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A note on the Hausdorff dimension of general sums of pulses graphs

Enrique de Amo · Imen Bhouri · Juan Fernández-Sánchez

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Abstract In this work we study the some general fractal sums of pulses defined in $\mathbb{R}$ by:

$$F(t)=\sum_{n=1}^{+\infty}a_nG(\lambda_n^{-1}(t-X_n))$$

where (a_n), (λ_n) two positive scalar sequences such that $\sum a_n$ is divergent, and (λ_n) is non-increasing to 0, G is an elementary bump and X_n are independent random variables uniformly distributed on a sufficiently large domain Ω. We investigate the Hausdorff dimension of the graph of G and in particular we answer a question given by Tricot in (Courbes et dimensions fractales, Springer, Berlin, 1995).

Keywords Hausdorff dimension · Sum of pulses · Box dimension

Mathematics Subject Classification (2000) 60D05 · 60G99 · 28A80

1 Introduction

Let $(\Gamma,\mathcal{F},\mathbb{P})$ be a probability space, we consider random functions of the type

$$F(t)=\sum_{n=1}^{+\infty}a_nG(\lambda_n^{-1}(t-X_n)) \tag{1}$$

E. de Amo · J. Fernández-Sánchez
University of Almeria, Almeria, Spain

E. de Amo
e-mail: edeamo@ual.es

I. Bhouri (✉)
University of Monastir, Monastir, Tunisia
e-mail: imen.bhouri@fsm.rnu.tn

where (a_n), (λ_n) are two positive scalar sequences such that $\sum a_n$ is divergent, and (λ_n) is non-increasing to 0. The function $G : \mathbb{R} \to \mathbb{R}$ is the elementary bump (i.e an even continuous function supported on $[-1, 1]$, decreasing on $[0, 1]$ and satisfying $G(0) = 1$) and X_n are continuous independent random variables uniformly distributed on a sufficiently large domain Ω.

In the particular case $a_n = \frac{1}{n^H}$, $H \in (0, 1)$ and $\lambda_n = \frac{1}{n}$, these functions have been introduced in [9] and [11] to generate measures associated to Poisson processes. In the same particular case and in higher dimension, the analysis of the fractal sums of pulses has been treated in [3] and [2]. The existence and regularity of functions defined by (1) have been studied in [1]. Notice that this kind of functions are important for the purpose of modeling strange phenomena which are known to exhibit multifractal behaviors. Such behaviors occur for instance in geophysics [5] when considering the spatial-temporal position and the intensity of seismic events, in telecommunications where the TCP Internet traffic is known to be multifractal [8], and also when studying financial time series [10]. This work was motivated by a question given in [13] about the Hausdorff dimension of the graph of functions defined by (1). In this paper, we investigate the Hausdorff dimension of their graphs which provides a measure of the irregularity of the process and gives a positive answer to the question of Tricot. In particular our result is an improvement of the result of [1] who gives only an upper bound of the upper box dimension of the graph of F.

The paper is organized as follows. In the next section we introduce some basic notions and properties. In Sect. 3 we state our main result giving the Hausdorff dimension of the pulse-sum functions. We prove Theorem 1 by using some potential theoretic methods for calculating the Hausdorff dimensions and some technical lemmas useful for our proof.

2 Preliminaries

Casually, we briefly recall some basic definitions and facts which will be used in subsequent developments.

Let A be a subset of $\mathbb{R}^2$. The s-dimensional Hausdorff measure Hausdorff of A is defined by

$$\mathcal{H}^s(A) = \lim_{\varepsilon \to 0} \mathcal{H}^s_\varepsilon(A)$$

where, for $\varepsilon > 0$,

$$\mathcal{H}^s_\varepsilon(A) = \inf\left\{ \sum_{i=0}^{\infty} |E_i|^s : E \subset \bigcup_{i=0}^{\infty} E_i \text{ and } |E_i| \le \varepsilon \right\},$$

with $|A|$ denoting the diameter of a set $A \in \mathbb{R}^2$. The Hausdorff dimension of A is given by

$$\dim(A) = \inf\{s : \mathcal{H}^s(A) = 0\} = \sup\{s : \mathcal{H}^s(A) = \infty\}$$

(see [4] and [12] for more details). When calculating the Hausdorff dimension of a set A, in general it is difficult to find a lower estimate of $\dim(A)$, and one approach is to relate Hausdorff dimension to certain energies.

