Mediterr. J. Math. 8 (2011), 431–450
DOI 10.1007/s00009-010-0073-9
1660-5446/11/030431-20, *published online* May 11, 2010

Mediterranean Journal of Mathematics

Measure-Preserving Functions and the Independence Copula

Enrique de Amo*, Manuel Díaz Carrillo and Juan Fernández-Sánchez

Abstract. We solve a problem recently proposed by Kolesárová et al. Specifically, we prove that a necessary and sufficient condition for a given copula to be the independence or product copula is for the pair of measure-preserving transformations representing the copula to be independent as random variables.

We provide examples of such pairs for the well-known Cantor, Peano, and Hilbert curves. Moreover, a general constructive method is given for the representation of copulae in terms of measure-preserving transformations. In particular, we apply numbers representation systems to the study of self-similar copulae properties.

Mathematics Subject Classification (2010). Primary 60E05; Secondary 28D05.

Keywords. measure-preserving transformations, independence or product copula, representation system, self-similarity.

1. Introduction

The notion of copula was introduced by Sklar [22] to express the relationship between multivariate distribution functions and their univariate marginals. Since he proved his celebrated theorem in 1959, the study of copulae and their applications has revealed itself to be a tool of great interest in several branches of mathematics. For an introduction to copulae, see [15].

It was proven, rather later, that it is possible to establish a correspondence between copulae and measure-preserving transformations in the unit interval ([16], [24]). In this paper, we investigate the most difficult aspect of this relation-that is, for a given copula C, let us find a pair of measure-preserving transformations (f, g) such that $C = C_{f,g}$. Another proof for the

*Corresponding author.

representation of a copula in terms of measure-preserving transformations can be found in [12].

In Section 3, we give a general method that allows the explicit building of functions f and g representing a copula C (Theorem 4). This Representation Theorem is applied in Section 4 to provide an answer to an open question given by Kolesárová et al. [12]: for the product or independent copula $\Pi(u,v) = uv$, find a sufficient condition for a pair of measure-preserving transformations (f,g) to be $C_{f,g} = \Pi$ (Theorem 5).

The above results allow us to present three examples for very well-known functions (Cantor, Hilbert, and Peano), where $C_{f,g} = \Pi$.

Recently, Fredricks et al. [9], using an iterated function system, have constructed families of copulae whose supports are fractals. In particular, they give sufficient conditions for the support of a self-similar copula to be a fractal whose Hausdorff dimension is between 1 and 2. Formulae for computing the Hausdorff dimension of these types of sets require that the iterated function system be similarities (see, for example, [5], [6], [7]). Towards that end, we impose that T be a transformation matrix that satisfies certain conditions to obtain invariant copulae C_T associated to T.

In Section 5, we study two representation systems of numbers in $[0,1]$, and we apply them to describe the family C_T of self-similar copulae in terms of measure-preserving transformations (Theorem 12). Finally, using techniques related with fractal, probability, and ergodic theories (see, for example, [1], [7], [8]), we study some properties for these numerical representation systems.

Section 2 contains background information.

2. Preliminaries

(2.1) Let $\mathbb{I} := [0,1]$ be the closed unit interval and $\mathbb{I}^2 := [0,1]^2$ be the unit square. A *two-dimensional copula* (or a *copula* for brevity) is a function $C : \mathbb{I}^2 \longrightarrow \mathbb{I}$ with the two following properties:

i) For every u, v in $\mathbb{I}$, $C(u,0) = 0 = C(0,v)$ and $C(u,1) = u$, $C(1,v) = v$.
ii) For every u_1, u_2, v_1, v_2 in $\mathbb{I}$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$,

$$C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \geq 0.$$

Alternatively, we can say that a copula is a bivariate distribution function in $\mathbb{I}^2$ whose marginal distribution functions are uniform distribution functions on $\mathbb{I}$. Therefore, each copula C induces a probability measure μ_C on $\mathbb{I}^2$, via the formula

$$\mu_C([a,b] \times [c,d]) = C(b,d) - C(b,c) - C(a,d) + C(a,c),$$

in a similar fashion to joint distribution functions; that is, the μ_C-measure of a set is its C-volume (i.e., its probability mass). Through standard measure-theoretic techniques, μ_C can be extended from the semi-ring of rectangles in $\mathbb{I}^2$ to the σ-algebra $\mathcal{B}(\mathbb{I}^2)$ of Borel sets in the unit square. We will denote by λ the standard Lebesgue measure on the Borel σ-algebra $\mathcal{B}(\mathbb{I}^2)$.

There is a one-to-one correspondence between copulae and doubly stochastic measures. For every copula C, the measure μ_C is doubly stochastic (that is, $\mu_C(A \times \mathbb{I}) = \mu_C(\mathbb{I} \times A) = \lambda(A)$, for every Borel set $A \subset \mathbb{I}$). Conversely, for every doubly stochastic measure μ, there exists a copula C given by $C(u, v) = \mu(([0, u]) \times ([0, v]))$. Therefore, we can translate some measure-theoretic concepts and results into the language of copulae. In particular, the Lebesgue Decomposition Theorem remains true (see [15, p. 27] and [20]).

(2.2) For any copula C, let $\mu_C = \mu_C^c + \mu_C^s$, where $\mu_C^c = \int\limits_D \frac{\partial^2 C}{\partial u \partial v} d\lambda$, and $\mu_C^s = \mu_C - \mu_C^c$, for all $D \in \mathcal{B}\left(\mathbb{I}^2\right)$. Then, $\mu_C^c \ll \lambda$ (i.e., μ_C^c is an *absolutely continuous measure* with respect to λ), and $\mu_C^s \perp \lambda$ (*mutually singular measures).* Because the marginals of C have uniform distributions, it follows that μ_C^s has no atoms.

(**2.3**) The independence (or product) copula Π, given by $\Pi(u, v) = uv$, distributes its probability mass uniformly on $\mathbb{I}^2$, and it is of particular importance. By Sklar's Theorem (see [22] or [15, sec. 2.4]), for given continuous random variables X, Y, we have $C_{X;Y} = \Pi$ if, and only if, X and Y are independent.

(**2.4**) Given a measure space (X, Ω, μ), a measurable function $F : X \to X$ is said to be *measure-preserving (*or *F preserves* μ) if $\mu\left(F^{-1}(A)\right) = \mu(A)$, for all $A \in \Omega$.

If the σ-algebra Ω is generated by a family P that is closed under finite intersections, then a sufficient condition for F being measurable and measure-preserving (see [1, p. 311]) is that $F^{-1}(A) \in \Omega$ and $\mu\left(F^{-1}(A)\right) = \mu(A)$, for all A in P. The system (X, Ω, μ, F) is said to be a *dynamical system*, and the central theorem in this context is the so-called Ergodic Theorem (see, for example, [10, p. 31] or [18, Chap. 10]).

The class of the measure-preserving transformations contains certain special ones. We recall that F is said to be *ergodic* if each *invariant set* A (i.e. $F^{-1}(A) = A$) is trivial in the sense of its measure being either 0 or 1. Alternatively, we say that the system (X, Ω, F, μ) is ergodic.

