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Characterization of copulas with given diagonal and opposite diagonal sections

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Abstract

In recent years special attention has been devoted to the problem of finding a copula, the diagonal section and opposite diagonal section of which are known. For given diagonal function and opposite diagonal functions, we provide necessary and sufficient conditions for the existence of a copula to have these functions as diagonal and opposite diagonal sections. We make use of techniques related to interpolation between the diagonals, the construction of checkerboard copulas, and linear programming. This result allows us to solve two open problems: to characterize the class of copulas where the knowledge of diagonal and opposite diagonal sections determines the copula in a unique way, and to formulate necessary and sufficient conditions for each pair of such functions to be the diagonal and opposite diagonal sections of a unique copula.

Keywords: Diagonal section; Opposite diagonal section; Checkerboard approximation

1. Introduction

Copulas are of interest in statistical analysis and modelling because they allow to build families of bivariate distributions with given marginals, and to study scale-free measures of dependence. The importance of copulas derives from Sklar's Theorem (see [1,29]), which states that the joint distribution function H of the random pair (X, Y) with respective marginals F and G can be expressed by $H(x, y) = C(F(x), G(y))$ for all (x, y), where C is a copula that is uniquely determined on $\mathrm{Ran}(F) \times \mathrm{Ran}(G)$. For details of copulas and their applications we refer to [14,25,28].

In recent years, in view of the fact that tail dependence is dictated by the behaviour of the copula on the diagonal and/or opposite diagonal of the unit square, special attention has been devoted in several papers to the problem of

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finding a copula C when its diagonal is known and to determining the best-possible bounds for the functions thus constructed. Specifically, there has been a growing interest in the determination of copulas with given values at some fixed sections, having pre-assigned values on some regions, or making use of a geometric nature on the copula, such as properties of the graphs of vertical, horizontal and diagonal section, which have specific tail dependences and asymmetries (see [2,9–12,15,18,26]).

As we have just observed, the diagonal and opposite diagonal sections of a copula have probabilistic interpretations and they can be used to study the upper–upper and lower–lower (upper–lower and lower–upper, respectively) tail dependence (see [5,19,25] and the references therein). On the other hand, in [13] a method is presented for constructing copulas based on the redefinition that a known copula assumes on some rectangles in the unit square ("rectangular patchwork" construction), and they provide copulas with a variety of tail dependence. In [18] the necessary and sufficient conditions are formulated for a function in the unit square to be the diagonal section of a multivariate absolutely continuous copula. Finally, we note that copulas are 1-Lipschitz aggregation operators and relevant applications of this fact are used in fuzzy logic theory. See, for instance, [22] and the references therein, where smallest and greatest 1-Lipschitz aggregation operators with given diagonal section and opposite diagonal section are determined.

At this point, to the best of our knowledge, the formulation of sufficient conditions that guarantee the generation of copulas with given diagonal section and opposite diagonal section is an open problem (see [4]).

The answer to this problem allows us to solve two other problems posed by Klement and Kolesárová in [23]:

(i) to characterize the class of copulas where the knowledge of its diagonal and opposite diagonal determines the copula in a unique way, and
(ii) to characterize all pairs of functions δ and ω such that there is a unique copula with diagonal section δ and opposite diagonal section ω.

The aim of this paper is to review these problems and to present a new method, based on interpolation between the diagonals, for creating copulas with given diagonal sections, and determining under which conditions (that are as weak as possible) the copulas with given diagonal and opposite diagonal can be obtained. Our method uses the concept of checkerboard copulas (see [24]) and the technique of linear programming (LP).

2. Preliminaries

A *bivariate copula* (briefly, a *copula*) is a function $C\colon \mathbb{I}^2 \to \mathbb{I}$ (with $\mathbb{I} = [0, 1]$) satisfying the following conditions:

(C1) $C(x, 0) = C(0, x) = 0$ for all $x \in \mathbb{I}$,
(C2) $C(x, 1) = C(1, x) = x$ for all $x \in \mathbb{I}$,
(C3) for all x_1, x_2, y_1, y_2 in $\mathbb{I}$ with $x_1 \le x_2$ and $y_1 \le y_2$, $V_C([x_1, x_2] \times [y_1, y_2]) = C(x_2, y_2) - C(x_1, y_2) - C(x_2, y_1) + C(x_1, y_1) \ge 0$.

V_C is called the *C-volume* of the rectangle $[x_1, x_2] \times [y_1, y_2]$. A copula C induces a probability measure μ_C on $\mathbb{I}^2$ defined for each rectangle $R \subseteq \mathbb{I}^2$ by $\mu_C(R) = V_C(R)$, and is extendible in a standard way to the σ-algebra of all Borel sets in $\mathbb{I}^2$.

Classical examples of copulas are $M(x, y) = \min\{x, y\}$, $\Pi(x, y) = xy$, and $W(x, y) = \max\{x + y - 1, 0\}$ expressing, respectively, *comonotonicity*, *independence* and *counter-monotonicity* between two random variables.

The *diagonal section* of a copula C is the function $\delta_C\colon \mathbb{I} \to \mathbb{I}$ defined by $\delta_C(t) = C(t, t)$. On the other hand, a *diagonal function* $\delta\colon \mathbb{I} \to \mathbb{I}$ is a function which satisfies the following conditions:

(D1) $\delta(1) = 1$,
(D2) $\delta(t) \le t$ for all $t \in \mathbb{I}$,
(D3) δ is increasing, and
(D4) $|\delta(v) - \delta(u)| \le 2|v - u|$ for all $u, v \in \mathbb{I}$.

Note that the diagonal section δ_C of any copula C is a diagonal function.