For $A \subset \mathbb{R}^2$, let

$$\mathcal{M}(A) = \{\mu : \mu \text{ is a finite Radon measure supported by A}\}.$$

For $\mu \in \mathcal{M}(A)$, we define the s-energy of μ by

$$I_s(\mu) = \iint \frac{d\mu(x)d\mu(y)}{|x-y|^s}. \tag{2}$$

Then

$$\dim(A) = \sup\{s \geq 0 : \exists \mu \in \mathcal{M}(A) \text{ with } I_s(\mu) < \infty\}$$

(see [4] and [12]). So, if we can construct a measure μ supported on A with finite s-energy then $\dim(A) \geq s$. For the graph $\Gamma_F \subset \mathbb{R}^2$ of a continuous function $F : [0, 1] \to \mathbb{R}$, there is a natural measure μ on Γ_F as follows. If $\mathcal{L}^1$ denotes the Lebesgue measure on [0, 1],

$$\mu(E) = \mathcal{L}^1\{t \in [0, 1] : (t, F(t)) \in E\} \quad \text{for all } E \subset \mathbb{R}^2.$$

If $x = (u, t) \in \mathbb{R}^2$, define $\|x\|_2 = (u^2 + t^2)^{1/2}$. We can rewrite (2) by

$$I_s(\mu) = \iint_{[0,1]^2} ((F(x) - F(y))^2 + |x - y|^2)^{-s/2} dxdy. \tag{3}$$

3 Results

The existence and regularity of bumps sums functions defined by (1) have been studied in [1]. In particular Abid proved the following results. We denote by

$$\Lambda_j = \{n : 2^{-j} \leq \lambda_n < 2^{-(j-1)}\},$$

and

$$H = \liminf_{n\to\infty} \left(\inf_{n\in\Lambda_j} \frac{\log a_n}{\log \lambda_n} \right). \tag{4}$$

Theorem (Ben Abid) 1 *Assume that* $\lambda_n = \frac{\alpha}{n}$, $\alpha > 0$ *and* $G \in C^1(\mathbb{R})$. *Then if* $H \in (0, 1]$ *we have, almost surely, for every* $\varepsilon \in (0, H)$, $F \in C^{H-\varepsilon}(\mathbb{R})$.

Denote by $\Gamma_F := \{(t, F(t)) : t \in [0, 1]\}$ the graph of the random function F. The Hölder estimates on F immediately give an upper bound for the upper box-counting dimension $\overline{\dim}_B \Gamma_F$ of the graph (see [4]).

Corollary (Ben Abid) 1 *We have*

$$\overline{\dim}_B \Gamma_F \leq 2 - H, \quad \textit{almost surely.}$$

From now on $\lambda_n = \frac{\alpha}{n}$ with $\alpha > 0$. Our main result is to calculate the Hausdorff dimension of the graph of F which improves the result of Ben Abid and gives an answer to a question given by Tricot in [13].

Theorem 1 *Assume that there exists a non-empty interval $I \subset [0,1]$ on which $G : I \to J$ is a C^1-diffeomorphism. Then we have*

$$\dim \Gamma_F = 2 - H, \quad \textit{almost surely.}$$

Since the Hausdorff dimension is less then its box dimension, due to Corollary 1, it is sufficient to prove that $\dim \Gamma_F \geq 2 - H$, almost surely. The proof is based on the potential theoretic method to calculate the Hausdorff dimension of graphs of many functions, such as the fractional Brownian motion [7] or the random Weierstrass function [6] and those given in the particular case $a_n = \frac{1}{n^H}$ and $\lambda_n = \frac{1}{n}$, $H \in (0,1)$ in [2]. The potential theoretic ideas are developed in the following section.