(2.5) A measure-preserving transformation F is said to be *mixing* (or *strongly mixing,* for other authors), if

$$\lim_{n\to\infty} \mu\left(A \cap F^{-n}B\right) = \mu(A)\,\mu(B)$$

holds for every pair of sets $A, B \in \Omega$ (or, equivalently, for $f, g \in L^2(X, \Omega, \mu)$,

$$\lim_{n\to\infty} \int_X f\left(F^n(x)\right) g(x) d\mu(x) = \int_X f(x) d\mu(x) \int_X g(x) d\mu(x)).$$

Let us observe that, if set B is invariant, then $\mu(A \cap B) = \mu(A)\,\mu(B)$, and, if we take $A = B$, then it follows that $\mu(B)$ is 0 or 1. Therefore, mixing implies ergodicity (see [18, Chap. 11]). If F is ergodic, then, as a consequence of the Ergodic Theorem, the orbit $\{F^n x : n \in N\}$ of x is a sort of replica of X itself for almost all x in X.

(2.6) Let $k \geq 2$ be an integer and $p_0, p_1, \ldots, p_{k-1}$ positive real numbers satisfying the relation $\sum_{i=0}^{k-1} p_i = 1$. Let $K = \{0, 1, \ldots, k-1\}$ and $P = 2^K$ be its power set. The triple (K, P, μ) is the probability space where $\mu(i) = p_i$. The space $\prod_{j=1}^{\infty} (K, P, \mu)$, together with the transformation $\sigma((x_0, x_1, x_2, \ldots)) = (x_1, x_2, \ldots)$, is called *the one-sided Bernoulli space.* One can check that the one-sided Bernoulli implies mixing (see [25, Sec. 4.9]).

Let us recall (see [18, Chap. 8]) that, for given (X_i, Ω_i, μ_i), $i = 1, 2$, probability spaces, an *isomorphism between measure-preserving transformations* $F_i : X_i \to X_i$ is a map $\phi : X_1 \to X_2$ such that:

(1) ϕ is a bijection (after removing sets of zero measure if necessary),
(2) both ϕ and ϕ^{-1} are measurable maps (i.e., $\phi^{-1}(\Omega_2) \subset \Omega_1$ and $\phi(\Omega_1) \subset \Omega_2$),
(3) $\mu_1\left(\phi^{-1}B\right) = \mu_2(B)$, for $B \in \Omega_2$ (also $\mu_2(\phi B) = \mu_1(B)$, for $B \in \Omega_1$), and
(4) $\phi \circ F_1 = F_2 \circ \phi$.

For the sake of simplicity, let us say that X_1 and X_2 are *isomorphic* if there exists such an isomorphism. If a space X is isomorphic to another one that is a one-sided Bernoulli, then X is said to be *Bernoulli.*

(2.7) A *transformation matrix* is a matrix T with non-negative entries, the sum of which is 1 and none of the rows or the columns of which sum up to zero.

Following the paper by Fredricks et al. [9], we recall that each transformation matrix T determines a subdivision of $\mathbb{I}^2$ into subrectangles $R_{ij} = [p_{i-1}, p_i] \times [q_{j-1}, q_j]$, where p_i (respect., q_j) denotes the sum of the entries in the first i columns (respect., j rows) of T. For a transformation matrix T and a copula C, $T(C)$ denotes the copula which, for each (i, j), spreads its mass on R_{ij} in the same way in which C spreads its mass on $\mathbb{I}^2$.

Theorem 2 in [9] shows that, for each transformation matrix $T \neq [1]$, there is a unique copula C_T for which $T(C_T) = C_T$.

(2.8) Let T be a transformation matrix. We now consider the following conditions on T:

i) T has at least one zero entry.
ii) For each non-zero entry of T, the row and column sums through for that entry are equal.
iii) There is at least one row or column of T with two non-zero entries.

Theorem 3 in [9] shows that, if T is a transformation matrix with i) in (2.8), then C_T is *singular* (that is, its support has either Lebesgue measure zero or $\mu_{C_T} \equiv \mu_{C_T}^s$).

We say that a copula C is *invariant* if $C = C_T$ for some transformation matrix T. An invariant copula C_T is said to be *self-similar* if T satisfies ii) in (2.8).

Theorem 6 in [9] shows that the support of a self-similar copula C_T for which T satisfies i) and iii) in (2.8), is a fractal with a Hausdorff dimension between 1 and 2.

(2.9) As a consequence of (2.4) we are interested in the case $(X, \Omega, \mu) = (\mathbb{I}, \mathcal{B}(\mathbb{I}), \lambda)$. There is a correspondence between copulae C and measure-preserving transformations f, g on the unit interval $\mathbb{I}$, via the formula

$$C_{f,g}(u, v) = \lambda\left(f^{-1}[0, u]\right) \cap \lambda\left(g^{-1}[0, v]\right).$$

(See [3], [4], [12], [16] or [24].)

(2.10) Finally, by using the probability measure induced by a copula, an important class of copulae is provided by the *Shuffles of Min* (see [4], [14], or [15, p. 68], for more details). Formally, a copula C is a shuffle of Min if there is a natural number n, two partitions $0 = s_0 < s_1 < ...s_n = 1$ and $0 = t_0 < t_1 < ...t_n = 1$ of $\mathbb{I}$, and a permutation σ of $\{1, ..., n\}$ such that each $[s_{i-1}, s_i] \times \left[t_{\sigma(i)-1}, t_{\sigma(i)}\right]$ is a square in which C distributes the mass $s_i - s_{i-1}$ uniformly spread along one of the diagonals. (Its support is the graph of a bijection but a finite number of points. Everyone of these bijections is said to be associated with the Shuffles of Min.)

3. Copulae Representations

Theorem 4 in [4] characterizes shuffles of Min in terms of measure-preserving transformations of $\mathbb{I}$ and push-forward of the doubly stochastic measure induced by the copula $M(u, v) = \min\{u, v\}$. On the other hand, for the general problem of determining just which curves in $\mathbb{I}^2$ can serve as the support of a copula, Mikusiński et al. [14] proved that, for every copula obtained as a shuffle of Min, there is a piece-wise linear function whose graph supports the probability mass. In this context, now, we have the following general result.

Proposition 1. *Let $f : \mathbb{I} \longrightarrow \mathbb{I}$ be a Borel measurable function. Then, there exists a copula C whose associated measure μ_C has its mass concentrated on the graph of f (with $\mu_C(G(f)) = 1$) if, and only if, the function f preserves the Lebesgue measure λ.*

Proof. On one hand, the condition is necessary: let C be a copula satisfying $\mu_C(G(f)) = 1$. Let $\mathcal{B}(G(f))$ be the σ-algebra of the Borel sets in the unit square that are subsets of $G(f)$. Let us denote by μ_f the restriction of μ_C to $\mathcal{B}(G(f))$. Let us recall that $f^* : \mathbb{I} \longrightarrow G(f)$ such that $x \rightarrow (x, f(x))$, is a bijection; in fact, it is an isomorphism from $(\mathbb{I}, \mathcal{B}(\mathbb{I}), \lambda)$ to $(G(f), \mathcal{B}(G(f)), \mu_C)$. In particular, f^* induces a bijection between σ-algebras. As a consequence, for each interval $[a, b] \subset \mathbb{I}$ (they generate the σ-algebra $\mathcal{B}(\mathbb{I})$), we have:

$$\lambda([a, b]) = \mu_C([a, b] \times \mathbb{I}) = \mu_C([a, b] \times \mathbb{I} \cap G(f)) = \mu_f([a, b] \times \mathbb{I} \cap G(f));$$

therefore, if $A \in \mathcal{B}\left(\mathbb{I}^2\right)$, then

$$\mu_C(A) = \lambda(\{t : (t, f(t)) \in A\}).$$

Thereby, for each $B \in \mathcal{B}(\mathbb{I})$, we have

$$\lambda(f^{-1}(B)) = \lambda(\{t : (t, f(t)) \in \mathbb{I} \times B\}) = \mu_C(\mathbb{I} \times B) = \lambda(B),$$

that is, f is measure-preserving.