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The *opposite diagonal section* ω_C of a copula C is a function ω_C: $\mathbb{I} \to \mathbb{I}$ defined by $\omega_C(t) = C(t, 1-t)$. An *opposite diagonal function* is a function ω: $\mathbb{I} \to \mathbb{I}$ satisfying the following conditions:

(W1) $\omega(t) \le \min(t, 1-t)$, for all $t \in \mathbb{I}$,
(W2) $|\omega(v) - \omega(u)| \le |v-u|$ for all $u, v \in \mathbb{I}$.

Then the opposite diagonal section of a copula is an opposite diagonal function. Conversely, for any opposite diagonal function there exists at least one copula with the function as opposite diagonal section (see [5]).

Given a diagonal function δ: $\mathbb{I} \to \mathbb{I}$, $\mathcal{C}_\delta$ denotes the class of all copulas having diagonal section equal to δ. $\mathcal{C}_\delta$ is a non-empty set; it contains the diagonal copula K_δ, defined by

$$K_\delta(x, y) = \min\left\{x, y, \frac{\delta(x) + \delta(y)}{2}\right\}. \tag{1}$$

Let C be a copula, then for any $0 < a_1 < a_2 \cdots < a_n < 1$ and $0 < b_1 < b_2 \cdots < b_m < 1$ there exists a copula D, named *checkerboard copula*, such that the measure μ_D distributes uniformly a mass equal to $V_C([a_i, a_{i+1}] \times [b_j, b_{j+1}])$ on the rectangle $[a_i, a_{i+1}] \times [b_j, b_{j+1}]$.

3. Copulas with given diagonal and opposite diagonal sections

In [5], a method for constructing copulas with given diagonal and opposite diagonal sections is presented. It makes use of a method for constructing cross-copulas with given horizontal and vertical sections.

Now, we solve an open problem concerning the minimal set of sufficient conditions on δ and ω that guarantees the existence of a copula with functions δ and ω as diagonal and opposite diagonal sections.

Let us consider the following assumptions: let δ be a diagonal function and ω an opposite diagonal function satisfying:

$$\begin{cases} (a) & \forall t \in [0, 1/2], \quad 0 \le \omega(t) - \delta(t); \qquad 0 \le \omega(1-t) - \delta(t), \\ (b) & \forall t \in [1/2, 1], \quad \delta(t) - \omega(t) \le 2t - 1; \qquad \delta(t) - \omega(1-t) \le 2t - 1, \\ (c) & \forall t, t' \in [0, 1/2], \quad t < t', \\ & \qquad \delta(t) + \delta(1-t) - \omega(t) - \omega(1-t) \ge \delta(t') + \delta(1-t') - \omega(t') - \omega(1-t'). \end{cases} \tag{2}$$

From (2)(a) and (2)(b), it follows that $\delta(\frac{1}{2}) = \omega(\frac{1}{2})$. Taking this equality into account and putting $t' = \frac{1}{2}$ in (2)(c), we have that $\delta(t) + \delta(1-t) - \omega(t) - \omega(1-t) \ge 0$, for all $t \in [0, 1/2]$.

Note that conditions (2) are necessary conditions, in the sense that for any copula C it holds that its diagonal and opposite diagonal sections satisfy these conditions. Conditions (a) and (b) essentially express that when one moves from a point on the diagonal (resp. opposite diagonal) horizontally or vertically to a point on the opposite diagonal (resp. diagonal), the values of any copula C in those two points cannot differ more in absolute value than the distance between the two points. In other words, conditions a) and b) follow from the 1-Lipschitz continuity property of copulas (which is itself a consequence of the defining properties (C1)–(C3)). Condition c), on the other hand, follows directly from the 2-increasingness property (C3) of copulas. It expresses that for any copula C, the C-volume of an arbitrary square centered on the main and opposite diagonal of the unit square (i.e. with two opposite corner points on the diagonal and the other two corner points on the opposite diagonal) is greater or equal than the C-volume of any other square that is centered on the diagonal and opposite diagonal and is entirely contained in the given square. As the midpoint of the square $(1/2, 1/2)$ can be regarded as a (degenerated) square with zero C-volume that is contained in any square centered on the diagonal and opposite diagonal, condition (c) also indirectly expresses that for any copula C and any square centered on the diagonal and opposite diagonal, it holds that the C-volume of that square is positive. Note that the latter condition alone is not equivalent with condition (c) as it does not guarantee the monotonicity of C-volumes on growing squares.

In what follows we will prove that the necessary conditions (2) are also sufficient conditions, in the sense that for any diagonal function δ and any opposite diagonal function ω satisfying (2), there exists at least one copula that has δ and ω, respectively as diagonal and opposite diagonal sections. This is the main result of our paper which is formally stated in Theorem 3.2. The major part of the paper is concerned with the proof of this theorem. Our proof is constructive, in the sense that given a diagonal function δ and opposite diagonal function ω satisfying (2), we

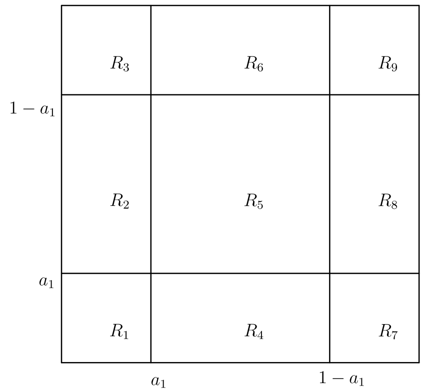

Fig. 1. Division in case $n = 1$.

construct (in a limiting procedure) a binary function defined on the unit square that has the boundary properties (C1) and (C2) and has δ and ω as diagonal and opposite diagonal section, respectively. To prove that this function also has the 2-increasingness property (C3) – proving that it is also a copula – we rely on an algebraic technique that has been used previously for proving the existence of ternary copulas with given values in some points of the unit cube [6]. The main idea is to bring the condition of 2-increasingness into a form such that, by means of Farkas' lemma, it suffices to show that the target function of some well-defined associated linear programming problem (LP-problem) has minimum value zero. Let us recall Farkas' lemma (see [16,27]) which is a theorem stating that of two systems, one or the other has a solution, but not both nor none. In the sequel, the notation $\mathbf{x} \geq \mathbf{0}$ denotes $x_i \geq 0$ for all the components of the vector $\mathbf{x}$.

