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FINITELY ADDITIVE INTEGRATION AND LOCAL INTEGRATION WITH RESPECT TO UPPER INTEGRALS

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Abstract. We study the integration theory for general integral metrics when restricted to upper integrals q, finding improvements in the relation between the classes of the q-integrable and the q_l-integrable functions. We give new results and notions which lead to the desirable characterizations of q-integrable functions as q_l-integrable f with $q(|f|) < \infty$, and of q_l-integrable functions via the integrability of their upper truncations, under natural conditions which are fulfilled in most finitely additive integration theories.

1. Introduction

It is well known the general integration theory for integral metrics q, developed by Aumann and Schäfke in [1] and [11]–[14].

With the aid of this theory, Díaz Carrillo and Günzler gave in [6] an extension process for an arbitrary Loomis system (X, B, I), without monotone continuity assumptions on the elementary integral I, and proved convergence theorems using a suitable local mean convergence which can be traced back to Loomis in [10].

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Moreover, a unified treatment of proper Riemann, Riemann-μ, abstract Riemann–Loomis, Daniell and Bourbaki integrals was given (doing proper choices of q). For a recent account of extension procedures we refer the reader to [4].

However, in many of such applications q is in fact an upper integral. Thus, we wish to investigate the benefits that can be obtained in general theory if we start from upper integrals instead of integral metrics.

Section 2 presents some preliminaries where we set up notation and terminology to be used throughout the paper.

In Section 3 we collect the most important results in integration theory with respect to upper integrals, giving natural conditions under which the q-integrable functions can be described as the class of q-regular functions.

Section 4 is devoted to the study of local integration. A partial subadditivity for the localized upper integral q_l is obtained (Proposition 4.6) and it is used to prove that $q_l(f) \leqq (q_l)_*(f)$ for all f q_l-integrable (under regularity) in Theorem 4.10.

In Section 5 we introduce and study the notions of upper and lower strong measurability. These new concepts will prove extremely useful to our purposes.

In Section 6 our main results (Corollary 6.2 and Theorem 6.5) are stated and proved. The basic idea is to apply the results of Sections 3 and 4 in order to relate strong measurability for q and q_l (see Theorem 6.1).

As an application of the developed theory we conclude in Section 7 by giving a unified treatment of the finitely additive integral extension theories. It makes evident that our viewpoint sheds some new light on the way of obtaining general proofs for facts which were known only in special cases.

2. General framework. Preliminaries

We extend the addition $+$ on $\mathbb{R}$ to $+$, $\dot{+}$ and $\underset{\cdot}{+}$ on $\overline{\mathbb{R}}$ by

$$a + b := 0, \quad a \dot{+} b := \infty, \quad a \underset{\cdot}{+} b := -\infty, \quad \text{if} \quad a = -b \in \{-\infty, \infty\}.$$

We also note

$$a - b := a + (-b),\ a \dot{-} b := a \dot{+} (-b),\ a \underset{\cdot}{-} b := a \underset{\cdot}{+} (-b),\ \forall a, b \in \overline{\mathbb{R}}.$$

The laws $+$, $\dot{+}$ and $\underset{\cdot}{+}$ are commutative, $+$ is distributive with $0 \cdot (\pm\infty) := 0$, but not associative, and $\dot{+}$ is associative.

The order $\leqq$ and absolute value $|\cdot|$ can also be extended to $\overline{\mathbb{R}}$ by an obvious way and, setting $a \wedge b := \inf\{a, b\}$, $a \vee b := \sup\{a, b\}$, $a^+ := a \vee 0$, $a^- := (-a) \vee 0$, $\forall a, b \in \overline{\mathbb{R}}$ we still have the Birkhoff inequalities

$$|a \wedge c - b \wedge c| \leqq |a - b|, \quad |a \vee c - b \vee c| \leqq |a - b|, \quad \forall a, b, c \in \overline{\mathbb{R}}$$

and these other properties which will be used without a further explicit reference:

$$a \leqq |a|, \quad |a| \leqq c \;\Leftrightarrow\; -c \leqq a \leqq c, \quad \forall a, b, c \in \overline{\mathbb{R}},\; c \geqq 0$$

$$a \leqq b \dotplus (a - b), \quad \forall a, b \in \overline{\mathbb{R}}$$

$$a \leqq b \dotplus c \;\Rightarrow\; a - b \leqq c, \quad \forall a, b, c \in \overline{\mathbb{R}},\; c \geqq 0$$

$$(a - b) \wedge c = \big[a \wedge (c + b)\big] - b, \quad \forall a, b, c \in \,]-\infty, \infty], \quad c \geqq 0$$

$$(a \dotplus b) \wedge c \leqq \big[a \wedge (c - b)\big] \dotplus b, \quad \forall a, b, c \in \overline{\mathbb{R}}$$

$$\big||a| - |b|\big| \leqq |a - b| \leqq \; |a - c| + |c - b|, \quad \forall a, b, c \in \overline{\mathbb{R}}.$$

For a nonempty set X, $\overline{\mathbb{R}}^X$ denotes the class of all $\overline{\mathbb{R}}$-valued functions defined on X. All operations and relations between functions are defined pointwise.

Given $M \subseteqq \overline{\mathbb{R}}^X$, let $+M := \{f \in M : f \geqq 0\}$. M is said to be a *vector lattice* if it is a real linear space under pointwise $+$, $\alpha\,\cdot$, such that $f \wedge g$, $f \vee g \in M$ for all $f, g \in M$; then $f \in M$ implies $|f| \in M$.

For each $\emptyset \neq A \subseteqq \overline{\mathbb{R}}$, $\inf A, \sup A \in \overline{\mathbb{R}}$, where we use $\inf \emptyset := +\infty$ and $\sup \emptyset = -\infty$.

3. Integration with respect to upper integrals

3.1. *Upper integrals.* We begin by introducing the abstract concept of upper integral, which will be the basic tool in all this paper.

Definition 3.1. A mapping $q : \overline{\mathbb{R}}^X \to \overline{\mathbb{R}}$ is said to be an *upper integral* if

(1) $q(0) = 0$,

(2) $q(f \dotplus g) \leqq q(f) \dotplus q(g) \;\; \forall f, g \in \overline{\mathbb{R}}^X$,

(3) $q(f) \leqq q(g) \;\; \forall f, g \in \overline{\mathbb{R}}^X$ with $f \leqq g$.

We also recall the notion of integral metric in the sense of Schäfke. Actually, this is a simplified version of Schäfke's definition [14, p. 120].

