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FUBINI-INTEGRAL METRICS

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In this paper, by using an integral extension of Lebesgue power with local integral metrics, we stablish abstract Fubini type theorems, which subsume most known situations of integration with respect to finitely additive measures.

Introduction.

Recently in [4] an integration theory (analogue to Daniell's extension process) was given which works for general integral metrics, without any continuity conditions. This is possible using a suitable local mean convergence, which can be traced back to Loomis [12].

In [4], for general local integral metrics convergence theorems are derived, extending results of Schäfke, and an unified treatment of proper Riemann-μ-, abstract Riemann-, Loomis-, Daniell- and Bourbaki-integrals is given. All this is specialized to integration with respect to finitely additive measure.

Since Fubini's theorem for finitely additive integration is in general false, (the existence of the abstract Riemann integral does not always imply the existence of the repeated integrals in the sense of

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Riemann), it seems therefore natural to ask for conditions under which the repeated integrals will exist. Conditions of this type were given by several authors in [7], [11] and [5].

The object of this paper is to study an abstract Fubini theorem in integration theory for general local integral metrics. The results, which generalize those of Elsner [7] and Hoffmann [11], are specialized and discussed for an abstract Riemann-integration theory for finitely additive set function as has been devolopped and used by Dunford-Schwartz [6], Aumann [2], Loomis [12] and Günzler [8], [9].

1. Notations and terminology.

Terminology and notations used are similar to that of [4] and will explained it whenever be necessary in order to make the paper self-contained.

On the set $\bar{\mathbb{R}}$ of extended real numbers we adopt the conventions $a + b := 0$, $a \dot{+} b := \infty$ if $a = -b \in \{-\infty, \infty\}$. We denote $a \vee b := \max(a, b)$, $a \wedge b := \min(a, b)$ and $a \cap t := (a \wedge t) \vee (-t)$ if $a, b \in \bar{\mathbb{R}}$, $0 \leq t \in \bar{\mathbb{R}}$.

For nonempty set X let $\bar{\mathbb{R}}^X$ consists of all functions $f : X \to \bar{\mathbb{R}}$. All operations and relations between functions are defined pointwise.

For each set $A \subset \bar{\mathbb{R}}$ one has $\inf A$, $\sup A \in \bar{\mathbb{R}}$, with the usual conventions $\inf \emptyset := \infty$ and $\sup \emptyset := -\infty$. We use the abbrevations $\bar{\mathbb{R}}^X_+$ for the set $f \geq 0$, and $A_- := \{f; -f \in A\}$.

A real linear space $B \subset \mathbb{R}^X$ is said to be a *vector lattice* if $h \in B$ implies $|h| \in B$ (then $h \wedge k$, $h \vee k \in B$, for all $h, k \in B$).

A functional $q : \bar{\mathbb{R}}^X \to \bar{R}$ is called an *upper integral* if $q(0) = 0$, $q(f \dot{+} g) \leq q(f) \dot{+} q(g)$ ($+$-subadditive) and $q(f) \leq q(k)$ (monotone) for all $f, g, k \in \bar{\mathbb{R}}^X$, $f \leq k$.

q_* denotes the functional defined on $\bar{\mathbb{R}}^X$ by $q_*(f) := -q(-f)$ for all $f \in \bar{R}^X$. One has that q_* is $+$-superadditive and monotone, $q_* \leq q$ and $(q_*)_* = q$ on $\bar{\mathbb{R}}^X_+$; q_* is said to be a *lower integral.*

If q is an upper integral then $q/\bar{\mathbb{R}}^X_+$ is an *integral metric* on $\bar{\mathbb{R}}^X_+$ in sense of [14] or [4], i.e. $q(0) = 0$, $q(f) \leq q(g) + q(k)$ if

$f \leq g + k,\ f, g, k \in \bar{\mathbb{R}}^X_+$.

Let $B \subset \mathbb{R}^X$ such that $0 \in B_- \subset B$, then an upper integral q for which $q(h) = q_*(h) \in \mathbb{R}$ for all $h \in B$, is said to be *regular* on B. If additionally B is a linear space of $\mathbb{R}^X$, q is regular on B if and only if q is linear on B.

2. q-Integrable functions.