On the other hand if μ is a measure such that

$$\mu(A) = \lambda(\{t : (t, f(t)) \in A\})$$

for each $A \in \mathcal{B}(\mathbb{I}^2)$, then it is doubly stochastic. And, hence, it is supported by the graph of f. Therefore, the condition is also sufficient. $\square$

It is well known that every copula can be represented through measure-preserving transformations on the unit interval (see, for example, [12] and [24]). The aim of this section is to present a new construction method to prove a Representation Theorem. In the sections below, we study the usefulness of these techniques for building new families of copulae.

Definition 2 (Unit square n-th division). *For each $i \in \{0, 1, 2, 3\}$, let $f_i : \mathbb{I}^2 \to \mathbb{I}^2$ be the functions defined as:*

$$\begin{cases} f_0(x, y) = (x/2, y/2), \\ f_1(x, y) = (x/2, y/2 + 1/2), \\ f_2(x, y) = (x/2 + 1/2, y/2 + 1/2), \\ f_3(x, y) = (x/2 + 1/2, y/2). \end{cases}$$

Set $A_i := f_i(\mathbb{I}^2)$ and $A_{ij} := f_i \circ f_j(\mathbb{I}^2)$, $i, j \in \{0, 1, 2, 3\}$. In general, $A_{i_1 i_2 \ldots i_n} = f_{i_1} \circ f_{i_2} \circ \ldots \circ f_{i_n}(\mathbb{I}^2)$, for $i_1, i_2, \ldots, i_n \in \{0, 1, 2, 3\}$. For a fixed natural n, the union of all the sets $A_{i_1 i_2 \ldots i_n}$, $i_1, i_2, \ldots, i_n \in \{0, 1, 2, 3\}$, is the unit square. Let us note that the interior of these squares are mutually disjoint and their two-by-two intersections are the empty set or they reduce to horizontal or vertical segments of length $1/2^n$.

The class $\{A_{i_1 i_2 \ldots i_n} : i_1, i_2, \ldots, i_n \in \{0, 1, 2, 3\}\}$ is termed the n-th division of the unit square $\mathbb{I}^2$.

Let $C : \mathbb{I}^2 \longrightarrow \mathbb{I}$ be a copula. We now consider the induced n-th division of the unit interval $\mathbb{I}$ by the copula C: define numbers $\beta_{i_1 i_2 \ldots i_n} := \mu_C(A_{i_1 i_2 \ldots i_n})$. Now, for each integer $0 \leq m < 4^n$, we allocate the index $i_1 i_2 \ldots i_n$, which comes from the 4-base representation for m; that is, $m = 4^{i_1} + 4^{i_2} + \cdots + 4^{i_n}$. We simplify notations thus: $A_{i_1 i_2 \ldots i_n} = A_{m,n}$ and $\beta_{i_1 i_2 \ldots i_n} = \beta_{m,n}$.

As a consequence, we obtain a division for $\mathbb{I}$ that is associated to the copula C. We say that

$$B_{m,n} = \left[\sum_{s=0}^{m-1} \beta_{s,n}, \sum_{s=0}^{m} \beta_{s,n} \right]$$

is the m-th interval of order n.

Let us note that it is possible for $B_{m,n}$ to be either the empty set or to be repeated for different subindexes. Consequently, it is easy to see that

$$B_{m,n-1} = B_{4m,n} \cup B_{4m+1,n} \cup B_{4m+2,n} \cup B_{4m+3,n}.$$

Hence, if the first set reduces to a single point, the others will as well.

With the above notations, we obtain a sequence of squares $\{D_n\}$, satisfying that D_n is an element in the n-th division, and $D_n \subset D_{n-1}$, which determines a unique point in the unit square.

By E_n, we denote the corresponding subintervals in $\mathbb{I}$ that are associated to their respective subsquares D_n. The sets $A_{m,n}$ and $B_{m,n}$ defined above are related one-to-one. Hence, for brevity, we can refer to E_n (in the unit interval) or to D_n (in the unit square), without confusion, as elements in the n-th division.

Let us recall that the measures associated to copulae have no atoms and, as a consequence, the sequence $\{\lambda(E_n)\}$ converges to zero. Therefore, the following result is true.

Lemma 3. *Let $\{E_n\}$ be a sequence of intervals. If, for each natural number n, E_n is in the n-th division of the unit interval, and $E_n \subset E_{n-1}$, then the sequence $\{E_n\}$ determines a unique point $\mathbb{I}$.*

Theorem 4 (of Representation). *If $f_1, f_2, \ldots, f_n : \mathbb{I} \longrightarrow \mathbb{I}$ are measure-preserving transformations, then the function $C_{f_1,f_2,\ldots,f_n} : \mathbb{I}^n \longrightarrow \mathbb{I}$ given by*

$$\begin{aligned} &C_{f_1,f_2,\ldots,f_n}(x_1, x_2, \ldots, x_n) \\ &:= \lambda\left(f_1^{-1}([0,x_1]) \cap f_2^{-1}([0,x_2]) \cap \cdots \cap f_n^{-1}([0,x_n])\right), \end{aligned}$$

is an n-copula. Conversely, for every n-copula C, there exist n measure-preserving functions $f_1, f_2, \ldots, f_n : \mathbb{I} \longrightarrow \mathbb{I}$ such that $C = C_{f_1,f_2,\ldots,.f_n}$.

Proof. We shall limit ourselves to proving the result in the case $n = 2$ and, thereby, to finding two measure-preserving functions f_1 and f_2 representing the given copula C. (A different proof of this theorem can be found in [12, Appendix].)

We define a correspondence between $\mathbb{I}$ and $\mathbb{I}^2$. For a given $x \in \mathbb{I}$, let us define A_x as the set of interval sequences $\{E_n\}$ such that E_n is in the n-th division, $E_n \subset E_{n-1}$, and $x \in E_n$ for all n. If $\{E_n\} \in A_x$, then, for each E_n, there is a unique m such that $E_n = B_{m,n}$. Therefore, there exists, in a natural way, a sequence of squares $\{D_n\}$ associated to $\{E_n\}$. The sequence $\{D_n\}$ satisfies that, for each n, the square D_n belongs to the n-th division and $D_n \subset D_{n-1}$. (The best way to achieve this goal is to define $D_n = A_{m,n}$.) Each sequence $\{D_n\}$ obtained in this way defines a unique point. Let us now set a correspondence $c : \mathbb{I} \longrightarrow \mathbb{I}^2$ as follows: $c(x)$ is the set of points determined by each of the sequences $\{D_n\}$ that are related to the elements in A_x.

This correspondence c determines another two:

$$c_1, c_2 : \mathbb{I} \longrightarrow \mathbb{I}$$

where $c_1(x)$ is the projection of $c(x)$ on the first coordinate and $c_2(x)$ is the corresponding projection on the second one. We are going to consider the following couple of functions:

$$f_1(x) := \inf c_1(x), \qquad f_2(x) := \inf c_2(x).$$

The aim now is to prove that they preserve the Lebesgue measure on Borel sets. But it is enough to consider intervals in the form of $\left[\frac{l}{2^k}, \frac{l+1}{2^k}\right]$.