Lemma 3.1 *(Farkas' lemma). Let M be an $m \times n$ matrix and $\mathbf{c}$ an m-dimensional vector. Then, exactly one of the following two statements is true:*

(i) *There exists a vector $\mathbf{y} \in \mathbb{R}^n$ such that $M\mathbf{y} = \mathbf{c}$ and $\mathbf{y} \geq \mathbf{0}$.*
(ii) *There exists a vector $\mathbf{x} \in \mathbb{R}^m$ such that $M^t\mathbf{x} \geq \mathbf{0}$ and $\mathbf{c}^t\mathbf{x} < 0$.*

We now state the main theorem of the present paper. As the proof contains many steps we reserve the whole Section 4 to it.

Theorem 3.2. *Let δ be a diagonal function and ω an opposite diagonal function for which (2) holds. Then there exists a copula C whose diagonal section and opposite diagonal section are equal to δ and ω, respectively.*

4. Proof of Theorem 3.2

4.1. Construction of a checkerboard function

We state that for m values $0 < a_1 < \cdots < a_m < 1/2$, a copula B exists and satisfies that

$$\begin{cases} B(a_i, a_i) = \delta(a_i), \\ B(1-a_i, 1-a_i) = \delta(1-a_i), \\ B(a_i, 1-a_i) = \omega(a_i), \\ B(1-a_i, a_i) = \omega(1-a_i), \end{cases}$$

for $i = 1, \ldots, m$. This copula is precisely a checkerboard copula.

First, we will prove that a copula exists while interpolating δ and ω at points a_1 and $1 - a_1$, with $0 < a_1 < 1/2$ (case $n = 1$). Let us divide the unit square $\mathbb{I}^2$ as Fig. 1 shows.

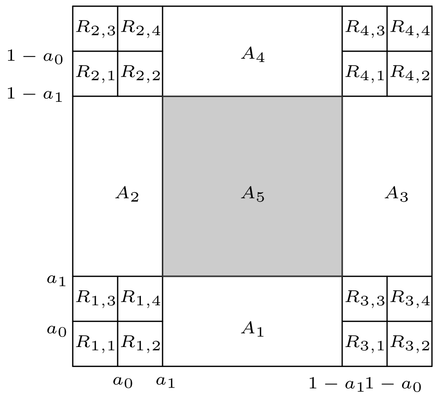

Fig. 2. Decomposition at the corners.

B_1 is the checkerboard copula satisfying

$$\begin{cases} V_{B_1}(R_1)=\delta(a_1), \\ V_{B_1}(R_2)=\omega(a_1)-\delta(a_1), \\ V_{B_1}(R_3)=a_1-\omega(a_1), \\ V_{B_1}(R_4)=\omega(1-a_1)-\delta(a_1), \\ V_{B_1}(R_5)=\delta(a_1)+\delta(1-a_1)-\omega(a_1)-\omega(1-a_1), \\ V_{B_1}(R_6)=1-2a_1+\omega(a_1)-\delta(1-a_1), \\ V_{B_1}(R_7)=a_1-\omega(1-a_1), \\ V_{B_1}(R_8)=1-2a_1+\omega(1-a_1)-\delta(1-a_1), \\ V_{B_1}(R_9)=-1+2a_1+\delta(1-a_1). \end{cases}$$

Conditions (2), with the fact that δ is a diagonal function and ω is an opposite diagonal function, ensure that $V_{B_1}(R_i)\geq 0$. For instance, as a consequence of Condition (2)(b), we have that $V_{B_1}(R_8)\geq 0$. If we add the B_1-volumes, then B_1 satisfies the boundary conditions, and B_1 is a copula. Furthermore, the choice of the allocations of values to $V_{B_1}(R_i)$ ensures us that

$$\begin{cases} B_1(a_1,a_1)=\delta(a_1), \\ B_1(1-a_1,1-a_1)=\delta(1-a_1), \\ B_1(a_1,1-a_1)=\omega(a_1), \\ B_1(1-a_1,a_1)=\omega(1-a_1). \end{cases}$$

Let us suppose that for a given n there exists a checkerboard copula that interpolates δ and ω at the points

$$0<a_1<\cdots<a_n<1/2<1-a_n<\cdots<1-a_1<1.$$

We now prove, by induction, that this is also true for $n+1$. Given the points

$$0<a_0<a_1<\cdots<a_n<1/2<1-a_n<\cdots<1-a_1<1-a_0<1,$$

using hypothesis induction, we build the interpolating checkerboard copula for

$$0<a_1<\cdots<a_n<1/2<1-a_n<\cdots<1-a_1<1.$$

We denote it by B_n. To build B_{n+1} we only modify the mass distribution on the rectangles in the form

$$[0,a_1]\times[\alpha,\beta], \qquad [1-a_1,1]\times[\alpha,\beta], \qquad [\alpha,\beta]\times[0,a_1], \qquad [\alpha,\beta]\times[1-a_1,1],$$

where $\alpha,\beta\in\{a_1,\dots,a_n,1-a_n,\dots,1-a_1\}$ and $]\alpha,\beta[\,\cap\{a_1,\dots,a_n,1-a_n,\dots,1-a_1\}=\emptyset$. For the case corresponding to $[0,a_1]^2$, we have a division by rectangles $R_{1,1}$, $R_{1,2}$, $R_{1,3}$ and $R_{1,4}$ shown in Fig. 2. In the other corners, we proceed in a similar way

The rectangle A_2 is divided in two parts as follows: the former is $A_2'=[0,a_0]\times[a_1,1-a_1]$ and the latter is $A_2''=[a_0,a_1]\times[a_1,1-a_1]$. The mass allocated to A_2' equals

$$V_{B_n}\big(A_2'\big) = \min\big\{V_{B_n}(A_2), \omega(a_0) - \delta(a_0)\big\} = \min\big\{\omega(a_1) - \delta(a_1), \omega(a_0) - \delta(a_0)\big\}.$$

On each of the subrectangles of A_2' and A_2'', the mass is proportionally distributed as it was in the corresponding rectangle in A_2.