In order to prove Theorem 1, we need some intermediate results. We use the following probability notations.

For each event $A \in \mathcal{F}$ with $\mathbb{P}(A) > 0$ we write $\mathbb{P}^A$ for the probability conditional on A. We have $\mathbb{P}^A$ is absolutely continuous with respect to $\mathbb{P}$ with density $\frac{d\mathbb{P}^A}{d\mathbb{P}} = \frac{1}{\mathbb{P}(A)}\chi_A$. We denote by $\mathbb{E}^A$ the expectation with respect to $\mathbb{P}^A$ to get for all random variables Y, $\mathbb{E}^A(Y) = \frac{1}{\mathbb{P}(A)}\mathbb{E}(Y\chi_A)$. Further, we write $\mathbb{P}_Y$ for the law of Y as a random variable on $(\Gamma, \mathcal{F}, \mathbb{P})$.

For $x,\ y \in [0,1]$ we define

$$Z = F(x) - F(y) = \sum_{n=1}^{\infty} Z_n$$

where

$$Z_n = a_n \left(G(\lambda_n^{-1}(x - X_n)) - G(\lambda_n^{-1}(y - X_n)) \right).$$

For this fixed x, we write A_n for the event $(x \in C_n')$ where

$$C_n' = \left\{ t \in \mathbb{R} : |t - X_n|\lambda_n^{-1} \in I \right\}.$$

The results of the following lemmas are similar to Lemma 3.1, Corollary 3.2 and Corollary 3.3 established in [2].

Lemma 1 *Let $x,\ y \in [0,1]$ be fixed. For all $p \geq 1$ such that $|x - y| > 2\lambda_p$, the random variable Z_p has a density conditional on A_p given by*

$$f_p(z) = \frac{\lambda_p}{a_p \mathbb{P}(A_p)} \left| h'\left(\frac{z}{a_p}\right) \right| \chi_J\left(\frac{z}{a_p}\right) \qquad \textit{for all } z \in \mathbb{R},$$

where $h : J \to I$ is the inverse of G.

Now denote by $S_p = \sum_{n \neq p} Z_n$ so that $Z = S_p + Z_p$. We condition on S_p and we regard Z as random variable on $(\Gamma, \mathcal{F}, \mathbb{P}^{\mathbb{A}_p})$.

Lemma 2 *Let $x, y \in [0,1]$ and $p \geq 1$ such that $|x - y| > 2\lambda_p$. Then Z has a density conditional on S_p given by*

$$f_z^{S_p = s}(z) = f_p(z - s) \quad \textit{for all } z \in \mathbb{R}, \tag{5}$$

where f_p is as in Lemma 1.

Lemma 3 $\forall n > m \geq 1$, $\forall x, y \in [0,1]$ *such that* $|x-y| > 2\lambda_m$ *and* $r > 0$, *we have*

$$\mathbb{P}((|F(x)-F(y)| < r) \cap (A_m \cup \cdots \cup A_n)) \leq C\frac{r}{a_n}$$

for some $C > 0$.

Lemma 4 *Let* $s > 1$. *For* $1 \leq m < n$, *let* $V = C_m \cup \cdots \cup C_n$. *For* $x, y \in [0,1]$ *such that* $|x-y| > 2\lambda_m$, *we have*

$$\mathbb{E}(((F(x)-F(y))^2 + |x-y|^2)^{-s/2}\chi_{(x\in V)}) \leq C|x-y|^{1-s}\frac{1}{a_n} \tag{6}$$

for some $C > 0$.

