\mathbb{I}) \subset \,]0, 1[$,

b) *if there exists* β *such that* $h_0(\beta) = 1/2$, *then*

$$h_0\big(r_0(\beta)\big) \neq 1/2 \neq h_0\big(r_1(\beta)\big).$$

Then, there exists a family of continuous functions satisfying that $f_\beta(0) = 0$; $f_\beta(1) = 1$; *and*

$$f_\beta\left(\frac{x}{2}\right) = h_0(\beta) f_{r_0(\beta)}(x), \quad f_\beta\left(\frac{1+x}{2}\right) = \big(1 - h_0(\beta)\big) f_{r_1(\beta)}(x) + h_0(\beta).$$

THEOREM 25. *The functions* f_β *are monotone, singular, and they do not admit non-zero derivatives.*

PROPOSITION 26. *The function* $m_x(\beta) := f_\beta(x)$ *is continuous.*

PROPOSITION 27. *If* r_0, r_1, h_0 *are monotone increasing, then* m_x *is monotone as well.*

PROPOSITION 28. *If* r_0, r_1, h_0 *are monotone increasing and* h_0 *is not constant, then* $f_\beta(x)$ *fills a surface that is surrounded by graphs of singular functions.*

4.1. Examples. We conclude by exhibiting two examples where our exposition above works.

EXAMPLE 29. The case $r_0(\beta) = r_1(\beta) = h_0(\beta) = a \neq 1/2$ corresponds to functions S_a.

EXAMPLE 30. Let us imitate the way harmonic functions were generated. Now, we consider weights a_0, a_1, instead of $\frac{2}{5}, \frac{1}{5}$ and $\frac{2}{5}$, satisfying $a_0 > a_1 \neq 0$ and $2a_0 + a_1 = 1$. Then we generate functions proceeding at the middle points of the sides, with the equalities:

$$H(p_0'') = a_1\alpha_0' + a_0\alpha_1' + a_0\alpha_2',$$

$$H(p_1'') = a_0\alpha_0' + a_1\alpha_1' + a_0\alpha_2',$$

$$H(p_2'') = a_0\alpha_0' + a_0\alpha_1' + a_1\alpha_2'.$$

In these conditions, the restriction to $\mathbb{I}$ is a singular function of this type where

$$h_0(\beta) = a_1\beta + a_2, \quad r_0(\beta) = \frac{a_2\beta + a_1}{a_1\beta + a_2}, \quad r_1(\beta) = \frac{(a_0 - a_1)\beta}{a_1(1 - \beta)}.$$

REMARK 31. In [27], the author generalizes the building of S_a, and characterizes the case when the functions are singular. In the case $r_0(\beta) = r_1(\beta)$, all the elements f_β in the family are members of those generalizations done by Salem.

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References

[1] E. de Amo, M. Díaz Carrillo and J. Fernández Sánchez, Singular functions with applications to fractals and generalised Takagi's functions, *Acta Appl. Math.*, **218** (2012), 8634–8644.

[2] E. de Amo, M. Díaz Carrillo and J. Fernández Sánchez, On the duality of aggregation operators and k-negations, *Fuzzy Sets and Systems*, **181** (2011), 14–27.

[3] E. de Amo, M. Díaz Carrillo and J. Fernández Sánchez, PCF self-similar sets and fractal interpolation, *Maths. Comput. Simul.*, **92** (2013), 28–39.

[4] B. Bernstein, T. Erber and M. Karamolengos, Sneaky plasticity and mesoscopic fractals, *Chaos, Solitons & Fractals*, **3** (1993), 269–277.

[5] L. Bibiloni, J. Paradís and P. Viader, Fibonacci, Van der Corput and Riesz–Nagy, *J. Math. Anal. Appl.*, **362** (2010), 401–414.

[6] P. Billingsley, *Probability and Measure*, 2nd ed., Wiley (New York, 1995).

[7] J. Bohnstengel and M. Kesseböhmer, Wavelets for iterated function systems, *J. Funct. Anal.*, **259** (2010), 583–601.

[8] G. Cantor, De la puissance des ensembles parfait de points, *Acta Math.*, **4** (1884), 381–392.