Definition 3.2. A mapping $q : +\overline{\mathbb{R}}^X \to +\overline{\mathbb{R}}$ is said to be an *integral metric* if

(1) $q(0) = 0$,

(2) $q(f) \leqq q(g) + q(h) \;\; \forall f, g, h \in +\overline{\mathbb{R}}^X$, $f \leqq g + h$.

We extend this definition to a mapping $q : \overline{\mathbb{R}}^X \to \overline{\mathbb{R}}$ saying that q is an *integral metric* if $q_{|+\overline{\mathbb{R}}^X}$ is an integral metric as above.

It is clear that if q is an upper integral then q is an integral metric, and for each integral metric q, $d_q : \overline{\mathbb{R}}^X \times \overline{\mathbb{R}}^X \to \overline{\mathbb{R}}$ given by $d_q(f,g) := q\big(|f-g|\big)$ is a distance on $\overline{\mathbb{R}}^X$, so one can do the next

DEFINITION 3.3. A function $f \in \overline{\mathbb{R}}^X$ is called *q-M-integrable* if it belongs to the closure of M in $\overline{\mathbb{R}}^X$ with respect to the distance d_q, i.e.

$$\forall \varepsilon > 0, \quad \exists h \in M : q\big(|f-h|\big) < \varepsilon.$$

The class of all q-M-integrable functions is denoted by M^q.

LEMMA 3.4. *If q is an integral metric, $0 \in M \subseteqq \overline{\mathbb{R}}^X$ and M is closed with respect to $*$, being $* \in \{+, -, \wedge, \vee, |\cdot|, \alpha\cdot$ with $\alpha \in \mathbb{R}\}$, then M^q is also closed with respect to $*$.*

We give now the basic property of the upper integrals and its principal consequences.

PROPOSITION 3.5. *If q is an upper integral then*

$$\big|q(f) - q(g)\big| \leqq q\big(|f-g|\big), \quad \forall f, g \in \overline{\mathbb{R}}^X.$$

PROOF. From $f \leqq g \dotplus (f-g)$ we deduce that

$$q(f) \leqq q(g) \dotplus q(f-g) \leqq q(g) \dotplus q\big(|f-g|\big)$$

and so $q(f) - q(g) \leqq q\big(|f-g|\big)$. Interchanging f and g gives $q(g) - q(f) \leqq q\big(|g-f|\big)$. Hence

$$-q\big(|f-g|\big) \leqq q(f) - q(g) \leqq q\big(|g-f|\big),$$

i.e. $\big|q(f) - q(g)\big| \leqq q\big(|f-g|\big)$. $\square$

REMARK 3.6. Notice that all properties (1)–(3) of Definition 3.1 have been used to prove this one.

COROLLARY 3.7. *Every upper integral q is q-continuous on $\overline{\mathbb{R}}^X$, i.e.*

$$q\big(|f-f_n|\big) \to 0 \;\Rightarrow\; q(f_n) \to q(f), \quad \forall f, f_n \in \overline{\mathbb{R}}^X.$$

COROLLARY 3.8. *Let be q an upper integral finite on $M \subseteqq \overline{\mathbb{R}}^X$. Then q is finite on M^q. Moreover, for each $f \in M^q$, and each sequence $\{h_n\}$ in M such as $q\big(|f-h_n|\big) \to 0$ (we say that $\{h_n\}$* defines *f in M^q) we have*

$$q(f) = \lim_{n\to\infty} q(h_n) \in \mathbb{R}.$$

3.2. *Regularity and bidetermination.* We are now concerned with additional conditions on q (fulfilled in most applications) yielding good properties on integration.

Definition 3.9. Given $q : \overline{\mathbb{R}}^X \to \overline{\mathbb{R}}$, we define q_* by

$$q_*(f) := -q(-f), \quad \forall f \in \overline{\mathbb{R}}^X.$$

If q is an upper integral then q_* is called the *lower integral* of q.

Lemma 3.10. *If q is an upper integral, then q_* satisfies:*

(i) $q_* \leqq q$ *on* $\overline{\mathbb{R}}^X$,

(ii) $q_*(f \dot{+} g) \geqq q_*(f) \dot{+} q_*(g) \quad \forall f, g \in \overline{\mathbb{R}}^X$,

(iii) $q_*(f) \leqq q_*(g) \quad \forall f, g \in \overline{\mathbb{R}}^X$ *with* $f \leqq g$.

Proof. From

$$0 = q(0) = q\big(f + (-f)\big) \leqq q\big(f \dot{+} (-f)\big) \leqq q(f) \dot{+} q(-f)$$

one deduces that $q_*(f) \leqq q(f)$. The rest is left to the reader. □

Remark 3.11. Notice that, once again, it is needed to use all properties (1)–(3) of Definition 3.1 to prove (i).

Definition 3.12. Let $q : \overline{\mathbb{R}}^X \to \overline{\mathbb{R}}$ and $f \in \overline{\mathbb{R}}^X$. We say that f is *q-regular* if $q_*(f) = q(f) \in \mathbb{R}$.

The class of all q-regular functions on X is denoted by $\operatorname{Reg}(q)$, i.e.

$$\operatorname{Reg}(q) := \left\{ f \in \overline{\mathbb{R}}^X : q_*(f) = q(f) \in \mathbb{R} \right\}.$$

It follows directly from the properties of q and q_* (Definition 3.1 and Lemma 3.10) that the class $\operatorname{Reg}(q)$ has a good behaviour with respect to addition $\dot{+}$ when q is an upper integral; exactly:

Proposition 3.13. *If q is an upper integral and $f, g \in \operatorname{Reg}(q)$ then $f \dot{+} g \in \operatorname{Reg}(q)$ and $q(f \dot{+} g) = q(f) + q(g)$.*

Definition 3.14. Given $M \subseteqq \overline{\mathbb{R}}^X$ and $q : \overline{\mathbb{R}}^X \to \overline{\mathbb{R}}$, we say that q is *M-regular* if $q(h) = q_*(h) \in \mathbb{R}, \ \forall h \in M$, i.e. if $M \subseteqq \operatorname{Reg}(q)$.

The regularity of q on M can be extended to M^q for every upper integral, as shown by the next proposition.

Proposition 3.15. *If q is an M-regular upper integral then q is M^q-regular; that is, $M^q \subseteqq \operatorname{Reg}(q)$.*

Proof. Let $g \in M^q$. There exists $\{h_n\}$ in M such that $\{h_n\}$ defines g in M^q, and so $q\big(\big|(-f) - (-h_n)\big|\big) \to 0$. Therefore, using Corollaries 3.7 and 3.8 and the M-regularity of q, we have

$$q_*(g) = -\lim_{n\to\infty} q(-h_n) = \lim_{n\to\infty} q_*(h_n) = \lim_{n\to\infty} q(h_n) = q(g) \in \mathbb{R}. \quad \square$$

Simple examples (for instance Example 7.5 below) make evident that this proposition becomes false without M-regularity.