In the present section we describe the integration with respect to an upper integral q or the associated localized functional q_ℓ, and, under some additional assumptions, we characterize q-integrability in terms of the equality of the upper and lower integrals.

The first notions was presented essentially by Aumann in [2], and more generally in [4] & 1.

DEFINITION 2.1. *Let q be an upper integral regular on B. A function $f \in \bar{\mathbb{R}}^X$ is said to be q-integrable if it belongs to the closure of B in $\bar{\mathbb{R}}^X$ with respect to the integral metric $q(|.|)$, i.e. for all $\varepsilon > 0$ there exists $h \in B$ such that $q(|f - h|) < \varepsilon$.*

The set of q-integrable functions will be denoted by B^q.

The following assertions are easy consequences of the definitions.

(1) If $f \in B^q$ and $(h_n) \subset B$ such that $q(|f - h_n|) \to 0$ as $n \to \infty((h_n)$ is called a defining sequence for f), then $q(f) = .lim q(h_n) \in \mathbb{R}$, as $n \to \infty$.

(2) If $f \in B^q$, $g \in \bar{\mathbb{R}}^X$ with $q(g) \in \mathbb{R}$, then $q(f \dot{+} g) = q(f) + q(g)$.

(Note that the inequalities needed here read: for $a, b, c \in \bar{\mathbb{R}}$, $a \leq b \dot{+} (a - b)$ and $a \leq b \dot{+} c$ if $a - b \leq c$).

LEMMA 2.1. *Let q be an upper integral regular on B. Then $q(f) = q_*(f) \in \mathbb{R}$ for all $f \in B^q$.*

Proof. Let $(h_n)_n \subset B$ a defining sequence for $f \in B^q$, by (1), $q(f) = lim q(h_n) = -lim q(-h_n) = -q(-f) = q_*(f) \in \mathbb{R}$, since

$q(|f - h_n|) = q(|(-f) - (-h_n)|) \to 0$ as $n \to \infty$, implies $-f \in B^q$ and $q(-h_n) \to q(-f)$ as $n \to \infty$. ■

(3) A functional $p : \bar{\mathbb{R}}^X \to \bar{\mathbb{R}}$ is said to be determined by a set of functions $M \subset \bar{\mathbb{R}}^X$ if $p(f) = inf\{p(g); f \leq g \in M\}$ for all $f \in \bar{\mathbb{R}}^X$.

Observe that if q is an upper integral regular in B, then q is determined by B iff $q_*(f) = \sup\{q(g); g \leq f, g \in B\}$ for all $f \in \bar{\mathbb{R}}^X$.

Standard assumption in this and the following are $B \subset \mathbb{R}^X$ such that $0 \subset B_- \subset B$ and q an upper integral regular on B.

LEMMA 2.2. *Let q be a regular upper integral determined by B.*

If $f \in \bar{\mathbb{R}}^X$ such that $q(f) = q_*(f) \in \mathbb{R}$, then $f \in B^q$.

Proof. By (3), given any $\varepsilon > 0$ there exists $h \in B$ such that $f \leq h$ and $0 \leq q(h) - q(f) < \varepsilon$. By (2), $q(|f-h|) = q(h) - q_*(f) = q(h) - q(f) < \varepsilon$, and the result follows. ■

In view of the above results, we state an useful q-integrability criterion.

COROLLARY 2.1. *Let q be a regular upper integral determined by B and $f \in \bar{\mathbb{R}}^X$, then the following assertions are equivalent:*

i) $f \in B^q$.

ii) $q(f) = q_*(f) \in \mathbb{R}$.

iii) Given any $\varepsilon > 0$ there exist $h, k \in B$ such that $-h \leq f \leq k$ and $q(h) + q(k) < \varepsilon$.

The following is a simplified version of Schäfke's definition [14] p. 120.

DEFINITION 2.3. *If $q : \bar{\mathbb{R}}^X \to \bar{\mathbb{R}}$ is any upper integral regular on B, the corresponding local upper integral is defined by*

$$q_\ell(f) := sup\{q(f \wedge h); 0 \leq h \in B\} \text{ for all } f \in \bar{\mathbb{R}}^X_+.$$

It is easy to check that q_ℓ is again an upper integral, $q_\ell = q$ on B, $q_\ell \leq q$ on $\bar{\mathbb{R}}_+^X$, $q_\ell(f) = q(f)$ if $f \leq$ some $h \in B^q$, and $(q_\ell)_\ell = q_\ell$ on $\bar{\mathbb{R}}^X$.