We proceed analogously with the rectangles A_1, A_3, and A_4, allocating to them corresponding masses $V_{B_n}(A_1')$, $V_{B_n}(A_3')$ and $V_{B_n}(A_4')$, in the same way in which $V_{B_n}(A_2')$ was defined.

4.2. 2-increasingness of the checkerboard function

Let $r_{ij} = V(R_{i,j})$. Considered as unknowns in the vector

$$(r_{12}, r_{13}, r_{14}, r_{21}, r_{31}, r_{42}, r_{24}, r_{34}, r_{22}, r_{33}, r_{43}, r_{44})^t,$$

where t means the transpose, we must prove that the system of linear equations with augmented matrix (i.e. the matrix obtained by adding to the matrix of the system the right-hand side of the system):

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_2 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_3 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & t_4 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & t_5 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & t_6 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & t_7 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & t_8
\end{pmatrix}$$

admits a positive solution where

$$\begin{aligned}
t_1 &= \max\big(0, \omega(1-a_0) - \omega(1-a_1) + \delta(a_1) - \delta(a_0)\big),\\
t_2 &= \max\big(0, \omega(a_0) - \omega(a_1) + \delta(a_1) - \delta(a_0)\big),\\
t_3 &= \delta(a_1) - \delta(a_0),\\
t_4 &= a_1 - a_0 + \omega(a_0) - \omega(a_1),\\
t_5 &= a_1 - a_0 + \omega(1-a_0) - \omega(1-a_1),\\
t_6 &= 2(a_1 - a_0) + \delta(1-a_1) - \delta(1-a_0),\\
t_7 &= \max\big(0, 2(a_1-a_0) + \omega(1-a_0) - \omega(1-a_1) + \delta(1-a_1) - \delta(1-a_0)\big),\\
t_8 &= \max\big(0, 2(a_1-a_0) + \omega(a_0) - \omega(a_1) + \delta(1-a_1) - \delta(1-a_0)\big),
\end{aligned}$$

and clearly $t_i \geq 0$ for all $i \in \{1, \dots, 8\}$.

By Gaussian elimination, we transform this system into the equivalent system

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_2 \\
0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_3 - t_1 - t_2 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & t_4 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & t_5 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & t_6 - t_7 - t_8 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & t_7 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & t_8
\end{pmatrix}$$

Note that only the third and sixth expression in the last column can be strictly negative and that the submatrix made from columns 1, 2, 3, 9, 10, 12, 6 and 11 is the unit matrix I_8.

Farkas' theorem implies that this system has a positive solution if and only if there does not exist a vector $\mathbf{x} \geq \mathbf{0}$ for which it holds that

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$t_1 x_1 + t_2 x_2 + (t_3 - t_1 - t_2) x_3 + t_4 x_4 + t_5 x_5 + (t_6 - t_7 - t_8) x_6 + t_7 x_7 + t_8 x_8 < 0.$$

We can reformulate this problem as an LP problem. Indeed, we must prove that the target function

$$t_1 x_1 + t_2 x_2 + (t_3 - t_1 - t_2) x_3 + t_4 x_4 + t_5 x_5 + (t_6 - t_7 - t_8) x_6 + t_7 x_7 + t_8 x_8$$

subject to the conditions

$$\begin{pmatrix} 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and the positivity conditions

$$x_j \geq 0 \quad \forall j \in \{1, 2, \ldots, 8\}$$

has minimum value 0 ($\mathbf{x} = \mathbf{0}$ being the minimizer).

4.3. Solving the LP-problem with the simplex method

Introducing slack variables x_9, x_{10}, x_{11} and x_{12}, the LP minimization problem is described by the following simplex table (the first column being associated with variable x_1, ... , and the last one with x_{12}):

$$\begin{array}{cccccccccccc|c} 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ \hline t_1 & t_2 & \begin{array}{c} t_3 - t_1 \\ -t_2 \end{array} & t_4 & t_5 & \begin{array}{c} t_6 - t_7 \\ -t_8 \end{array} & t_7 & t_8 & 0 & 0 & 0 & 0 & 0 \end{array}$$

Since the last column contains only zeros, we change signs on all rows to obtain the equivalent table

$$\begin{array}{cccccccccccc|c} 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline t_1 & t_2 & \begin{array}{c} t_3 - t_1 \\ -t_2 \end{array} & t_4 & t_5 & \begin{array}{c} t_6 - t_7 \\ -t_8 \end{array} & t_7 & t_8 & 0 & 0 & 0 & 0 & 0 \end{array}$$

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which is the canonical simplex table associated with the basic variables $x_9, x_{10}, x_{11}, x_{12}$. The entries on the last row (below the solid line) are called reduced cost coefficients; they must be zero for the basic variables (if they are not zero, they must be made zero by elementary row operations) and they are in general non-zero for the other variables. If all reduced cost coefficients are positive, then the basic solution is an optimal basic solution. Such basic solution has all its components corresponding to columns that are not basis vectors, equal to zero. Hence $x_2 = x_3 = x_4 = x_5 = x_6 = x_7 = x_8 = 0$. Furthermore the components corresponding to basis vectors can be directly read off the last column, but since this column contains only zeros, it follows that $x_9 = x_{10} = x_{11} = x_{12} = 0$, whence $\mathbf{x} = \mathbf{0}$. Moreover, the minimum value of the target function is 0.