Combining the two previous propositions we obtain the $\dot{+}$-additivity of q on M^q.

COROLLARY 3.16. *If q is an M-regular upper integral and $f, g \in M^q$, then $f \dot{+} g \in M^q$ and $q(f \dot{+} g) = q(f) + q(g)$.*

The opposite relation between the classes M^q and $\operatorname{Reg}(q)$ depends on the following property of q.

DEFINITION 3.17. $q : \overline{\mathbb{R}}^X \to \overline{\mathbb{R}}$ is said to be *determined by* $M \subseteqq \overline{\mathbb{R}}^X$ if

$$q(f) = \inf \left\{ q(h) : f \leqq h \in M \right\}, \quad \forall f \in \overline{\mathbb{R}}^X.$$

PROPOSITION 3.18. *If q is determined by M and $f \in \overline{\mathbb{R}}^X$ with $q(f) = q_*(f) \in \mathbb{R}$ then $f \in M^q$.*

PROOF. Since $q(f) = \inf \left\{ q(h) : f \leqq h \in M \right\} \in \mathbb{R}$, given $\varepsilon > 0$, there exists $h \in M$, $f \leqq h$ such that $q(f) + \varepsilon > q(h)$, and we have

$$q\big(|f - h|\big) = q(h - f) \leqq q(h) + q(-f) = q(h) - q_*(f) = q(h) - q(f) < \varepsilon$$

which gives $f \in M^q$. $\square$

Example 7.6 shows that this is no longer true if the determination by M is dropped.

Therefore, for M-regular upper integrals q determined by M, the classical characterization of q-M-integrability in terms of the equality of upper and lower integrals, follows by combining the preceding results, i.e.

COROLLARY 3.19. *If q is an M-regular upper integral determined by M then*

$$M^q = \left\{ f \in \overline{\mathbb{R}}^X : q_*(f) = q(f) \in \mathbb{R} \right\}.$$

We have generalized this result to the case of an upper integral determined by a class M' bigger than the class M where it is regular.

DEFINITION 3.20. Given $M, M' \subseteqq \overline{\mathbb{R}}^X$ with $M \subseteqq M'$ and $q : \overline{\mathbb{R}}^X \to \overline{\mathbb{R}}$, q is said to be *bidetermined by* (M, M') if q is determined by M', $q_{*|-M'}$ is determined by M and $q > -\infty$ on M'.

Note that, under M-regularity and with $-M \subseteqq M$, to be bidetermined by (M, M) is equivalent to be determined by M.

THEOREM 3.21. *If q is an M-regular upper integral bidetermined by (M, M') and $-M \subseteqq M$ then*

$$M^q = \left\{ f \in \overline{\mathbb{R}}^X : q_*(f) = q(f) \in \mathbb{R} \right\}.$$

PROOF. $M^q \subseteqq \operatorname{Reg}(q)$ by Proposition 3.15. To prove the opposite inclusion, let $f \in \overline{\mathbb{R}}^X$ with $q_*(f) = q(f) \in \mathbb{R}$ and let $\varepsilon > 0$. From $q(f) = \inf\{q(h) : f \leqq h \in M'\} \in \mathbb{R}$, we deduce that there exists $h \in M'$, $f \leqq h$ such that $q(h) < q(f) + \frac{\varepsilon}{2}$. Since $q_{*|-M'}$ is determined by M and $-M \subseteqq M$, an easy computation shows that $q(h) = \sup\{q(u) : u \leqq h, u \in M\}$. Clearly $q(h) \in \mathbb{R}$ and consequently there exists $u \in M$, $u \leqq h$ such that $q(h) - \frac{\varepsilon}{2} < q(u)$. Thus, we have

$$\begin{aligned} q\big(|f-u|\big) &\leqq q\big(|f-h| + |h-u|\big) \leqq q\big(|f-h|\big) + q\big(|h-u|\big) \\ &\leqq q(h-f) + q(h-u) \leqq q(h) - q_*(f) + q(h) - q_*(u) \\ &= q(h) - q(f) + q(h) - q(u) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \square \end{aligned}$$

Taking $M' = M$ with $-M \subseteqq M$ we reobtain Corollary 3.19. This generalized version will be needed for some applications.

Thus, regularity and bidetermination turn out to be natural conditions for the integration with respect to upper integrals, because under such conditions integrability can be decided by an intrinsic criterion for this type of functionals: to check that upper and lower integrals are equal and finite.

4. Local integration with respect to upper integrals

DEFINITION 4.1. Let q be a functional defined on $\overline{\mathbb{R}}^X$. We define the functional q_l on $\overline{\mathbb{R}}^X$ by the formula

$$q_l(f) := \sup\{q(f \wedge h) : h \in +M\}, \quad \forall f \in \overline{\mathbb{R}}^X$$

and, when q is an upper integral, q_l is called the *localized integral* of q.

LEMMA 4.2. *If q is an upper integral then q_l is an integral metric.*

PROOF. We only note that the inequality

$$(f+g) \wedge h \leqq f \wedge h + g \wedge h$$

is valid for $f, g \in +\overline{\mathbb{R}}^X$ and $h \in +M$. $\square$

DEFINITION 4.3. The class of all *q_l-M-integrable functions* with respect to the distance $d_{q_l}(f,g) := q_l\big(|f-g|\big)$ on $\overline{\mathbb{R}}^X$ is denoted by M^{q_l}.

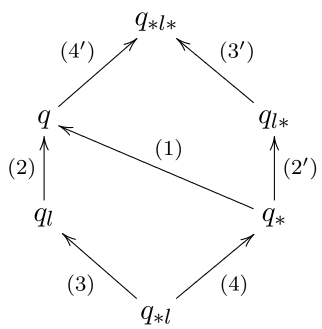
We collect some elementary properties of q_l that we will use without explicit reference.

PROPOSITION 4.4. *If q is a nondecreasing functional, then*
(i) $q_l \leqq q$. *In particular,* $M^q \subseteqq M^{q_l}$.
(ii) *If* $\exists h \in +M$, $f \leqq h$ *then* $q_l(f) = q(f)$.
(iii) *If* $|M| \subseteqq M$ *and* $\exists h \in M$, $f \leqq h$ *then* $q_l(f) = q(f)$.