If $f \in B^{q_\ell}$, $g \in \bar{\mathbb{R}}_+^X$ with $q_\ell(g) \in \mathbb{R}$ then $q_\ell(f+g) = q_\ell(f) + q_\ell(g)$.

B^{q_ℓ} denotes the set of q_ℓ-integrable functions, i.e. the closure of B in $\bar{\mathbb{R}}^X$ with respect to the integral metric $q_\ell(|\cdot|)$.

(4) With definition 2.3. one has for all $f \in \bar{R}^X$

$$(q_\ell)_*(f) := -q_\ell(-f) = inf\{q_*(f \vee (-h)); 0 \leq h \in B\}.$$

Remarks 1. (see [3], [4]).

For later reference and the benefit of the reader, we collect some results and examples mostly given in [3], [4] and [10].

1.1. Let B be a vector lattice in $\mathbb{R}^X$ and $I : B \to \mathbb{R}$ linear functional with $I(h) \geq 0$ if $h \geq 0$, $h \in B$, which is uniformly continuous on B with respect to an upper integral q, then, theorem 1 in [4] gives that B^q is closed with respect to $\pm$, $\alpha\cdot$ $(\alpha \in \mathbb{R})$, $|\cdot|, \vee, \wedge, \cap$; and there exists an unique I^q monotone, linear and q-continuous extension of I to B^q. $B \subset B^q \subset B^{q_\ell}$ and $I^q = I^{q_\ell}$ =on B^q.

With $q(f) = I^-(f) := inf\{I(g); f \leq g \in B\}$ one has $B^q = R_{prop}(B, I)$ (proper Riemann-I-integrable functions or the "two-sided completion" $\mathcal{R}$ of Loomis [12] p. 170), and $q = q_*$ on B^q.

1.2. Starting with B, I, q and $q_\ell = I_\ell^-$ as above, one gets $R_1(B, I) := B^{q_\ell}$ =closure of B in $\bar{\mathbb{R}}^X$ with respect to the distance $d(f, g) = (I^-)_\ell(|f - g|)$ =abstract Riemann-I-integrable functions of [3], containing the "one-sided completion" of Loomis [12] p. 178.

We recall that I_l is the "essential upper functional" associated with I^- in sense of Anger and Portenier [1], so that, $R_1(B, I)$ is the set of all the essentially integrable functions (w.r.t. I^-).

For $R_1(B, I)$ the study of convergence concepts related to the integrability yields results similar to the classical ones (e.g. the I^--closedeness property of $R_1(B, I)$ and the Lebesgue convergence theorems).

Finally, it is interesting to note that if B and I are as above and I is σ-continuous (or with Daniell's continuity condition) i.e. $I(h_n) \to 0$ whenever $0 \leq h_n \in B$, $h_n \geq h_{n+1} \to 0$ pointwise on X, by Aumann [2],

$$q(f)I^\sigma(f) := inf \left\{ \sum_{n=1}^{\infty} I(h_n); f \leq \sum_{n=1}^{\infty} h_n, 0 \leq h_n \in B \right\},$$

defines a σ-subadditive integral metric on $\bar{\mathbb{R}}_+$. Then, $B^{q_\ell} = L^1(B, I)$ (usual Daniell-I-integrable functions).

1.3. We consider now B, I arising from finitely additive set function μ, with arbitrary set X.

Ω is a semiring of sets from X, $\mu : \Omega \to \mathbb{R}_+$ is finitely additive on Ω, B_Ω =real-valued step functions on Ω and $I_\mu(h) := \int hd_\mu$, $h \in B_\Omega$, where B_Ω contains all $h = \sum_{i=1}^{n} a_i \chi_{A}$, $n \in \mathbb{N}$, $a_i \in \mathbb{R}$ $A_i \in \Omega$ and $\int hd_\mu = \sum_{i=1}^{n} a_i \mu(A_i)$.