Let us recall some well-known facts from linear programming and the simplex method (notations and conventions are not uniformly fixed, we use those from [3]). If at least one reduced cost coefficient is strictly negative, then the actual basic solution may not be optimal. Two situations can be distinguished. If all entries above the line in a column with strictly negative reduced cost coefficient are negative, then the LP problem is unbounded, and the target function can be made as negative as one wishes by letting one or more of the variables grow indefinitely. If, however, some of the entries in that column are strictly positive, then the column can be transformed into a unit vector and the reduced cost coefficient can be made 0 by elementary row operations, such that the associated variable becomes a basic variable. The element of the table that is selected to become the 1-component of the new unit vector is called the pivot element. Hence, the above LP problem is reduced to the problem of proving that for all possible values of $t_1, t_2, \ldots, t_8$, it is always possible to find a sequence of pivot elements such that all reduced cost coefficients become non-negative.

Now, if $t_3 - t_1 - t_2 \geq 0$ and $t_6 - t_7 - t_8 \geq 0$ then the actual basic solution is optimal, which proves that the target function has minimal value 0.

We further distinguish the three mutual exclusive cases

a. $t_3 - t_1 - t_2 < 0$ and $t_6 - t_7 - t_8 \geq 0$.
b. $t_3 - t_1 - t_2 \geq 0$ and $t_6 - t_7 - t_8 < 0$.
c. $t_3 - t_1 - t_2 < 0$ and $t_6 - t_7 - t_8 < 0$.

In the following subsections, we treat each of these cases separately.

4.4. *The case $t_3 - t_1 - t_2 < 0$ and $t_6 - t_7 - t_8 \geq 0$*

In the last simplex table we retain element $(1, 3)$ as pivot; hence we make the third column equal to the basis vector $\vec{e}_1$ and the associated cost coefficient equal to 0 by means of appropriate row operations. We obtain the new table

$$
\begin{array}{cccccccccccc|c}
0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\
\hline
t_1 & t_3 - t_1 & 0 & \begin{array}{c} t_4 + t_3 \\ -t_1 - t_2 \end{array} & t_5 & \begin{array}{c} t_6 - t_7 \\ -t_8 \end{array} & t_7 & t_8 & \begin{array}{c} t_1 + t_2 \\ -t_3 \end{array} & 0 & 0 & 0 & 0
\end{array}
$$

Since by assumption $t_1 + t_2 - t_3 \geq 0$, there remain two cost coefficients that can be strictly negative, namely $t_3 - t_1$ and $t_4 + t_3 - t_1 - t_2$. We distinguish the following mutual exclusive subcases: A: $t_3 - t_1 < 0$ (no condition on the sign of $t_4 + t_3 - t_1 - t_2$) and B: $t_3 - t_1 \geq 0$, $t_4 + t_3 - t_1 - t_2 < 0$ (hence also $t_4 - t_2 < 0$).

4.4.1. *Subcase A: $t_3 - t_1 < 0$*

We must take element $(2, 4)$ as pivot and obtain the table

$$
\begin{array}{cccccccccccc|c}
-1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\
\hline
t_3 & 0 & 0 & t_4 - t_2 & \begin{array}{c} t_5 + t_3 \\ -t_1 \end{array} & \begin{array}{c} t_6 - t_7 \\ -t_8 \end{array} & t_7 & t_8 & t_2 & 0 & 0 & 0 & 0
\end{array}
$$

Since $t_3 \geq 0$ and $t_2 \geq 0$ and

$$t_5 + t_3 - t_1 \geq a_1 - a_0 \geq 0$$

only the cost coefficient $t_4 - t_2$ can still be strictly negative. If $t_4 - t_2 \geq 0$, then the actual basic solution is optimal and yields minimum value 0.

We must take element $(2, 2)$ as pivot and obtain the table

$$\begin{array}{cccccccccccc|c}
-1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & -1 & 1 & 0 & -1 & -1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\
\hline
t_3 + t_4 & t_2 - t_4 & 0 & 0 & t_5 + t_3 & t_6 - t_7 & t_7 & t_8 & t_4 & t_2 - t_4 & 0 & 0 & 0 \\
-t_2 & & & & -t_1 + t_4 & -t_8 & & & & & & & \\
 & & & & -t_2 & & & & & & & &
\end{array}$$

Now $t_2 - t_4 > 0$. Columns 1 and 5 have only negative entries; therefore, the corresponding cost coefficients must be positive or zero as otherwise the claim of the theorem would fail. However

$$t_3 + t_4 - t_2 \geq a_1 - a_0 \geq 0$$

and

$$t_3 + t_4 + t_5 - t_1 - t_2 \geq 2(a_1 - a_0) - \left[\delta(a_1) - \delta(a_0)\right] \geq 0,$$

and therefore the actual basic solution is optimal and yields minimum value 0.

4.4.2. *Subcase B: $t_3 - t_1 \geq 0$ and $t_3 + t_4 - t_1 - t_2 < 0$*

As stated before, this case also implies that $t_4 - t_2 < 0$.

We can only choose as pivot the element $(2, 4)$. The table is transformed into the equivalent table

$$\begin{array}{cccccccccccc|c}
-1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & -1 & 1 & 0 & -1 & -1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\
\hline
t_3 + t_4 & t_2 - t_4 & 0 & 0 & t_5 + t_3 & t_6 - t_7 & t_7 & t_8 & t_4 & t_1 + t_2 & 0 & 0 & 0 \\
-t_2 & & & & +t_4 - t_1 & -t_8 & & & & -t_3 - t_4 & & & \\
 & & & & -t_2 & & & & & & & &
\end{array}$$

We have shown before that $t_3 + t_4 - t_2 \geq 0$ and $t_3 + t_4 + t_5 - t_1 - t_2 \geq 0$. Hence, also in this case the actual basic solution is optimal and yields minimum value 0.