It is clear that $q_{**} = q$ and $q_{ll} = q_l$, but we can also consider the functionals given by iterated applications of the operations "$_*$" and "$_l$" on q: $q_{l*}, q_{*l}, q_{*l*}, \ldots$. For instance,

$$q_{l*}(f) := (q_l)_*(f) = -q_l(-f) = \inf\left\{q_*\big(f \vee (-k)\big) : k \in +M\right\}, \quad \forall f \in \overline{\mathbb{R}}^X.$$

PROPOSITION 4.5. *Let q be an upper integral. The general relations among the functionals q, q_l, q_*, q_{l*}, q_{*l} and q_{*l*} are given by the following diagram (arrows mean $\leqq$):*



PROOF. (1) and (2) are already known. (3) follows from $q_*(f \wedge h) \leqq q(f \wedge h)$, $\forall h \in +M$ and (4) from $q_*(f \wedge h) \leqq q_*(f)$, $\forall h \in +M$, both taking suprema. (2′), (3′) and (4′) are obtained by applying $*$ to (2), (3) and (4) respectively. $\square$

We cannot expect as good behavior to local integration as to integration with respect to upper integrals, because the process of localization does not preserve upper integrals.

However, we have proved that localization of upper integrals preserves some amount of additivity, which will be the key to find the relationship between the classes M^q and M^{q_l}; exactly:

PROPOSITION 4.6. *Let q be an upper integral and M be a vector lattice. Then*

$$q_l(f \dot{+} h) \leqq q_l(f) \dot{+} q_l(h), \quad \forall f \in \overline{\mathbb{R}}^X, \quad \forall h \in M.$$

PROOF. If $q_l(f \dot{+} h) = -\infty$ then the inequality is trivial.

If $q_l(f \dot{+} h) \in \mathbb{R}$ then, given $\varepsilon > 0$, there exists $k \in +M$ such that

$$q_l(f \dot{+} h) - \varepsilon \leqq q\big((f \dot{+} h) \wedge k\big) \leqq q(f \wedge (k - h) \dot{+} h)$$

$$\leqq q\big(f \wedge (k-h)\big) \dotplus q(h) \leqq q\big(f \wedge |k-h|\big) \dotplus q(h) \leqq q_l(f) \dotplus q_l(h).$$

Since $\varepsilon > 0$ is arbitrary, it follows that $q_l(f \dotplus h) \leqq q_l(f) \dotplus q_l(h)$. Finally, if $q_l(f \dotplus h) = \infty$ then, given $\alpha \in \mathbb{R}$ there exists $k \in +M$ such that $\alpha \leqq q\big((f \dotplus h) \wedge k\big)$ and, as above, we have $\alpha \leqq q_l(f) \dotplus q_l(h)$ so $q_l(f) \dotplus q_l(h) = \infty$. □

COROLLARY 4.7. *Let M be a vector lattice and q be an M-regular upper integral. Then*

$$q_l(f \dotplus h) = q_l(f) + q_l(h), \quad \forall f \in \overline{\mathbb{R}}^X, \quad \forall h \in M.$$

PROOF. The reverse inequality uses the one just proved:

$$q_l(f) \leqq q_l\big(\big(f-(-h)\big) \dotplus (-h)\big) \leqq q_l(f+h) \dotplus q_l(-h)$$

$$\leqq q_l(f+h) \dotplus q(-h) \leqq q_l(f+h) - q(h)$$

where $q(h) = -q(-h) \in \mathbb{R}$. □

As for the partial subadditivity of q_l let us consider the following property, which is the best approximation to Proposition 3.5 that we get for q_l:

COROLLARY 4.8. *If q is an upper integral and M is a vector lattice then*

$$q_l(f) - q(h) \leqq q_l\big(|f-h|\big), \quad \forall f \in \overline{\mathbb{R}}^X, \quad \forall h \in M.$$

PROOF. It follows immediately from

$$q_l(f) \leqq q_l\big((f-h) \dotplus h\big) \leqq q_l(f-h) \dotplus q_l(h) \leqq q_l\big(|f-h|\big) \dotplus q(h). \quad \square$$

We will now use this property to prove that $q_l(f) \leqq (q_l)_*(f)$ for all f in M^{q_l} when q is an M-regular upper integral. The following lemma is needed.

LEMMA 4.9. *Let q be an upper integral. Assume that M is a vector lattice, q is finite on M, $f \in M^{q_l}$ and $\{h_n\}$ defines f in M^{q_l}. Then $\big\{q(h_n)\big\}$ is a Cauchy sequence in $\mathbb{R}$.*

PROOF. Since $\{h_n\}$ defines f, given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\forall n \geqq n_0, \quad q_l\big(|f-h_n|\big) < \varepsilon.$$

Thus, for $n, m \geqq n_0$ we have, by Proposition 3.5

$$\big|q(h_n) - q(h_m)\big| \leqq q\big(|h_n - h_m|\big) = q_l\big(|h_n - h_m|\big)$$

$$\leqq q_l\big(|h_n - f|\big) + q_l\big(|f-h_n|\big) < \varepsilon. \quad \square$$

THEOREM 4.10. *Let M be a vector lattice and q be an M-regular upper integral. If $f \in M^{q_l}$ and $\{h_n\}$ defines f in M^{q_l} then*

$$q_l(f) \leqq \lim_{n\to\infty} q(h_n) \leqq (q_l)_*(f).$$

In particular

$$q_l(f) \leqq (q_l)_*(f), \quad \forall f \in M^{q_l}.$$

PROOF. Let $f \in \overline{\mathbb{R}}^X$ and $\{h_n\}$ in $+M$ such that $q_l\big(|f-h_n|\big) \to 0$. Since $\big\{q(h_n)\big\}$ is a Cauchy sequence in $\mathbb{R}$ (Lemma 4.9), there exists $\alpha \in \mathbb{R}$ such that $q(h_n) \to \alpha$. By Corollary 4.8 we have

$$q_l(f) - q(h_n) \leqq q_l\big(|f-h_n|\big), \quad \forall n \in \mathbb{N}$$

and, letting $n \to \infty$ yields $q_l(f) - \alpha \leqq 0$, that is, $q_l(f) \leqq \alpha$. Since $q(-h_n) = -q(h_n) \to -\alpha$ (q is M-regular), by using once again Corollary 4.8, we have

$$q_l(-f) - q(-h_n) \leqq q_l\big(|h_n - f|\big), \quad \forall n \in \mathbb{N}$$

and we now deduce that $q_l(-f) + \alpha \leqq 0$, and so $\alpha \leqq (q_l)_*(f)$. Hence $q_l(f) \leqq \alpha \leqq (q_l)_*(f)$. □

What can be said for the localized integral q_l when it is also assumed that q is bidetermined by (M, M')? The next proposition addresses this question for the particular case of determination by M. It gives conditions that make equality for $(2')$ in the diagram of Proposition 4.5.