[9] D. Celik, S. Koçak and Y. Özdemir, Fractal interpolation on the Sierpinski Gasket, *J. Math. Anal. Appl.*, **337** (2008), 343–347.

[10] K. Dalrymple, R. Strichartz and J. Vinson, Fractal differential equations on the Sierpiński gasket, *J. Fourier Anal. Appl.*, **5** (1999), 203–284.

[11] G. De Rham, Sur quelques courbes definies par des equations fonctionelles, *Rend. Sem. Mat. Univ. Politec. Torino*, **16** (1956), 101–113.

[12] B. Demir, V. Dzhafarov, Ş. Koçak and M. Üreyen, Derivatives of the restrictions of harmonic functions on the Sierpinski gasket to segments, *J. Math. Anal. Appl.*, **333** (2007), 817–822.

[13] A. Denjoy, Sur une fonction de Minkowski, *C.R. Acad. Sci. Paris*, **194** (1932), 44–46.

[14] D. E. Dutkay and P. E. T. Jorgensen, Wavelets on fractals, *Rev. Mat. Iberoam.*, **22** (2006), 131–180.

[15] E. Hewitt and K. R. Stromberg, *Real and Abstract Analysis*, Springer-Verlag (New York, 1965).

[16] J. I. Kigami, *Analysis on Fractals*, Cambridge University Press (2001).

[17] H. Lebesgue, *Leçons sur l'intégration et la recherche des fonctions primitives*, Gauthier-Villars (Paris, 1904).

[18] G. Mayor and J. Torrens, Duality for binary operations, *Fuzzy Sets and Systems*, **47** (1992), 77–80.

[19] H. Minkowski, Verhandlungen des III internationalen Mathematiker-kongresses in Heidelberg, 1904; and in *Gesammelte Abhandlungen,* **2** (1991), 50–51.

[20] J. Paradís, P. Viader and L. Bibiloni, A total order in $(0, 1]$ defined through a 'next' operator, *Order*, **16** (1999), 207–220.

[21] J. Paradís, P. Viader and L. Bibiloni, The derivative of Minkowski's $?(x)$ function, *J. Math. Anal. Appl.*, **253** (2001), 107–125.

[22] J. Paradís, P. Viader and L. Bibiloni, Riesz–Nagy singular functions revisited, *J. Math. Anal. Appl.*, **329** (2007), 592–602.

[23] C. Puente Baliarda, J. Romeu Robert, R. Pous Andrés and A. Cardama Aznar, On the behavior of the Sierpinski multiband fractal antenna, *IEEE Trans. on Antennas and Propagation*, **46** (1998), 517–524.

[24] C. Puente, J. Romeu, R. Pous, X. García and F. Beníte, Fractal multiband antenna based on the Sierpinski gasket, *Electr. Letters*, **32** (1996), 1–2.

[25] F. Riesz and B. Sz.-Nagy, *Leçons d'analyse fonctionnelle*, Gauthier-Villars (Paris, 1968).

[26] W. Rudin, *Real and Complex Analysis*, 3rd ed., McGraw-Hill Inc. (1987).

[27] R. Salem, On singular monotone functions, *Trans. Amer. Math. Soc.*, **53** (1943), 427–439.

[28] L. Takács, An increasing continuous singular function, *Amer. Math. Monthly*, **85** (1978), 35–36.

[29] P. Viader, J. Paradís and L. Bibiloni, A new light on Minkowski's $?(x)$ function, *J. Number Theory*, **73** (1998), 212–227.

[30] L. Scheefer, Allgemeine Untersuchungen über Rektification den Kurven, *Acta Math.*, **5** (1884), 49–82.

[31] W. Sierpiński, Sur une courbe dont tout point est un point de ramification, *C.R. Acad. Sci. Paris*, **160** (1915), 302–305.

[32] L. Wen, An approach to construct the singular monotone functions by using Markov chains, *Taiwanese J. Math.*, **2** (1998), 361–368.

[33] M. Yamaguti, M. Hata and J. I. Kigami, *Mathematics of Fractals*, University of Bangalore Press (Bangalore, 1997).

[34] A. Zygmund, *Trigonometric Series, Vol. I*, Cambridge U. P. (Cambridge, 1959).