First, we demonstrate the assertion $\lambda\left(f_1^{-1}\left(\left\{\frac{l}{2^k}\right\}\right)\right) = 0$. Clearly,

$$f_1^{-1}\left(\left\{\frac{l}{2^k}\right\}\right) \subseteq f_1^{-1}\left(\left]\frac{2^s l-1}{2^{k+s}}, \frac{2^s l+1}{2^{k+s}}\right[\right) \subseteq c^{-1}\left(\left]\frac{2^s l-1}{2^{k+s}}, \frac{2^s l+1}{2^{k+s}}\right[\times \mathbb{I}\right).$$

Moreover, the way we built c yields

$$\lambda\left(f_1^{-1}\left(\left\{\frac{l}{2^k}\right\}\right)\right) \le \lambda\left(c^{-1}\left(\left]\frac{2^s l-1}{2^{k+s}}, \frac{2^s l+1}{2^{k+s}}\right[\times \mathbb{I}\right)\right) \le \frac{1}{2^{k+s-1}}$$

(where the last inequality follows from the fact that C is a copula). By making $s \to \infty$, we conclude the assertion.

Finally, from the definition of c and the assertion just proven above, we can write

$$\lambda\left(f_1^{-1}\left(\left[\frac{l}{2^k}, \frac{l+1}{2^k}\right]\right)\right) = c^{-1}\left(\left[\frac{l}{2^k}, \frac{l+1}{2^k}\right]\times \mathbb{I}\right) = \frac{1}{2^k}. \qquad \square$$

4. The independence or product copula Π

Kolesárová et al. [12] have recently proposed the following problem: Given the copula Π, what is the pair of measure-preserving transformations f and g such that $\Pi = C_{f,g}$?

The building method used in the proof of the Representation Theorem allows us to find these functions and we use it below. However, now we are going to provide a general answer to the problem they propose.

Theorem 5. *A necessary and sufficient condition for the copula $C_{f,g}$ to be equal to Π is that functions f and g be independent as random variables.*

Proof. If f and g are independent as random variables in $\mathbb{I}$, then

$$\begin{aligned} C_{f,g}((x,y)) &= \lambda\left\{f^{-1}([0,x]) \cap g^{-1}([0,y])\right\} \\ &= \lambda\left\{f^{-1}([0,x])\right\}\lambda\left\{g^{-1}([0,y])\right\} = xy = \Pi(x,y). \end{aligned}$$

Conversely, if

$$\Pi(x,y) = xy = C_{f,g}((x,y)) = \lambda\left\{f^{-1}([0,x]) \cap g^{-1}([0,y])\right\}$$

then

$$\lambda\left\{f^{-1}([0,x]) \cap g^{-1}([0,y])\right\} = xy = \lambda\left\{f^{-1}([0,x])\right\}\lambda\left\{g^{-1}([0,y])\right\},$$

and the independence of f and g follows. $\square$

Corollary 6. *If $f : \mathbb{I} \longrightarrow \mathbb{I}$ is a bijective measure-preserving transformation, then there does not exist another measure-preserving function g such that f and g, as random variables, are independent.*

Proof. First, we prove this for the identity map, that is, $f(x) = i(x) = x$. If g is a measure-preserving transformation such that it is independent with respect to f, then we consider the set $A := g^{-1}([0,1/2])$. The independence guarantees that

$$\begin{aligned}\lambda\,(A\cap[0,t]) &= \lambda\left\{g^{-1}([0,1/2])\cap i^{-1}\left([0,t]\right)\right\}\\ &= \lambda\left\{g^{-1}([0,1/2])\right\}\lambda\left\{i^{-1}\left([0,t]\right)\right\} = t/2.\end{aligned}$$

Let us now define $h(t) = \int_0^t \chi_A(x)dx$ (where χ_A denotes the characteristic or indicator function of A.) Because

$$\lambda\,(A\cap[0,t]) = \int_0^t \chi_A(x)dx,$$

then, for each t, there exists the derivative $h'(t) = 1/2$.

On the other hand, we have $h' = \chi_A$ on a set of measure 1. However, this contradicts the existence of g.

In the general case, if f is a measure-preserving bijection and g having this property, is independent with respect to f (as random variables), we have

$$\Pi = C_{f,g} = C_{i,g\circ f^{-1}}.$$

But, in this case, because $g\circ f^{-1}$ is measure-preserving, we have just found an independent random variable with respect to the identity map i. Contradiction. □

Remark 7. *As a particular case, there does not exist any measure-preserving function g being independent, as a random variable, with respect to any (function which is associated with a) shuffle of Min.*

The remainder of this section is devoted to showing three examples of pairs of functions where $C_{f,g}$ is Π. (See [11], and [17]). The book of Sagan [21] is a unifying framework for the generation of the space-filling curves of Peano, Hilbert, and Cantor types.

4.1. Cantor example

In this case, the copula satisfies that, for each rectangle with a non-empty interior, its associated measure is positive. Concretely, the rectangles in the form $\left[\frac{l}{2^k},\frac{l+1}{2^k}\right]\times\left[\frac{m}{2^k},\frac{m+1}{2^k}\right]$ have a measure equal to $\frac{1}{2^{2k}}$. This guarantees, via the correspondence introduced in the proof of the Representation Theorem, that there exists a unique point $c\,(x)$ to each one of those numbers x in $\mathbb{I}$ that does not have a finite 4-base expansion, and two points for each of the others.

For the product copula Π, the induced divisions on $\mathbb{I}$ are intervals of length $1/4^n$. Hence, it would seem to suggest the use of the 4-base representation system. Nevertheless, we shall use the 2-base representation system to describe functions f and g. Let $x = \sum \frac{a_n}{2^n}$ be a number in the unit interval.

Then, we define:

$$\begin{cases} f(x) = \sum \frac{a_{2n-1}}{2^n} \\ g(x) = \sum \frac{a_{2n}}{2^n}. \end{cases}$$

The rest of the points x are in a set of null measure, and therefore it is unimportant from the point of view of measure theory how their images are defined.

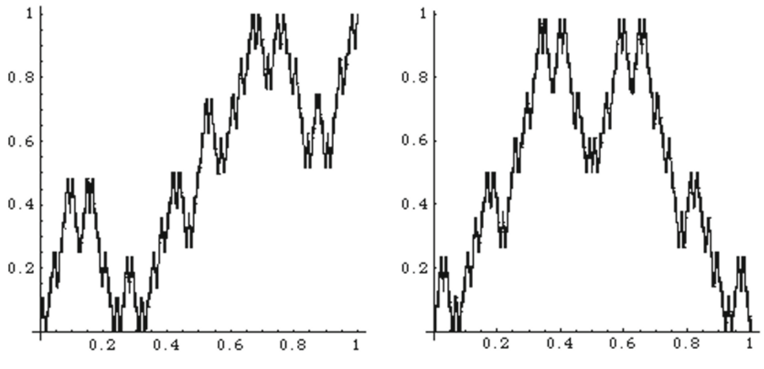
This example can be extended to other dimensions without any difficulty.