PROPOSITION 4.11. *If q is an upper integral determined by M, $|M| \subseteqq M$ and $f \in \overline{\mathbb{R}}^X$ with $q_*(f) > -\infty$ then $(q_l)_*(f) = q_*(f)$.*

PROOF. Since $q(-f) = -q_*(f)$, that is, $\inf\big\{q(h) : -f \leqq h \in M\big\} = -q_*(f) < \infty$, there exists $h \in M$ such that $-f \leqq h$. Therefore $q_l(-f) = q(-f)$, i.e., $(q_l)_*(f) = q_*(f)$. □

In order to generalize this proposition to the case of bidetermined upper integrals we need the following fact:

PROPOSITION 4.12. *Let $M \subseteqq \overline{\mathbb{R}}^X$ with $|M| \subseteqq M$ and q be an upper integral. If $f \in \overline{\mathbb{R}}^X$ such that $f \leqq g \in M^q$ then $q_l(f) = q(f)$.*

PROOF. We can assume that $g \in +M^q$. Given $\varepsilon > 0$, there exists $h \in +M$ such that $q\big(|g-h|\big) < \varepsilon$. Since

$$f \leqq f \wedge h \dotplus (f - f \wedge h) = f \wedge h \dotplus (f \wedge g - f \wedge h) \leqq f \wedge h \dotplus |f \wedge g - f \wedge h|$$

we have

$$q(f) \leqq q(f \wedge h) \dotplus q\big(|f \wedge g - f \wedge h|\big) \leqq q(f \wedge h) \dotplus q\big(|g-h|\big) < q_l(f) + \varepsilon$$

and, letting $\varepsilon \to 0$, it follows $q(f) \leqq q_l(f)$. □

THEOREM 4.13. *Let M be a vector lattice. If q is an M-regular upper integral bidetermined by (M, M') and $f \in \overline{\mathbb{R}}^X$ with $q_*(f) > -\infty$ then $(q_l)_*(f) = q_*(f)$.*

PROOF. Since $\inf\left\{q(h) : -f \leqq h \in M\right\} = -q_*(f) < \infty$, there exists $h \in M'$ such that $-f \leqq h$ and $q(h) < \infty$. From $-q(h) = q_*(-h) = \inf\left\{q_*(u) : -h \leqq u \in M\right\} \in \mathbb{R}$ we deduce that there exists $u \in M$, $-h \leqq u$ such that $q_*(u) < q_*(-h) + \varepsilon$. Thus, we have

$$q\big(\big|(-h) - u\big|\big) = q\big(u - (-h)\big) \leqq q(u) - q_*(-h) = q_*(u) - q_*(-h) < \varepsilon.$$

Therefore $-h \in M^q$ and so $h \in M^q$. Since $-f \leqq h \in M^q$, it follows by Proposition 4.12 that $q_l(-f) = q(-f)$, that is, $(q_l)_*(f) = q_*(f)$. □

Summarizing, the situation for local integration with respect to upper integrals is the following:

COROLLARY 4.14. *Let M be a vector lattice. If q is an M-regular upper integral bidetermined by (M, M') then*

(i) $M^{q_l} \subseteqq \left\{f \in \overline{\mathbb{R}}^X : q_l(f) \leqq (q_l)_*(f)\right\}$.

(ii) $(q_l)_* = q_*$ *on* $\left\{f \in \overline{\mathbb{R}}^X : q_*(f) > -\infty\right\}$.

These two properties will be sufficient to our purposes.

5. Upper and lower strong measurability

From now on M will be a vector lattice in $\overline{\mathbb{R}}^X$ and, in this section, q will be a general integral metric.

Following the notion of measurability in the sense of Stone (i.e. $(f \wedge h) \vee (-h) \in M^q$, $\forall h \in +M$), we introduce new concepts of upper and lower strong measurability and derive their basics properties.

DEFINITION 5.1. Let q be an integral metric and $f \in \overline{\mathbb{R}}^X$. The classes of the *upper strongly q-M-measurable functions* and of the *lower strongly q-M-measurable functions* are defined, respectively, as:

$$M^q_\wedge := \{f \in \overline{\mathbb{R}}^X : f \wedge h \in M^q,\ \forall h \in +M\}$$

and

$$M^q_\vee := \left\{f \in \overline{\mathbb{R}}^X : f \vee (-k) \in M^q,\ \forall k \in +M\right\}.$$

PROPOSITION 5.2. *Let q be an integral metric. Then*

(i) $f \in M^q_\wedge$ *if and only if* $-f \in M^q_\vee$.

(ii) *If* $f \in M^q$ *then* $f \in M^q_\wedge \cap M^q_\vee$.
(iii) *If* $f \in M^q_\wedge \cup M^q_\vee$ *then* f *is* q*-measurable.*
(iv) $M^q_\wedge$, $M^q_\vee$ *are lattices.*

PROOF. (i) $(-f) \vee (-k) = -(f \wedge k)$, $(-f) \wedge h = -\big(f \vee (-k)\big)$, $\forall h, k \in +M$ and M^q is closed for the operation $-$.

(ii) It is easily seen, by using the Birkhoff inequalities, that if f is in M^q then so are $f \wedge h$ and $f \vee (-k)$, for all $h, k \in +M$. Hence $f \in M^q_\wedge \cap M^q_\vee$.

(iii) If $f \in M^q_\wedge$, that is, $f \wedge h \in M^q, \forall h \in +M$, then, by (ii), $f \wedge h \in M^q_\vee$. Therefore $(f \wedge h) \vee (-h) \in M^q$, $\forall h \in +M$ which gives the q-measurability of f.

Analogously, if $f \in M^q_\vee$ then f is q-measurable, keeping in mind that $(f \wedge h) \vee (-h) = \big(f \vee (-h)\big) \wedge h$.

(iv) For all $f, g \in M^q_\wedge$ and $h \in +M$ one has

$$(f \wedge g) \wedge h = (f \wedge h) \wedge (g \wedge h) \in M^q,$$

$$(f \vee g) \wedge h = (f \vee h) \wedge (g \vee h) \in M^q,$$

and dual arguments apply to $M^q_\vee$. □

The next results make evident the relation between these new concepts and q-integrability.

COROLLARY 5.3. *For each integral metric* q, $M^q = M^q_\wedge \cap M^q_\vee$.