This finishes the discussion of the case $t_3 - t_1 - t_2 < 0$ and $t_6 - t_7 - t_8 \geq 0$.

4.5. *The case $t_3 - t_1 - t_2 \geq 0$ and $t_6 - t_7 - t_8 < 0$*

This situation is completely analogous as the previous one. In fact we can formally perform the analogous steps by making use of the following substitutions:

$$t_1 \leftrightarrow t_8, \qquad t_2 \leftrightarrow t_7, \qquad t_3 \leftrightarrow t_6, \qquad t_4 \leftrightarrow t_5.$$

The subcase $t_3 - t_1 < 0$ becomes $t_6 - t_8 < 0$ and we have to check the positivity of $t_4 + t_6 - t_8$, i.e.

$$t_4 + t_6 - t_8 \geq a_1 - a_0 \geq 0.$$

Next $t_4 - t_2 < 0$ becomes $t_5 - t_7 < 0$ and we must check the positivity of $t_5 + t_6 - t_7$ and of $t_4 + t_5 + t_6 - t_7 - t_8$:

$$t_5 + t_6 - t_7 \geq a_1 - a_0 \geq 0,$$
$$t_4 + t_5 + t_6 - t_7 - t_8 \geq \delta(1 - a_0) - \delta(1 - a_1) \geq 0.$$

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4.6. *The case $t_3 - t_1 - t_2 < 0$ and $t_6 - t_7 - t_8 < 0$*

Now we consecutively use the elements (1, 3) and (4, 6) as pivots, to obtain the following table:

$$
\begin{array}{cccccccccccc|c}
0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\
\hline
t_1 & t_3 - t_1 & 0 & \substack{t_4 + t_3 \\ -t_1 - t_2} & \substack{t_5 + t_6 \\ -t_7 - t_8} & 0 & t_6 - t_8 & t_8 & \substack{t_1 + t_2 \\ -t_3} & 0 & 0 & \substack{t_7 + t_8 \\ -t_6} & 0
\end{array}
$$

There remain four cost coefficients that can become negative, namely $t_3 - t_1$, $t_6 - t_8$, $t_4 + t_3 - t_1 - t_2$ and $t_5 + t_6 - t_7 - t_8$. We will consider the following mutual exclusive subcases:

A. $t_3 - t_1 < 0, t_6 - t_8 \geq 0, t_5 + t_6 - t_7 - t_8 \geq 0$.
B. $t_3 - t_1 \geq 0, t_3 + t + 4 - t_1 - t_2 < 0, t_6 - t_8 \geq 0, t_5 + t_6 - t_7 - t_8 \geq 0$.
C. $t_3 - t_1 \geq 0, t + 3 + t_4 - t_1 - t_2 \geq 0, t_6 - t_8 < 0$.
D. $t_3 - t_1 \geq 0, t_3 + t + 4 - t_1 - t_2 \geq 0, t_6 - t_8 \geq 0, t_5 + t_6 - t_7 - t_8 < 0$.
E. $t_3 - t_1 < 0, t_6 - t_8 < 0$.

4.6.1. *Subcase A: $t_3 - t_1 < 0, t_6 - t_8 \geq 0, t_5 + t_6 - t_7 - t_8 \geq 0$*

Note that we make no a priori statement on the sign of $t_3 + t_4 - t_1 - t_2$. We choose element (2, 2) as pivot to obtain the following table:

$$
\begin{array}{cccccccccccc|c}
-1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\
\hline
t_3 & 0 & 0 & t_4 - t_2 & \substack{t_5 + t_6 \\ -t_7 - t_8 \\ +t_3 - t_1} & 0 & t_6 - t_8 & t_8 & t_2 & t_1 - t_3 & 0 & \substack{t_7 + t_8 \\ -t_6} & 0
\end{array}
$$

There are still two coefficients that could be negative, namely $t_4 - t_2$ and $t_5 + t_6 - t_7 - t_8 + t_3 - t_1$.
First, assume that $t_4 - t_2 < 0$, then we use element (2, 4) as pivot to obtain:

$$
\begin{array}{cccccccccccc|c}
-1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\
\hline
\substack{t_3 + t_4 \\ -t_2} & t_2 - t_4 & 0 & 0 & \substack{t_5 + t_6 \\ -t_7 - t_8 \\ +t_3 - t_1 \\ +t_4 - t_2} & 0 & t_6 - t_8 & t_8 & t_4 & \substack{t_1 - t_3 \\ +t_2 - t_4} & 0 & \substack{t_7 + t_8 \\ -t_6} & 0
\end{array}
$$

Clearly, all reduced cost coefficients are positive, except for possibly the coefficient $t_3 + t_4 + t_5 + t_6 - t_1 - t_2 - t_7 - t_8$, of which it is required that it is positive, for otherwise there would exist an unbounded solution to the LP-problem. Hence it should hold that

$$
\begin{aligned}
& t_3 + t_4 + t_5 + t_6 - t_1 - t_2 - t_7 - t_8 \\
& \quad \geq -\omega(a_0) + \omega(a_1) - \delta(a_1) + \delta(a_0) \\
& \qquad - \omega(1 - a_0) + \omega(1 - a_1) - \delta(1 - a_1) + \delta(1 - a_0)
\end{aligned}
$$

be positive. This is guaranteed by (2)(c).