4.2. Hilbert example

The functions in the preceding example are discontinuous. But it is possible to obtain continuous functions if we introduce some changes (on the divisions). Now, the functions will be:

$$\begin{cases} f_0 : \mathbb{I}^2 \longrightarrow \mathbb{I}^2; (x, y) \to (y/2, x/2) \\ f_1 : \mathbb{I}^2 \longrightarrow \mathbb{I}^2; (x, y) \to (x/2, y/2 + 1/2) \\ f_2 : \mathbb{I}^2 \longrightarrow \mathbb{I}^2; (x, y) \to (x/2 + 1/2, y/2 + 1/2) \\ f_3 : \mathbb{I}^2 \longrightarrow \mathbb{I}^2; (x, y) \to (1 - y/2, x/2 + 1/2) \end{cases}$$

In this case, there is no possibility of double definition for any point. And the correspondence is, in fact, a function. The corresponding graphs for f and g are:



Proposition 8. *The coordinate functions in the Hilbert curve are independent random variables and measure-preserving.*

4.3. Peano example

The division for the unit square given in definition 2 it is not essential for the proof of the Representation Theorem since other divisions could be done. An example follows:

$$\begin{cases} f_0 : \mathbb{I}^2 \longrightarrow \mathbb{I}^2; (x,y) \to (y/3, x/3) \\ f_1 : \mathbb{I}^2 \longrightarrow \mathbb{I}^2; (x,y) \to (-x/3+1/3, y/3+1/3) \\ f_2 : \mathbb{I}^2 \longrightarrow \mathbb{I}^2; (x,y) \to (x/3, y/3+2/3) \\ f_3 : \mathbb{I}^2 \longrightarrow \mathbb{I}^2; (x,y) \to (x/3+1/3, -y/3+1) \\ f_4 : \mathbb{I}^2 \longrightarrow \mathbb{I}^2; (x,y) \to (-x/3+2/3, -y/3+2/3) \\ f_5 : \mathbb{I}^2 \longrightarrow \mathbb{I}^2; (x,y) \to (x/3+1/3, -y/3+1/3) \\ f_6 : \mathbb{I}^2 \longrightarrow \mathbb{I}^2; (x,y) \to (x/3+2/3, y/3) \\ f_7 : \mathbb{I}^2 \longrightarrow \mathbb{I}^2; (x,y) \to (-x/3+1, y/3+1/3) \\ f_8 : \mathbb{I}^2 \longrightarrow \mathbb{I}^2; (x,y) \to (x/3+2/3, y/3+2/3) \end{cases}$$

Proposition 9. *The coordinate functions in the Peano curve are independent random variables and measure-preserving.*

The book of Sagan [21] is an excellent reference for the extension of Hilbert and Peano examples to higher dimensions.

5. Representation systems for self-similar copulae

In the literature reviewed, all the examples of singular copulae we have found are supported by sets of Hausdorff dimension 1. However, it is implicit in the literature, for example in [21, Chaps. II and III], that the well-known examples of Peano and Hilbert curves, endowed with the measure $v(F) := \lambda(\pi(F))$, yield self-similar copulae with fractal support because the Hausdorff dimension of their graphs is 3/2 (see [13] and [23]).

Recently, Fredricks et al. [9], using an iterated function system, construct families of self-similar copulae whose supports are fractals with Hausdorff dimensions between 1 and 2. We recall that any formulae for computing the Hausdorff dimension of these types of sets require that the iterated function system be given by similarities (see, for example, [5] or [6]). Therefore, in order to obtain invariant copulae C_T associated to T, we consider transformation matrixes T satisfying certain conditions (see (2.7) and (2.8) above).

5.1. Representation systems

In this section we build on the work presented by Fredricks et al. [9]. We study copulae whose supports are fractals and consider the problem of finding an explicit expression for two measure-preserving transformations f and g representing a family of self-similar copulae, in the same way we proved the Representation Theorem.

Let us note that the general method described in Section 3 is not always the most suitable for every situation. In the particular case of self-similar copulae, we can use tools better suited to the properties of these copulae. (See [9], and (2.7), (2.8) above.)

(**5.11**) Set $0 < r < 1$. Let us consider the family of copulae C determined by self-similar measures such that on the squares

$$\begin{cases} C_0 \text{ of vertexes } (0,0),(0,r),(r,0),(r,r) \\ C_1 \text{ of vertexes } (0,1-r),(0,1),(r,1-r),(r,1) \\ C_2 \text{ of vertexes } (1-r,0),(0,1),(1-r,r),(1-r,1) \\ C_3 \text{ of vertexes } (1-r,1-r),(1-r,r),(1-r,1),(1,1) \\ C_4 \text{ of vertexes } (r,r),(1-r,r),(r,1-r),(1-r,1-r) \end{cases}$$

have masses respectively given by $r/2, r/2, 1-2r, r/2, r/2$; and null for the rest. Matricially, all the preceding can be expressed, with $r \in \left]0, \frac{1}{2}\right[$, in the following way

$$T_r = \begin{bmatrix} r/2 & 0 & r/2 \\ 0 & 1-2r & 0 \\ r/2 & 0 & r/2 \end{bmatrix}$$

Now, we introduce two number representation systems in $]0,1]$. Let us divide the unit interval $[0,1]$ into five subintervals $[0,r/2],[r/2,r],[r,1-r]$, $[1-r,1-r/2]$, and $[1-r/2,1]$. If $x \in [0,1]$, then we can write $x = \alpha_0 + \delta_1 x_1$, with $\alpha_0 \in \{0, r/2, r, 1-r, 1-r/2\}$, $x_1 \in [0,1]$, and

$$\delta_1 = \begin{cases} r/2, & \text{if } \alpha_0 = 0 \\ r/2, & \text{if } \alpha_0 = r/2 \\ 1-2r, & \text{if } \alpha_0 = r \\ r/2, & \text{if } \alpha_0 = 1-r \\ r/2, & \text{if } \alpha_0 = 1-r/2 \end{cases}$$

If we repeat the above iteration process for x_1, instead of x, then: $x = \alpha_0 + \delta_1\alpha_1 + \delta_2 x_2$, with $\alpha_1 \in \{0, r/2, r, 1-r, 1-r/2\}$, $x_2 \in [0,1]$, and

$$\delta_2 = \begin{cases} \delta_1 r/2, & \text{if } \alpha_1 = 0 \\ \delta_1 r/2, & \text{if } \alpha_1 = r/2 \\ \delta_1(1-2r), & \text{if } \alpha_1 = r \\ \delta_1 r/2, & \text{if } \alpha_1 = 1-r \\ \delta_1 r/2, & \text{if } \alpha_1 = 1-r/2 \end{cases}$$

Hence, for each positive integer n, we can write $x = \alpha_0 + \delta_1\alpha_1 + \delta_2\alpha_2 + \cdots + \delta_n x_n$, with $x_n \in [0,1]$ and $|\delta_n| = (\min\{r/2, 1-2r\})^n$. The next result is a consequence of this recursive process.