PROOF. $M^q \subseteqq M^q_\wedge \cap M^q_\vee$ is (ii) of Proposition 5.2.

To prove the other inclusion, let $f \in M^q_\wedge \cap M^q_\vee$. Then $f^- = -(f \wedge 0) \in M^q$ and $f^+ = f \vee 0 \in M^q$ and so $f = f^+ - f^- \in M^q$. □

Theorem 2 of [6] on measurability can be split in two strong measurability counterparts. These results gain interest if we realize that a hypothesis is strengthened whereas the companion one is weakened, so they are far from the measurable version.

THEOREM 5.4. *For each integral metric* q *the following are equivalent:*
(i) $f \in M^q$,
(ii) $f \in M^q_\wedge$ *and* $f^+ \in M^q$,
(iii) $f \in M^q_\wedge$ *and* $\exists \varphi \in M^q$ *such that* $f \leqq \varphi$.

PROOF. (i) $\Rightarrow$ (ii). $f \in M^q_\wedge$ and $f^+ = f \vee 0 \in M^q$ because $f \in M^q_\wedge \cap M^q_\vee$.

(ii) $\Rightarrow$ (iii). It is sufficient to take $\varphi := nf^+$.

(iii) $\Rightarrow$ (i). Since M is a vector lattice we can assume that $\varphi \geqq 0$. Thus, given $\varepsilon > 0$, there exists $h \in +M$ such that $q\big(|\varphi - h|\big) < \varepsilon$, and from

$$|f - f \wedge h| = |f \wedge \varphi - f \wedge h| \leqq |\varphi - h|$$

we deduce that

$$q\big(|f-f\wedge h|\big) \leqq q\big(|\varphi-h|\big) < \varepsilon.$$

Moreover, $f\wedge h\in M^q$ and this leads to $f\in (M^q)^q=M^q$. □

COROLLARY 5.5. *For each integral metric q the following are equivalent:*

(i) $f\in M^q$,

(ii) $f\in M^q_\vee$ *and* $f^-\in M^q$,

(iii) $f\in M^q_\vee$ *and* $\exists\varphi\in M^q$ *such that* $\varphi\leqq f$.

As a consequence, we give a characterization of strong measurability by truncations with, not only functions in $+M$, but integrable functions of arbitrary sign.

COROLLARY 5.6. *Let q be an integral metric. Then*

(i) $M^q_\wedge=\{f\in\overline{\mathbb{R}}^X : f\wedge u\in M^q,\ \forall u\in M^q\}$,

(ii) $M^q_\vee=\{f\in\overline{\mathbb{R}}^X : f\vee v\in M^q,\ \forall v\in M^q\}$.

PROOF. (i) Let $f\in M^q_\wedge$. Given $u\in M^q$, $f\wedge u\in M^q_\wedge$ by Proposition 5.2 (items (ii) and (iv)), and $f\wedge u\leqq u\in M^q$. Therefore, Theorem 5.4 gives $f\wedge u \in M^q$. The opposite inclusion is trivial.

(ii) Analogously. □

The last corollary enables us to show that the classes $M^q_\wedge$ and $M^q_\vee$ are closed with respect to $+$.

THEOREM 5.7. *Let q be an integral metric. If $f,g\in M^q_\wedge$ (resp. $f,g\in M^q_\vee$) then $f+g\in M^q_\wedge$ (resp. $f+g\in M^q_\vee$).*

PROOF. Given $f,g\in M^q_\wedge$, set $\varphi:=f^++g^+$ and $\psi:=f^-+g^-$. It is clear that $\varphi,\psi\geqq 0$, $\psi\in M^q$ and $f+g=\varphi-\psi$. Fix $h\in +M$. A trivial verification shows that

$$\varphi\wedge h=(f^++g^+)\wedge h=\big[(f^+\wedge h)+(g^+\wedge h)\big]\wedge h$$

which gives $\varphi\in M^q_\wedge$. Thus

$$(f+g)\wedge h=(\varphi-\psi)\wedge h=\varphi\wedge(h+\psi)-\psi\in M^q,$$

that is, $f+g\in M^q_\wedge$. □

6. Relations between M^q and M^{q_l}

We are ready to establish some general descriptions for M^q and M^{q_l} in terms of each other. The results in Section 4 are now used to find conditions to derive upper (resp. lower) strongly q-measurability from upper (resp. lower) strongly q_l-measurability. M keeps on being a vector lattice in $\overline{\mathbb{R}}^X$.

THEOREM 6.1. *Let q be an upper integral. Assume that q is M-regular and bidetermined by (M, M').*

(i) *If $f \in M_\wedge^{q_l}$ and $q(f^-) < \infty$ then $f \in M_\wedge^q$,*

(ii) *If $f \in M_\vee^{q_l}$ and $q(f^+) < \infty$ then $f \in M_\vee^q$.*

PROOF. (i) Let $h \in +M$. We have $f \wedge h \in M^{q_l}$ and we want to prove that, in fact, $f \wedge h \in M^q$. Since q is an M-regular upper integral bidetermined by (M, M'), Theorem 3.21 says that this is equivalent to prove that $q_*(f \wedge h) = q(f \wedge h) \in \mathbb{R}$.

Clearly $q_*(f \wedge h) \leqq q(f \wedge h) \leqq q(h) < \infty$. Furthermore, from $f \wedge h \geqq -f^-$ we deduce that

$$q_*(f \wedge h) \geqq q_*(-f^-) = -q(f^-) > -\infty$$

and now, using Corollary 4.14 (both items), it follows that

$$q(f \wedge h) = q_l(f \wedge h) \leqq (q_l)_*(f \wedge h) = q_*(f \wedge h).$$

Therefore

$$-\infty < q_*(f \wedge h) = q(f \wedge h) < \infty$$

that is, $f \wedge h \in M^q$ as claimed.

(ii) We have only to apply (i) to $-f$, since $f^+ = (-f)^-$. $\square$

COROLLARY 6.2. *If q is an M-regular upper integral bidetermined by (M, M'), then*

$$M^q = \left\{ f \in M^{q_l} : q\big(|f|\big) < \infty \right\}.$$

PROOF. On the one hand, it is clear that $f \in M^q$ implies $f \in M^{q_l}$ and $q\big(|f|\big) < \infty$. On the other hand, if $f \in M^{q_l}$ with $q\big(|f|\big) < \infty$ then $f \in M_\wedge^{q_l}$ and $f \in M_\vee^{q_l}$ (Theorem 5.3) and $q(f^-), q(f^+) < \infty$. Therefore, Theorem 6.1 guarantees that $f \in M_\wedge^q$ and $f \in M_\vee^q$, that is, $f \in M^q$ (Theorem 5.3 once again). $\square$

We now study the relation between the functionals q_* and q_l in order to get a characterization of M^{q_l}.