Next, assume that $t_4 - t_2 \geq$ but $t_5 + t_6 - t_7 - t_8 + t_3 - t_1 < 0$. Taking $(3, 5)$ as pivot, we obtain

$$
\begin{array}{cccccccccccc|c}
-1 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 1 & 1 & -1 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 1 & -1 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\
\hline
t_3 & 0 & 0 & \begin{array}{c} t_4 - t_2 \\ +t_5 + t_6 \\ -t_7 - t_8 \\ +t_3 - t_1 \end{array} & 0 & 0 & \begin{array}{c} t_1 + t_7 \\ -t_3 - t_5 \end{array} & \begin{array}{c} t_3 + t_5 \\ -t_7 - t_8 \\ +t_6 \end{array} & t_2 & t_1 - t_3 & \begin{array}{c} t_7 + t_8 \\ +t_1 - t_3 \\ -t_5 - t_6 \end{array} & \begin{array}{c} t_3 + t_5 \\ -t_1 \end{array} & 0
\end{array}
$$

Now, as we have seen before

$$t_3 + t_4 + t_5 + t_6 - t_1 - t_2 - t_7 - t_8 \geq 0.$$

Furthermore from the assumption $t_5 + t_6 - t_7 - t_8 + t_3 - t_1 < 0$, it follows that

$$t_1 + t_7 - t_3 - t_5 > t_6 - t_8 \geq 0,$$

the last inequality also being one of the assumptions. Next, since by assumption $t_5 + t_6 - t_7 - t_8 \geq 0$, it follows that

$$t_3 + t_5 + t_6 - t_7 - t_8 \geq t_3 \geq 0.$$

Finally, we have already shown that

$$t_5 + t_3 - t_1 \geq a_1 - a_0 \geq 0.$$

Hence, case A is completed, with the conclusion that there always exists an optimal basic solution yielding zero as minimum value of the target function.

4.6.2. *Subcase B:* $t_3 - t_1 \geq 0, t_3 + t_4 - t_1 - t_2 < 0, t_6 - t_8 \geq 0, t_5 + t_6 - t_7 - t_8 \geq 0$

We must take $(2, 4)$ as pivot and obtain the following simplex table:

$$
\begin{array}{cccccccccccc|c}
-1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\
\hline
\begin{array}{c} t_3 + t_4 \\ -t_2 \end{array} & t_2 - t_4 & 0 & 0 & \begin{array}{c} t_5 + t_6 \\ -t_7 - t_8 \\ +t_4 + t_3 \\ -t_1 - t_2 \end{array} & 0 & t_6 - t_8 & t_8 & t_4 & \begin{array}{c} t_1 + t_2 \\ -t_3 - t_4 \end{array} & 0 & \begin{array}{c} t_7 + t_8 \\ -t_6 \end{array} & 0
\end{array}
$$

It is easily verified that all reduced cost coefficients are positive (either by the assumptions or the stated conditions of the theorem). Hence this case is immediately completed.

4.6.3. *Subcase C:* $t_3 - t_1 \geq 0, t_3 + t_4 - t_1 - t_2 \geq 0, t_6 - t_8 < 0$

Due to the inbuilt symmetry (one can simultaneously interchange t_1 and t_8, t_2 and t_7, t_3 and t_6, and t_4 and t_5, the steps required in this subcase are similar to those of Subcase A. Hence, we can immediately conclude that there exists an optimal basic solution yielding zero as minimum value of the target function.

4.6.4. *Subcase D:* $t_3 - t_1 \geq 0, t_3 + t_4 - t_1 - t_2 \geq 0, t_6 - t_8 \geq 0, t_5 + t_6 - t_7 - t_8 < 0$

For the same reasons of symmetry, this subcase is completely analogous to Subcase B, leading to the same conclusion.

4.6.5. Subcase E: $t_3 - t_1 < 0$, $t_6 - t_8 < 0$

We take as consecutive pivots the elements (2, 2) and (3, 7) and find the following simplex table:

$$
\begin{array}{cccccccccccc|c}
-1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\
\hline
t_3 & 0 & 0 & \begin{array}{c} t_4 - t_2 \\ +t_6 - t_8 \end{array} & \begin{array}{c} t_5 - t_7 \\ +t_3 - t_1 \end{array} & 0 & 0 & t_6 & t_2 & t_1 - t_3 & t_8 - t_6 & t_7 & 0
\end{array}
$$

The cost coefficients that can still be negative are $t_4 + t_6 - t_2 - t_8$ and $t_3 + t_5 - t_1 - t_7$. Assume that $t_3 + t_5 - t_1 - t_7 < 0$. Then we should take (3, 5) as pivot and we obtain:

$$
\begin{array}{cccccccccccc|c}
-1 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & -1 & 1 & 1 & -1 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 1 & -1 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 1 & -1 & -1 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\
\hline
t_3 & 0 & 0 & \begin{array}{c} t_4 - t_2 \\ +t_6 - t_8 \\ +t_3 + t_5 \\ -t_1 - t_7 \end{array} & 0 & 0 & \begin{array}{c} t_1 + t_7 \\ -t_3 - t_5 \end{array} & \begin{array}{c} t_6 + t_3 \\ +t_5 - t_1 \\ -t_7 \end{array} & t_2 & t_1 - t_3 & \begin{array}{c} t_8 - t_6 \\ -t_3 - t_5 \\ +t_1 + t_7 \end{array} & \begin{array}{c} t_3 + t_5 \\ -t_1 \end{array} & 0
\end{array}
$$

It is easy to check that all reduced cost coefficients are positive. In particular

$$t_3 + t_5 + t_6 - t_1 - t_7 \geq a_1 - a_0 - \omega(1 - a_0) + \omega(1 - a_1) \geq 0$$

Also since $t_8 - t_6 \geq 0$ and $t_1 + t_7 - t_3 - t_5 \geq 0$ by assumption, it holds that their sum is positive too.

In case $t_4 + t_6 - t_2 - t_8 < 0$, a similar computation leads to the same conclusions.

4.7. *Finalizing the proof*

Finally, let C_n denote a copula that interpolates the points $\frac{i}{2n+1}$, with $1 \leq i \leq n$. By compactness arguments v.g. the class of all copulas is compact w.r.t. to the norm of supremum (see [8]), the sequence (C_n) has a convergent subsequence, and the limit of this subsequence is a copula having δ as diagonal and ω as opposite diagonal sections. This observation finalizes our proof of Theorem 3.2.