Proposition 10. *(***System** $\boldsymbol{A}$ **with parameter** r*) All the points in* $]0,1]$ *can be written as:*

$$\alpha_0 + \delta_1\alpha_1 + \delta_2\alpha_2 + \cdots + \delta_i\alpha_i + \cdots,$$

where $\alpha_i \in \{0, r/2, r, 1-r, 1-r/2\}$, *and* δ_i *is defined from* δ_{i-1} *as follows:*

$$\delta_i = \begin{cases} \delta_{i-1} r/2, & \text{if } \alpha_{i-1} = 0 \\ \delta_{i-1} r/2, & \text{if } \alpha_{i-1} = r/2 \\ \delta_{i-1}(1-2r), & \text{if } \alpha_{i-1} = r \\ \delta_{i-1} r/2, & \text{if } \alpha_{i-1} = 1-r \\ \delta_{i-1} r/2, & \text{if } \alpha_{i-1} = 1-r/2 \end{cases}$$

The above representation is realized via the dynamical system $(\mathbb{I}, \mathcal{B}(\mathbb{I}), \lambda, F)$ (see (2.4) above), where

$$F(x) = \begin{cases} 2x/r, & \text{if } x \in [0, r/2] \\ 2(x - r/2)/r, & \text{if } x \in [r/2, r] \\ (x - r)/(1 - 2r), & \text{if } x \in [r, 1 - r] \\ 2(x - 1 + r)/r, & \text{if } x \in [1 - r, 1 - r/2] \\ 2(x - 1 + r/2)/r, & \text{if } x \in [1 - r/2, 1] \end{cases}$$

because $x = \alpha_0 + \delta_1 F(x)$; and, in general, $x = \alpha_0 + \delta_1\alpha_1 + \delta_2\alpha_2 + \cdots + \delta_n F^n(x)$.

We can use the intervals $[0, r], [r, 1 - r]$ and $[1 - r, 1]$, and the same ideas to obtain this result:

Proposition 11. *(***System *B* with parameter** r**)** *All the points in* $]0, 1]$ *can be written as:*

$$a_0 + d_1a_1 + d_2a_2 + \cdots + d_ia_i + \cdots ,$$

where $a_i \in \{0, r, 1 - r\}$, *and* d_i *is defined from* d_{i-1} *as follows:*

$$d_i = \begin{cases} d_{i-1}r, & \textit{if } a_{i-1} = 0 \\ d_{i-1}(1 - 2r), & \textit{if } a_{i-1} = r \\ d_{i-1}r, & \textit{if } a_{i-1} = 1 - r \end{cases}$$

We now consider the dynamical system $(\mathbb{I}, \mathcal{B}(\mathbb{I}), \lambda, G)$, where

$$G(x) = \begin{cases} x/r, & \text{if } x \in [0, r] \\ (x - r)/(1 - 2r), & \text{if } x \in [r, 1 - r] \\ (x - 1 + r)/r, & \text{if } x \in [1 - r, 1] \end{cases}$$

Let us note that the representation in each system is unique, but on a denumerable set where the representation has two possible forms: one is finite and the other stationary.

The relationship between the systems $\boldsymbol{A}$ and $\boldsymbol{B}$ provides a useful tool to obtain an explicit expression of the pair of measure-preserving transformations (f, g) representing the family of copulae C_{T_r} given in (5.11). We shall proceed as follows:

(5.12) Let x be a point with unique representation

$$x = \alpha_0 + \delta_1\alpha_1 + \delta_2\alpha_2 + \cdots + \delta_i\alpha_i + \cdots$$

in the system $\boldsymbol{A}$. Let us define

$$f(x) = a_0 + d_1a_1 + d_2a_2 + \cdots + d_ia_i + \cdots ,$$

where the coefficients a_i are determined by the corresponding α_i :

$$a_i = \begin{cases} 0, & \text{if } \alpha_{i-1} \in \{0, r/2\} \\ r, & \text{if } \alpha_{i-1} = r \\ 1 - r, & \text{if } \alpha_{i-1} \in \{1 - r, 1 - r/2\} \end{cases}$$

Analogously, we define a function g, where the relation is given by

$$a_i = \begin{cases} 0, & \text{if } \alpha_{i-1} \in \{0, 1 - r\} \\ r, & \text{if } \alpha_{i-1} = r \\ 1 - r, & \text{if } \alpha_{i-1} \in \{r/2, 1 - r/2\} \end{cases}$$

In case of double representation for x, we can choose the one we want. The result is correct in every case.

Theorem 12. *For every copula C given for the matrix T_r, there exist two functions f and g (given by (5.12)) that are λ-measure-preserving and such that $C = C_{f,g}$.*

Proof. The proof runs identically to that in the representation theorem, but we now use the self-similarity and singularity of the copula, and its support is determined by five contractions. □

5.2. Representation systems properties

If we use probabilistic, ergodic and fractal techniques (see, for example, [1], [7], [8], and [18, Chap. 8]), then we obtain the results that follow for the two representation systems we have just introduced in the above subsection.

It is easy to check that the systems $\boldsymbol{A}$ and $\boldsymbol{B}$ are given by generalized Lüroth series (see [2, sec. 2.3]). Hence, the next result is true.

Theorem 13. *The systems $\boldsymbol{A}$ and $\boldsymbol{B}$ are Bernoulli.*

Proof. We prove the statement for $\boldsymbol{A}$; the other case is analogous. With notation as in (2.6), let $k = 5$ and

$$p_0 = r/2, p_1 = r/2, p_2 = 1 - r, p_3 = r/2, p_4 = r/2.$$

Let $K = \{0, 1, 2, 3, 4\}$ and $P = 2^K$ be its power set. Then, the triple (K, P, μ) is a probability space where $\mu(i) = p_i$; and the space $\prod_{j=1}^{\infty}(K, P, \mu)$, together with the transformation $\sigma((x_0, x_1, x_2, \ldots)) = (x_1, x_2, \ldots)$, is isomorphic to $(\mathbb{I}, \mathcal{B}(\mathbb{I}), \lambda, F)$. Therefore, the representation system $\boldsymbol{A}$ is Bernoulli. □

As a consequence, the next result is true:

Proposition 14. *In the system $\boldsymbol{A}$ (resp., the system $\boldsymbol{B}$), the $\alpha_i(x)$'s (resp., the $a_i(x)$'s) are independent random variables that are identically distributed.*

Proposition 15. *There exists a set of measure 1 where:*

i. *For the system $\boldsymbol{A}$, the limits that follow are valid:*

$$\begin{cases} \lim_{n\to\infty} \frac{Card\{\alpha_i = 0 : 0 \le i \le n\}}{n} = r/2 \\ \lim_{n\to\infty} \frac{Card\{\alpha_i = r/2 : 0 \le i \le n\}}{n} = r/2 \\ \lim_{n\to\infty} \frac{Card\{\alpha_i = r : 0 \le i \le n\}}{n} = 1 - 2r \\ \lim_{n\to\infty} \frac{Card\{\alpha_i = 1-r : 0 \le i \le n\}}{n} = r/2 \\ \lim_{n\to\infty} \frac{Card\{\alpha_i = 1-r/2 : 0 \le i \le n\}}{n} = r/2 \end{cases}$$

ii. *In the system $\boldsymbol{B}$, these limits are true:*

$$\begin{cases} \lim_{n\to\infty} \frac{Card\{\alpha_i = 0 : 0 \le i \le n\}}{n} = r \\ \lim_{n\to\infty} \frac{Card\{\alpha_i = r : 0 \le i \le n\}}{n} = 1 - 2r \\ \lim_{n\to\infty} \frac{Card\{\alpha_i = 1-r : 0 \le i \le n\}}{n} = r \end{cases}$$

Proof. i) The case

$$\lim_{n\to\infty} \frac{Card\{\alpha_i = 0 : 0 \leq i \leq n\}}{n} = r/2$$

follows from the ergodic theorem applied to the function

$$\chi_{\boldsymbol{A},0}(x) = \begin{cases} 1, & \text{if } x \in [0, r/2] \\ 0, & \text{otherwise} \end{cases}$$

(See, for example, [18, Chap. 10].) The other cases, and ii), are shown in a similar way. □