Proof of Lemma 4 Denote $h = |x-y|$, for $r > 0$ due to Lemma 3 we have

$$p(r) := \mathbb{P}((|Z| < r) \cap (x \in V)) \leq C\frac{r}{a_n}.$$

So,

$$\begin{aligned}\mathbb{E}^{(x\in V)}((|Z|^2+h^2)^{-s/2}) &= \int_0^\infty (r^2+h^2)^{-s/2} d(\mathbb{P}^{(x\in V)}(|Z|<r)) \\ &= \frac{1}{\mathbb{P}(x\in V)}\int_0^\infty (r^2+h^2)^{-s/2} dp(r).\end{aligned}$$

As a consequence,

$$\mathbb{E}^{(x\in V)}((|Z|^2+h^2)^{-s/2})\chi_{(x\in V)} = \int_0^\infty (r^2+h^2)^{-s/2} dp(r).$$

Integrating by parts we get,

$$\begin{aligned}\int_0^\infty (r^2+h^2)^{-s/2} dp(r) &\leq \int_0^h h^{-s} dp(r) + \int_h^\infty r^{-s} dp(r) \\ &\leq h^{-s}p(h) + [r^{-s}p(r)]_{r=h}^\infty + s\int_h^\infty r^{-s-1}p(r)dr \\ &\leq Ch^{-s}\frac{h}{a_n} + Cs\int_h^\infty r^{-s-1}\frac{r}{a_n}dr \leq \frac{C}{a_n}h^{1-s}\end{aligned}$$

and (6) yields.

Next we want to prove that for given x, y, the quantity $|F(x)-F(y)|$ is of high probability of being suitably large, for x in a large random subset of $[0,1]$. □

Remark 1 Recall that H is defined by (4) so for all $\varepsilon \in (0, H)$, there exists $k_\varepsilon \geq 1$ such that for all $k \geq k_\varepsilon$, for all $n \in \Lambda_k$, $a_n \geq 2^{-kH(1+\varepsilon/2)}$.

The following result is a straightforward consequence of Lemma 4 by considering the random set $V_k = C'_{n_{k^2}} \cup \cdots \cup C'_{n_{(k+1)^2-1}}$ with $n_j \in \Lambda_j$ for $(k+1)^2 - 1 \leq j \leq k^2$.

Corollary 1 *Let* $s > 1$, $\varepsilon > 0$ *and* $x, y \in [0,1]$ *such that* $|x-y| < 2\lambda_{k_\varepsilon - 1}$. *Let* $k \geq 1$ *be the unique integer satisfying* $2\lambda_k < |x-y| \leq 2\lambda_{k-1}$. *Then*

$$\mathbb{E}(((F(x)-F(y))^2 + |x-y|^2)^{-s/2}\chi_{(x\in V_k)}) \leq C|x-y|^{1-s-H(1+\varepsilon/2)}$$

for some $C > 0$.

Further we will estimate the measure of V_k.

Lemma 5 *There exists a constant* $\delta > 0$ *such that for all* $1 \leq m < n$, *we have*

$$\mathbb{E}\left(\mathcal{L}^1\left([0,1]\setminus\bigcup_{p=m}^{n} C_p\right)\right) \leq \left(\frac{m}{n}\right)^{\delta}.$$

For the proof of this lemma see Lemma 3.6 in [2].

3.1 Proof of Theorem 1

Let $1 < s < 2-H$. Choose $\varepsilon > 0$ such that $(1+\varepsilon/2)H < 2-s < 1$ with $k_\varepsilon \geq 1$ the associated integer. Fix $k_0 \geq k_\varepsilon$, we define $W = [0,1]\cap(\bigcap_{k=k_0}^{\infty} V_k)$. The proof of Theorem 1 splits in two steps. Denote by $\mathcal{L}^1{}_W$ the restriction of Lebesgue measure to W and $R_k = \{(x,y)\in[0,1]\times[0,1] : 2\lambda_k < |x-y| \leq 2\lambda_{k-1}\}$.