5. Characterization of copulas with given diagonal and opposite diagonal sections

Given a diagonal and opposite diagonal function such that conditions (2) are satisfied, we have constructed in the proof of Theorem 3.2 a sequence of checkerboard copulas whose limit yields a copula that has the given diagonal and opposite diagonal functions as diagonal and opposite diagonal sections, respectively. This construction method clearly has its theoretical merit, but for actually constructing copulas with the given sections it is highly unpractical (especially due to the limiting procedure). Easier techniques for constructing such copulas have been established recently, amongst which we mention the methods based on linear or quadratic interpolation between points on the diagonal and opposite diagonal [5,20,21]. Also, in general there is not a unique copula with given diagonal and opposite diagonal sections, as is illustrated by the following example.

Example 5.1. Consider as given diagonal and opposite diagonal function the diagonal and opposite diagonal section of the product copula Π, i.e.:

$$\delta(x) = x^2, \qquad \omega(x) = x(1 - x).$$

Obviously, since the product copula Π has δ as diagonal and ω as opposite diagonal section, conditions (2) are satisfied. If we apply the technique of linear interpolation [20], then whether we construct the so-called orbital semilinear copula or the so-called radial semilinear copula, we always retrieve the product copula Π itself. However, if we apply the technique of quadratic interpolation [21], then we obtain orbital semiquadratic and radial semiquadratic copulas that differ from Π. For instance, it has been proven that all copulas in the parametrized family

$$C_\lambda(x,y)=\begin{cases} xy-\lambda(y-x)(x+y-1)\min(y,1-y), & \text{if } (x\le y\wedge x+y\ge 1)\vee(x\ge y\wedge x+y\le 1)\\ xy-\lambda(x-y)(x+y-1)\min(x,1-x), & \text{otherwise}\end{cases}$$

with $\lambda\in[-1,1]$, are (orbital) semiquadratic copulas that have the same diagonal section and opposite diagonal section as Π. Hence, in this case there is even an infinity of copulas with the same diagonal and opposite diagonal section.

An obvious question that arises from the previous example is to characterize the diagonal and opposite diagonal functions for which there exists a unique copula that has these functions as diagonal and opposite diagonal section, respectively. This is exactly one of the two problems posed by Klement and Kolesárová in [23]. We are now able to solve both these problems.

Theorem 5.2. *Let C be a copula, and let δ and ω be its diagonal and opposite diagonal sections, respectively. Then C is the unique copula with these sections if and only if*

$$\begin{array}{ll}(a) & \delta(x)=\omega(x)\\ (b) & \delta(x)=\delta(1-x)-1+2x\\ (c) & \omega(x)=\omega(1-x)\end{array}\qquad (3)$$

for all $x\in[0,1/2]$.

Proof. Following the ideas in [7], based on the use of a rectangular patchwork (see for instance [13]), we obtain that the copula C is unique if and only if μ_C is concentrated on the diagonals of the unit square.

If μ_C is concentrated on the diagonals of the unit square, then conditions (3) are necessary.

Conversely, conditions (3) ensure that there is no mass out of the diagonals. □

Finally, we characterize all pairs of functions δ and ω such that there is a unique copula with diagonal δ and opposite diagonal ω. In fact, it is only necessary for δ and ω to be "compatible", and to satisfy the conditions imposed in Theorem 5.2.

Theorem 5.3. *Let δ be a diagonal function and ω be an opposite diagonal function. Then there exists a unique copula having δ and ω as diagonal and opposite diagonal sections, respectively, if and only if δ and ω satisfy conditions (2) and (3).*

Note that the uniqueness of the copula implies that its support is in the union of the diagonal and opposite diagonal of the unit square. Copulas that have the latter property have been previously called X-copulas and it has been proven that any X-copula is a Bertino copula [17]. It follows that the unique copula with diagonal sections satisfying conditions (2) and (3) is a Bertino copula, hence a copula that can be written as:

$$C(x,y)=\min(x,y)-\min_{t\in[\{x,y\}]}\big(t-\delta(t)\big),$$

where $[\{x,y\}]$ denotes the closed interval with endpoints x and y. It is no surprise that this construction only entails δ, as ω is entirely fixed by the given δ by means of conditions (3a) and (3c).

Example 5.4. Consider as given diagonal and opposite diagonal functions

$$\delta(x)=x^2,\qquad \omega(x)=\min\big(x^2,(1-x)^2\big).$$

They satisfy conditions (2) and (3). Note that δ is again the diagonal section of Π. It follows that the unique copula that has δ and ω as diagonal and opposite diagonal section, respectively, is the Bertino-copula

$$C(x,y)=\min(x,y)-\min\big(x-x^2,y-y^2\big).$$

6. Conclusions

We hereby provide the necessary and sufficient conditions for the existence of a copula to have given functions as diagonal and opposite diagonal sections. The techniques we use are interpolation, construction of checkerboard copulas and linear programming.

Among other results, we have found the answer to some open problems. For example that proposed by Klement and Kolesárová [23]:

(i) To characterize the class of copulas where the knowledge of their diagonal and opposite diagonal determines the copula in a unique way, and
(ii) To characterize all pairs of functions δ and ω that there is a unique copula with a diagonal section δ and opposite diagonal section ω.

The potential use of the result concerns the study of the relation of statistical data situated around the diagonal and opposite diagonal sections and located near the corners, and the correlation between extreme events. We need to highlight the fact that the construction method has theoretical interest but hardly any practical use.

Finally, an open problem to be the subject of further work can be established as:

Given a diagonal function and opposite diagonal function that satisfy all the properties that guarantee the existence of at least one copula with these functions as diagonal and opposite diagonal sections, respectively, does there exist a (pointwise) smallest (resp. greatest) copula with these sections, and if so, can we construct this copula?

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