Proposition 16. *There exists a set of measure 1 such that:*

i. *for the system* $\boldsymbol{A}$ *there exists the limit*

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} \alpha_i}{n} = 2r - \frac{5}{2}r^2.$$

ii. *for the system* $\boldsymbol{B}$ *there exists the limit*

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} a_i}{n} = 2r - 3r^2.$$

Proof. i) It is a new application of the Ergodic Theorem. We now consider the function:

$$\Sigma_{\boldsymbol{A}}(x) = \begin{cases} 0, & \text{if } x \in [0, r/2] \\ r/2, & \text{if } x \in [r/2, r] \\ r, & \text{if } x \in [r, 1-r] \\ 1-r, & \text{if } x \in [1-r, 1-r/2] \\ 1-r/2, & \text{if } x \in [1-r/2, 1] \end{cases}$$

ii) The function we consider here is:

$$\Sigma_{\boldsymbol{B}}(x) = \begin{cases} 0, & \text{if } x \in [0, r] \\ r, & \text{if } x \in [r, 1-r] \\ 1-r, & \text{if } x \in [1-r, 1] \end{cases}$$

□

Remark 17. *In the above propositions, the same results can be obtained if we use the proposition* 14 *and the Strong Law of Large Numbers.*

It is possible to go beyond, and using the Law of the Iterated Logarithm, we can improve the preceding result introducing an optimal error term.

Proposition 18. *There exists a set of measure 1 such that:*

i. *for the system* $\boldsymbol{A}$ *there exists the limit*

$$\frac{\sum_{i=1}^{n} \alpha_i}{n} = 2r - \frac{5}{2}r^2 + O\left(\frac{\sqrt{\ln \ln n}}{\sqrt{n}}\right);$$

ii. *for the system* $\boldsymbol{B}$ *there exists the limit*

$$\frac{\sum_{i=1}^{n} a_i}{n} = 2r - 3r^2 + O\left(\frac{\sqrt{\ln \ln n}}{\sqrt{n}}\right).$$

Proposition 19. *There exists a set of measure 1 such that:*

i. *for the system* $\boldsymbol{A}$ *there exists the limit*

$$\lim_n \lambda\left(a < 2\frac{\sum_{i=1}^n \alpha_i - n\left(2r - \frac{5}{2}r^2\right)}{\sqrt{n\left(4r - 18r^2 + 35r^3 - 25r^4\right)}} < b\right) = \frac{1}{\sqrt{2\pi}}\int_a^b e^{-\frac{x^2}{2}}\,dx;$$

ii. *for the system* $\boldsymbol{B}$ *there exists the limit*

$$\lim_n \lambda\left(a < \frac{\sum_{i=1}^n \alpha_i - n\left(2r - 3r^2\right)}{\sqrt{n\left(\left(r - 5r^2 + 11r^3 - 9r^4\right)\right)}} < b\right) = \frac{1}{\sqrt{2\pi}}\int_a^b e^{-\frac{x^2}{2}}\,dx.$$

Proof. They are consequences of the Central Limit Theorem under the assumption of the Lindeberg condition (see [1, sec. 27]).

For further considerations, we need the following useful two results: □

Lemma 20. *Let us suppose that a function f has (finite) derivative on a point x. Then*

$$\lim_{u\to x^-, v\to x^+} \frac{f(u) - f(v)}{u - v} = f'(x).$$

Proof. See [1, p. 404]. □

Theorem 21. *Monotone functions (on real intervals) have derivatives almost everywhere.*

Proof. See, for example, [1, th. 31.2] or [19, p. 11]. □

Now, let $r, r' \in \,]0, 1/2[$. Let us consider the system of representation $\boldsymbol{A}$ (with parameter r). Let us define a function h_1 in the following way: if a point x has representation

$$x = \alpha_0 + \delta_1\alpha_1 + \delta_2\alpha_2 + \cdots + \delta_i\alpha_i + \cdots,$$

in the system $\boldsymbol{A}$ (with parameter r), then

$$h_1(x) = \alpha'_0 + \delta'_1\alpha'_1 + \delta'_2\alpha'_2 + \cdots + \delta'_i\alpha'_i + \cdots,$$

where the prime symbol means that we have substituted r for r'.

Proposition 22. *The function h_1 is well defined (i.e., it does not depend on the selected representation for those points with double representation). Moreover, it is a continuous and strictly monotone increasing function with null derivatives on a set of measure 1 (i.e., it is singular).*

Proof. In case that the point has one and only one representation, is easy to prove that h_1 is well-defined, continuous, and monotone. If the point x has two representations, then we divide in two subcases: $y \to x^-$ or $y \to x^+$.

The proposition 15 ensures that there exists a set, say P_r, where the variables α_i are in the same proportions. If $x \in P_r$, then

$$x = \alpha_0 + \delta_1\alpha_1 + \delta_2\alpha_2 + \cdots + \delta_i\alpha_i + \cdots;$$

and we consider numbers

$$x_n = \alpha_0 + \delta_1\alpha_1 + \delta_2\alpha_2 + \cdots + \delta_{n-1}\alpha_{n-1}$$

and

$$x_n^* = \alpha_0 + \delta_1\alpha_1 + \delta_2\alpha_2 + \cdots + \delta_{n-1}\alpha_{n-1} + \delta_n.$$

Hence, $x_n \le x \le x_n^*$; and

$$\begin{aligned} h_1(x) &= \alpha_0' + \delta_1'\alpha_1' + \delta_2'\alpha_2' + \cdots + \delta_i'\alpha_i' + \cdots \\ h_1(x_n) &= \alpha_0' + \delta_1'\alpha_1' + \delta_2'\alpha_2' + \cdots + \delta_{n-1}'\alpha_{n-1}' \\ h_1(x_n^*) &= \alpha_0' + \delta_1'\alpha_1' + \delta_2'\alpha_2' + \cdots + \delta_{n-1}'\alpha_{n-1}' + \delta_n'. \end{aligned}$$

Now, by the proportions in the proposition 15, we have:

$$0 \le \frac{h_1(x_n^*) - h_1(x_n)}{x_n^* - x_n} = \frac{\delta_n'}{\delta_n} = \left(\frac{(r/2)^{r+o(1)} (1-2r)^{(1-2r)+o(1)}}{(r'/2)^{r+o(1)} (1-2r')^{(1-2r)+o(1)}} \right)^n .$$

If $r \neq r'$, then $\frac{(r/2)^r (1-2r)^{(1-2r)}}{(r'/2)^r (1-2r')^{(1-2r)}} < 1$. Therefore, there exists the limit

$$\lim_{n\to\infty} \frac{h_1(x_n^*) - h_1(x_n)}{x_n^* - x_n} = 0.$$

The lemma 20 implies that in the case that there exists a derivative at a point in P_r, it is zero. On the other hand, the theorem 21, ensures that h_1 has derivatives on a set of measure 1. Therefore, the subset of points in P_r where there exist derivatives, is of measure 1, and we conclude that h_1 has null derivatives on a set of measure 1. □

Proposition 23. *The function h_1 does not admit non-null derivatives.*

We enunciate a Frostman-type lemma (for a proof, see [7, pp. 60-61]) that is useful for our final results.

Lemma 24. *Let μ be a finite measure on the Euclidean space $\mathbb{R}^N$, let F be a Borel set, and let $c \in \mathbb{R}^+$.*

- **i.** *If $\limsup_{r\to 0} \frac{\mu(B_r(x))}{r^s} < c$ for all $x \in F$, then $\dim_{\mathcal{H}} F \ge s$.*
- **ii.** *If $\liminf_{r\to 0} \frac{\mu(B_r(x))}{r^s} > c$ for all $x \in F$, then $\dim_{\mathcal{H}} F \le s$.*

This lemma will be used with intervals instead of balls, but the proof already cited works in the same way.