PROPOSITION 6.3. *Let q be an M-regular upper integral. Then*

$$q_l(f) \leqq q_*(f), \quad \forall f \in M_\wedge^q.$$

PROOF. Let $h \in +M$. Since $f \wedge h \in M^q$ and q is M-regular we deduce that $q_*(f \wedge h) = q(f \wedge h)$, by Proposition 3.15, and hence $q_{*l}(f) = q_l(f)$ by taking suprema. A look at the diagram of Proposition 4.5 gives $q_l(f) \leqq q_*(f)$. □

PROPOSITION 6.4. *If q is an upper integral bidetermined by (M, M'), then $q_* \leqq q_l$.*

PROOF. An easy computation shows that bidetermination by (M, M') leads to

$$q_*(f) = \sup \left\{ q(-k) : k \in M',\ -k \leqq f \right\}, \quad \forall f \in \overline{\mathbb{R}}^X$$

and

$$q_*(-k) = \inf \left\{ q_*(u) : -k \leqq u \in M \right\}, \quad \forall k \in M'$$

We can obviously assume that $q_*(f) > -\infty$. If $q_*(f) \in \mathbb{R}$, given $\varepsilon > 0$, there exists $k \in M'$ with $-k \leqq f$ such that $q_*(f) - \varepsilon < q_*(-k)$. Moreover, there exists $u \in +M$ with $-k \leqq u$. Therefore

$$q_*(f) - \varepsilon < q_*(-k) \leqq q_*(f \wedge u) \leqq q(f \wedge u) \leqq q_l(f)$$

and letting $\varepsilon \to 0$ it follows that $q_*(f) \leqq q_l(f)$.

If $q_*(f) = +\infty$, given $\alpha \in \mathbb{R}$, there exists $k \in M'$ with $-k \leqq f$ such that $\alpha < q_*(-k)$. Moreover, there exists $u \in +M$ with $-k \leqq u$. Therefore

$$\alpha < q_*(-k) \leqq q_*(f \wedge u) \leqq q(f \wedge u) \leqq q_l(f)$$

and consequently $q_l(f) = +\infty = q_*(f)$. □

THEOREM 6.5. *Let $M \subseteqq \mathbb{R}^X$ be a vector lattice and q an M-regular upper integral bidetermined by (M, M'). Then*

$$+M^{q_l} = \{ f \in +\overline{\mathbb{R}}^X : f \in M^q_\wedge,\ q_*(f) < \infty \}$$

and therefore

$$M^{q_l} = \left\{ f \in \overline{\mathbb{R}}^X : f^\pm \in M^q_\wedge,\ q_*\big(|f|\big) < \infty \right\}.$$

PROOF. Let us assume $f \in +M^q_\wedge$ with $q_*(f) < \infty$. In particular f is q-measurable and, by Proposition 6.3, $q_l\big(|f|\big) = q_l(f) \leqq q_*(f) < \infty$. Theorem 5 in [6] guarantees that $f \in M^{q_l}$ (q is M-semiadditive because q is, in fact, M-additive).

To prove the opposite inclusion, let $f \in +M^{q_l}$. Proposition 5.2 (item (ii)) says that $f \in M^{q_l}_\wedge$ and, since $q(f^-) = q(0) = 0 < \infty$, Theorem 6.1 provides $f \in M^q_\wedge$.

Moreover, $q_*(f) \leqq q_l(f) < \infty$, by Proposition 6.4 and Theorem 4.10. □

7. Applications and examples

We now present a standard procedure to construct upper integrals from functionals on an $M \subseteqq \overline{\mathbb{R}}^X$.

DEFINITION 7.1. For $\emptyset \neq M \subseteqq P \subseteqq \overline{\mathbb{R}}^X$ and $T : P \to \overline{\mathbb{R}}$ we define $T^M : \overline{\mathbb{R}}^X \to \overline{\mathbb{R}}$ by

$$T^M(f) := \inf\left\{T(h) : f \leqq h \in M\right\}, \ \forall f \in \overline{\mathbb{R}}^X.$$

With a slightly modified version of Lemma 11 in [6] and some related results, a unified treatment of the finitely additive integral extension theories can still be obtained.

LEMMA 7.2. *Let $M \subseteqq \overline{\mathbb{R}}^X$ with $0 \in M$ and $M \dot{+} M \subseteqq M$. If $T : M \to \overline{\mathbb{R}}$ is nondecreasing, $T(0) = 0$, and*

$$T(u \dot{+} v) \leqq T(u) \dot{+} T(v), \quad \forall u, v \in M,$$

then T^M is an upper integral, $T^M = T$ on M, and $(T^M)^M = T^M$ on $\overline{\mathbb{R}}^X$. Therefore, T^M is an upper integral determined by M which extends T.

Moreover, T^M is M-regular under closely related conditions:

LEMMA 7.3. *Let q be an upper integral and $M \subseteqq \mathbb{R}^X$ with $-M \subseteqq M$. The following are equivalent:*

(i) *q is M-regular.*

(ii) *q is $+$-additive and finite on M.*

PROOF. (i) $\Rightarrow$ (ii). By Corollaries 3.16 and 3.8.

(ii) $\Rightarrow$ (i). $0 = q\big(h + (-h)\big) = q(h) + q(-h) \Rightarrow q(h) = -q(-h) = q_*(h) \in \mathbb{R}$. $\square$

COROLLARY 7.4. *Let $M \subseteqq \mathbb{R}^X$ be a vector space. Assume $T : M \to \mathbb{R}$ is nondecreasing, $T(0) = 0$, and additive for $+$, i.e.*

$$T(u + v) = T(u) + T(v), \quad \forall u, v \in M.$$

Then T^M is an M-regular upper integral determined by M which extends T.

A. *Proper Riemann Integration.* Given (X, Ω, μ), where Ω is a semiring of sets from X and $\mu : \Omega \to [0, \infty[$ is additive, let us consider $B_\Omega := S(\Omega, \mathbb{R})$, the step functions, and $I_\mu := \int \cdot \, d\mu$ on Ω, [9]. With $q = I_\mu^- := (I_\mu)^{B_\Omega}$, Corollary 7.4 says that q is a B_Ω-regular upper integral determined by B_Ω and so Corollary 3.19 yields

$$(B_\Omega)^{I_\mu^-} = \left\{f \in \overline{\mathbb{R}}^X : I_\mu^-(f) = I_\mu^+(f) \in \mathbb{R}\right\} = R_e^1(\mu, \mathbb{R}) \quad [9].$$

B. *Abstract Riemann-μ-integration.* With $B := B_\Omega$, $I := I_\mu$, $q = I_\mu^-$ the B_Ω-regular upper integral determined by B_Ω of A, and $R_1(\mu, \overline{\mathbb{R}}) := B^{q_l}$ it follows that $R_1(\mu, \overline{\mathbb{R}}) \cap \mathbb{R}^X = R_1(\mu, \mathbb{R})$ ([9, p. 70–144]) $\supseteqq L(X, \Omega, \mu, \mathbb{R})$ ([8, p. 112]), the latter inclusion being an equality if $X \in \Omega$.