Step 1. From the definition of W and due to Corollary 1 we have

$$\begin{aligned}
&\mathbb{E}\left(\iint_{\{x,y\in[0,1]:\ |x-y|\leq 2\lambda_{k_0}\}} ((F(x)-F(y))^2+|x-y|^2)^{-s/2} d\mathcal{L}^1{}_W(x) d\mathcal{L}^1{}_W(y)\right)\\
&\leq \mathbb{E}\left(\iint_{\{x\in W,\ y\in[0,1]:|x-y|\leq 2\lambda_{k_0}\}} ((F(x)-F(y))^2+|x-y|^2)^{-s/2} dxdy\right)\\
&\leq \mathbb{E}\left(\sum_{k=k_0}^{\infty}\iint_{R_k\cap W\times[0,1]} ((F(x)-F(y))^2+|x-y|^2)^{-s/2} dxdy\right)\\
&\leq \mathbb{E}\left(\sum_{k=k_0}^{\infty}\int\int_{R_k} ((F(x)-F(y))^2+|x-y|^2)^{-s/2}\chi_{(x\in V_k)} dxdy\right)\\
&\leq \sum_{k=k_0}^{\infty}\left(\iint_{R_k}\mathbb{E}(((F(x)-F(y))^2+|x-y|^2)^{-s/2}\chi_{(x\in V_k)}) dxdy\right)\\
&\leq C\sum_{k=k_0}^{\infty}\left(\iint_{R_k}|x-y|^{1-s-H(1+\varepsilon/2)} dxdy\right)\\
&\leq C\iint_{\{x\in W,\ y\in[0,1]:|x-y|\leq 2\lambda_{k_0}\}}|x-y|^{1-s-H(1+\varepsilon/2)} dxdy,
\end{aligned}$$

since $1 - s - H(1+\varepsilon) > -1$, this last integral converges, therefore the integral

$$\iint_{\{x,y\in[0,1]:|x-y|\le 2\lambda_{k_0}\}} ((F(x)-F(y))^2 + |x-y|^2)^{-s/2} d\mathcal{L}^1{}_W(x) d\mathcal{L}^1{}_W(y)$$

is finite almost surely and so

$$\iint_{[0,1]\times[0,1]} ((F(x)-F(y))^2 + |x-y|^2)^{-s/2} d\mathcal{L}^1{}_W(x) d\mathcal{L}^1{}_W(y) < \infty$$

almost surely.

Step 2. Let μ_W be the finite Borel measure on $\mathbb{R}^2$ defined by $\mu_W(E) = \mathcal{L}^1\{t \in W : (t, F(t)) \in E\}$ for all $E \subset \mathbb{R}^2$. Notice that μ_W is supported on Γ_F and of finite s-energy. Hence, to conclude that $\dim \Gamma_F \ge s$ it is sufficient to prove that μ_W is positive which is equivalent to show that $\mathcal{L}^1(W) > 0$.

We have $[0,1] \setminus W = \bigcup_{k=k_0}^{\infty} ([0,1] \setminus \bigcup_{p=n_{k^2}}^{n_{(k+1)^2}-1} C_k)$ so by Lemma 5

$$\mathbb{E}(\mathcal{L}^1([0,1]\setminus W)) \le \sum_{k=k_0}^{\infty} \left(\frac{n_{k^2}}{n_{(k+1)^2} - 1}\right)^{\delta}.$$

Since $n_{k^2} \in \Lambda_{k^2}$ then $\alpha 2^{k^2-1} \le n_{k^2} < \alpha 2^{k^2}$.

Hence

$$\mathbb{E}(\mathcal{L}^1([0,1]\setminus W)) \le \sum_{k=k_0}^{\infty} 2^{-2k\delta} = \frac{2^{-2k_0\delta}}{1-2^{-2\delta}}.$$

Using Markov's inequality we have,

$$\mathbb{P}(\mathcal{L}^1(W) < 1/2) = \mathbb{P}(\mathcal{L}^1([0,1]\setminus W) \ge 1/2) \le 2\frac{2^{-2k_0\delta}}{1-2^{-2\delta}}.$$

Let $0 < \eta < 1$ and choose k_0 large enough such that $\frac{2^{1-2k_0\delta}}{1-2^{-2\delta}} < \eta$, so $\mathcal{L}^1(W) \ge 1/2$ with probability greater than $1-\eta$. From the previous two steps we conclude that $\dim \Gamma_F \ge s$ with probability at least $1-\eta$. The arbitrariness on s and η implies that $\dim \Gamma_F \ge 2 - H$ almost surely.

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