Proposition 25. *The function h_1 maps a null set with fractal dimension*

$$\frac{2r \ln \frac{r'}{2} + (1-2r)\ln(1-2r')}{2r \ln \frac{r}{2} + (1-2r)\ln(1-2r)}$$

onto another of measure 1.

Proof. If $x = \alpha_0 + \delta_1\alpha_1 + \delta_2\alpha_2 + \cdots + \delta_i\alpha_i + \cdots$, we then consider the interval $[x_n, x_n^*]$ (the points x_n and x_n^* are defined by the proposition 22); and the corresponding image under h_1 is the interval $[h_1(x_n), h_1(x_n^*)]$. The respective lengths of both intervals are given by the numbers δ_n and δ_n'.

Therefore, the Hausdorff dimension of the set of these points is given by the number

$$\sup\left\{\beta>0:\lim_{n\to\infty}\frac{\delta_n'}{\delta_n^{\beta}}=\lim_{n\to\infty}\frac{(r/2)^{rn+o(n)}\,(1-2r)^{(1-2r)n+o(n)}}{\left((r'/2)^{rn+o(n)}\,(1-2r')^{(1-2r)n+o(n)}\right)^{\beta}}<+\infty\right\}.$$

Taking logs, because it must be finite:

$$\lim_{n\to+\infty}\frac{\ln\delta_n'-\beta\ln\delta_n}{n}n<+\infty;$$

and this gives rise to $\frac{2r\ln\frac{r'}{2}+(1-2r)\ln(1-2r')}{2r\ln\frac{r}{2}+(1-2r)\ln(1-2r)}$, the desiderated dimension. □

Proposition 26. *The function h_1 maps a set of measure 1 onto another of null measure and fractal dimension*

$$\frac{2r'\ln\frac{r}{2}+(1-2r')\ln(1-2r)}{2r'\ln\frac{r'}{2}+(1-2r')\ln(1-2r')}.$$

If we use the representation system $\boldsymbol{B}$ (instead of $\boldsymbol{A}$), then we will obtain, as we have above, a new function h_2 with the same properties as h_1. However, in this case, the Hausdorff dimensions of the sets obtained are $\frac{2r\ln r'+(1-2r)\ln(1-2r')}{2r\ln r+(1-2r)\ln(1-2r)}$ and $\frac{2r'\ln r+(1-2r')\ln(1-2r)}{2r'\ln r'+(1-2r')\ln(1-2r')}$, respectively.

Acknwoledgement:. We are greatful to the anonymous referee for his or her careful reading and useful suggestions that improved the paper.

References

[1] Billingsley, P.: Probability and Measure (3rd ed.), John Wiley & Sons, Inc., 1995

[2] Dajani, K.; and Kraaikamp, C.: Ergodic Theory of Numbers, The Mathematical Association of America Monographs **29**, Washington, 2002

[3] Durante, F.; Klement, E.P.; Quesada-Molina, J.J.; and Sarkoci, P.: *Remarks on two product-like constructions for copulas*, Kybernetika (Prague) **43** (2007) 235-244

[4] Durante, F.; Sarkoci, P.; and Sempi, C.: *Shuffles of copulas*, J. Math. An. Appl. **352** (2009) 914-921

[5] Edgar, G.A.: Measure, Topology, and Fractal Geometry. Undergraduate Texts in Mathematics, Springer. Berlin, Heidelberg, New York, 1990

[6] Edgar, G.A.: Integral, Probability and Fractal Measures, Springer Verlag, 1998

[7] Falconer, K.J.: Fractal Geometry: Mathematical Foundations and Applications (2nd Ed.), Wiley, 1990

[8] Feller, Introduction to Probability Theory and Its Applications (3rd ed), John Wiley & Sons, New York, 1968

[9] Fredricks, G.A.; Nelsen, R.B; and Rodríguez-Lallena, J.A.: *Copulas with fractal supports*, Insurance Math. Econom. **37** (1) (2005) 42-48

[10] Halmos, P.R.: Ergodic Theory, Chelsea, New York, 1956

[11] Hilbert, D.: *Ueber die stetige Abbildung einer Line auf ein Flächenstück*, Mathematische Annalen **38** (1891) 459-460

[12] Kolesárová, A.; Mesiar, R; and Sempi, C.: *Measure-preserving transformations, copula and compatibility*, Mediterr. J. Math. **5** (2008) 325-339

[13] McClure, M.: *The Hausdorff dimension of Hilbert's coordinate functions*, The Real Analysis Exchange, **24** (1998/99) 875-884

[14] Mikusiński, P.; Sherwood, H.; and Taylor, M.D.: *Shuffles of Min*, Stochastica **12** (1992) 61-72

[15] Nelsen, R.B.: An introduction to copulas (2nd ed.), Springer Series in Statistics, New York, 2006

[16] Olsen, E.T.; Darsow, W.F.; and Nguyen, B.: *Copulas and Markov operators*, in: Distributions with Fixed Marginals and Related Topics, Seattle, WA, 1993, in: IMS Lecture Notes Monogr. Ser. **28** Inst. Math. Statist., Hayward, CA (1996) 244-259

[17] Peano, G.: *Sur une courbe, qui remplit toute une aire plane*, Mathematische Annalen **36** (1) (1890) 157-160

[18] Pollicott, M.; and Yuri, M.: *Dynamical systems and ergodic theory*, London Mathematical Society Student Texts, 40, Cambridge University Press, Cambridge, 1998

[19] Riesz, F.; and Sz-Nagy, B.: Leçons d'analyse fonctionnelle, Gauthier-Villars, Paris, 1968

[20] Rudin, W.: Real and complex analysis (3rd ed.), McGraw-Hill, New York, 1966

[21] Sagan, H.: Space-Filling Curves, Springer-Verlag, New York, 1994

[22] Sklar, A.: *Fonctions de répartition à n dimensions et leurs marges*, Publ. Inst. Statidt. Univ. Paris **8** (1959) 229-231

[23] Urbański, M.: *The Hausdorff dimension of the graphs of continuous self-affine functions*, Proc. Amer. Math. Soc. **108** (4) (1990) 921-930

[24] Vitale, R.A.: *Parametrizing doubly stochastic measures*, in Distributions with Fixed Marginals and Related Topics, Seattle, WA, 1993, in: IMS Lecture Notes Monogr. Ser. **28**, Inst. Math. Statist., Hayward, CA (1996) 358-364

[25] Walters, P.: An Introduction to Ergodic Theory,Springer-Verlag, New York, Heidelberg, Berlin, 1980

Enrique de Amo
Departamento de Álgebra y Análisis Matemático
Universidad de Almería
La Cañada de San Urbano
04120-Almería
Spain
e-mail: `edeamo@ual.es`

Manuel Díaz Carrillo
Departamento de Análisis Matemático
Universidad de Granada
Campus de Fuente Nueva s/n
18071-Granada
Spain
e-mail: madiaz@ugr.es

Juan Fernández-Sánchez
Departamento de Álgebra y Análisis Matemático
Universidad de Almería
La Cañada de San Urbano
04120-Almería
Spain
e-mail: jfsjufesa@gmail.com

Received: December 9, 2009.
Revised: February 23, 2010.
Accepted: March 29, 2010.