Corollary 6.2 and Theorem 6.5 say that

$$R_e^1(\mu, \mathbb{R}) = \left\{ f \in R_1(\mu, \mathbb{R}) : \, I_\mu^-\big(|f|\big) < \infty \right\},$$

$$R_1(\mu, \mathbb{R}) = \left\{ f \in \overline{\mathbb{R}}^X : \, f^\pm \wedge h \in R_e^1(\mu, \mathbb{R}), \forall h \in +B, \; I_\mu^+\big(|f|\big) < \infty \right\}.$$

C. *Loomis completions.* Given a Loomis system (X, B, I) ($B \subseteqq \mathbb{R}^X$ vector lattice, $I : B \to \mathbb{R}$ linear and positive) and $q = I^- := I^B$, Corollary 7.4 guarantees that q is a B-regular upper integral determined by B and therefore Theorem 3.19 gives

$$B^q \cap \mathbb{R}^X = R_e^1 \;\; \text{two-sided completion } R \text{ of Loomis [10, p. 170]}$$

$$= R_{\mathrm{prop}}(B, I) \quad [7].$$

Moreover, it is known that $B^{q_l} \supseteqq R_e^1$ (one-sided completion U of Loomis [10, p. 178]), $B^{q_l} = R_1(B, I)$ (I-integrable functions of [7, p. 147]).

A and B above are special cases.

Corollary 6.2 and Theorem 6.5 here give the familiar descriptions in [7, Corollaries 1.7, 1.8]

$$R_{\mathrm{prop}}(B, I) = \left\{ f \in R_1(B, I) : \, I^-\big(|f|\big) < \infty \right\},$$

$$R_1(B, I) = \left\{ f \in \overline{\mathbb{R}}^X : \, f^\pm \wedge h \in R_{\mathrm{prop}}(B, I), \; \forall h \in +B, \; I^+\big(|f|\big) < \infty \right\}$$

but nothing is said about the classes $B_\wedge^q$ and $B_\vee^q$ in this case. These results are obtained by other methods (using local convergence).

D. *Finitely-additive Daniell extension.* For a Loomis system (X, B, I), the following are introduced in [2]:

$$I^+(f) := \sup \left\{ I(h) : \, h \leqq f, \; h \in B \right\},$$

$$B^+ := \left\{ f \in \overline{\mathbb{R}}^X : \, f = \sup \{ h : \, f \geqq h \in B \}, \; I^+(f) > -\infty \right\},$$

$$B_+ := \left\{ f \in B^+ : \, I^+(f + g) \leqq I^+(f) + I^+(g), \; \forall g \in B^+ \right\}.$$

By Lemmas 7.2 and 7.3, $q = \overline{I} := (I^+)^{B_+}$ is a B-regular upper integral determined by B_+. In fact, q is a B-regular upper integral bidetermined by (B, B_+) and, therefore, applying Theorem 3.21, we still have

$$\overline{B} := B^q = \left\{ f \in \overline{\mathbb{R}}^X : \, \overline{I}(f) = \underline{I}(f) \in \mathbb{R} \right\},$$

the summable functions B_0 of [2].

Moreover, Corollary 5.6 guarantees that $B^q_\wedge = B^*_+ = L^\wedge$, $B^q_\vee = B^*_- = L^\vee$ of [2] and [3], respectively. Section 5 generalizes several properties, known for these classes B^*_+ and B^*_-, to our general setting on upper integrals.

E. *Localization of the Daniell-analogue.* For a Loomis system (X, B, I), $q = \overline{I}$ of D leads to $B^{q_l} = L$ of [5].

We are able to apply all results in Section 6, yielding the new descriptions

$$\overline{B} = \big\{ f \in L : \overline{I}\big(|f|\big) < \infty \big\}$$

and

$$L = \big\{ f \in \overline{\mathbb{R}}^X : f^\pm \wedge h \in \overline{B}, \forall h \in +B,\ \underline{I}\big(|f|\big) < \infty \big\}.$$

Since $\overline{I} \leqq I^-$, one has $\overline{B} \subseteqq L$ and $R_{\text{prop}}(B, I) \subseteqq R_1(B, I) \subseteqq L$, with coinciding integrals.

Example 7.5. For $q : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ given by $q(x) = x$ if $x \geqq 0$ and $q(x) = 0$ elsewhere, it is clear that $\{0\} = \operatorname{Reg}(q) \subset \mathbb{R} = \mathbb{R}^q$.

Example 7.6. Let $X = \{1, 2, 3\}$ and $q : \overline{\mathbb{R}}^X \to \overline{\mathbb{R}}$ defined by $q(x) := x_1 \dotplus x_2 \dotplus x_3$, $\forall x = (x_1, x_2, x_3) \in \overline{\mathbb{R}}^X = \overline{\mathbb{R}}^3$.

It is clear that $q(0) = 0$, q is monotone and $q(x \dotplus y) = q(x) \dotplus q(y)$ (since $\dotplus$ is associative). Thus, q is an upper integral (it is even a Bourbaki-continuous integral norm). For $\Omega := \big\{ \emptyset, \{1\}, \{2, 3\}, X \big\}$ (a σ-algebra) and taking $B := B_\Omega$, the step functions with respect to Ω (see A above), one has $B^q = B$ whereas $\operatorname{Reg}(q) = \mathbb{R}^X$. Obviously q is B-regular (see Lemma 7.3), but q is not determined by B. For instance, with $x_0 := (0, 0, 1)$, $q(x_0) = 1 < 2 = q(0, 1, 1) = \inf \big\{ q(u) : x_0 \leqq u \in B \big\}$. Moreover, q is not bidetermined by (B, B') for any $B' \supseteqq B$. If there exists an $x_0 \in B'$ such that $x_0 \notin B$ then $x_0(2) \neq x_0(3)$ and therefore $q_*(x_0) \neq \inf \big\{ q_*(u) : x_0 \leqq u \in B \big\}$.

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