For B_Ω, I_μ, $q = I_\mu^-$ and $q_\ell = (I_\mu^-)_\ell$ one has $B_\Omega^q = R_{prop}(\mu, \Omega)$ (abstract proper Riemann-μ-integrable functions of Loomis [12]) and $B_\Omega^{q_\ell} = R_1(B_\Omega, I_\Omega) = R_1(\mu, \bar{\mathbb{R}})$ (Riemann-μ-integrable functions of Günzler [8]), which contains $L(X, \Omega, \mu, \mathbb{R})$ of Dunford-Schwartz [6].

In particular, $X = \mathbb{R}$, $\Omega = \{[a, b[; -\infty < a \leq b < \infty\}$ and $\mu([a, b[) = b - a$ gives the classical proper Riemann-integrable functions.

One has $R_{prop}(\mu, \mathbb{R}) \subset R_1(\mu, \bar{\mathbb{R}}) \cap \mathbb{R}^X$, and if $X \in \Omega$ then $R_1(\mu, \mathbb{R}) = L(X, \Omega, \mu, \mathbb{R})$. Finally, if μ is σ-additive and Ω is a σ-ring, then $R_1(\mu, \bar{\mathbb{R}})$ =Lebesgue-μ-integrable functions $L^1(\mu, \bar{\mathbb{R}})$ modulo null functions ([9], A. 146), and one gets the usual Lebesgue convergence theorems.

3. Product systems.

In this section we can apply the general theory of the section 2, to discuss product systems and Fubini's theorem in an abstract setting, considering the results peculiar to abstract Riemann integration.

We shall assume that X_1, X_2 are arbitrary sets and $X_3 := X_1 \times X_2$.

For $j = 1, 2, 3$, B_j is a vector lattice $\subset \mathbb{R}^{X_j}$ and $q_j : \bar{\mathbb{R}}^{X_j} \to \bar{\mathbb{R}}$ is an upper integral.

If $f \in \mathbb{R}^{X_3}$ and $x \in X_1$, we define $f_x(y) := f(x, y)$ for each $y \in X_2$ and $(q_2 f)(x) := q_2(f_x)$.

Let $I_1 := B_1 \to \mathbb{R}$ be a nonnegative linear functional which is $q_1(|\cdot|)$-continuous, and $I_2 : B_2 \to \mathbb{R}$ a nonnegative linear functional such that $|I_2(f)| \leq q_2(|f|)$ for all $f \in B_2$.

A system (X_3, B_3) is called a *product system* with respect to (X_1, B_1) and (X_2, B_2), whenever for each $f \in B_3$ the following conditions are satisfied:

i) $f_x \in B_2$ for each $x \in X_1$.

ii) $I_2 f \in B_1$, where $(I_2 f)(x) := I_2(fx)$, $x \in X_1$.

In all that follows (X_3, B_3) will be a product system.

We define a nonnegative linear functional on B_3 by the rule $I_3(f) := (I_1 \circ I_2)(f) = I_1(I_2 f)$ for each $f \in B_3$.

In view of the definitions involved, it is easily checked that

(5) If p, q are upper integral on $\bar{\mathbb{R}}^X$, then $q_* = (q_*)_*$, $q_* \leq p_*$ if $p \leq q$ and $q_* \leq (q_\ell)_* \leq q_\ell \leq q$ on $\bar{R}^X$.

For $j = 2, 2, 3$, if $q_j : \bar{\mathbb{R}}$ is an upper integral, then $q_1 \circ q_2$ is an upper integral, $(q_1 \circ q_2)_* = (q_1)_* \circ (q_2)_*$ and $(q_3)_* \leq (q_1)_* \circ (q_2)*$ if $q_3 \geq q_1 \circ q_2$.

As we have seen in the section 2, B^{X_j}, $j = 1, 2, 3$, denotes the set of all the q_j-integrable functions and I^{q_j} is the q_j-continuous extension of I_j to $B_j^{q_j}$, again denoted by I_j.

$A \subset X_j$ is called an q_j*-null set* if $q_j(\chi_A) = 0$.

(6) If $f \in B_j^{q_j}$, $g \in \bar{\mathbb{R}}^{X_j}$ and $q_j(|f - g|) = 0$, then $g \in B_j^{q_j}$ and

$I_j(f) = I_j(g)$.

In all the follows we assume, for $j = 1, 2, 3$, that

(7) $q_j : \bar{\mathbb{R}}^{X_j} \to \bar{\mathbb{R}}$ is a regular upper integral determined by $B_j \subset \mathbb{R}^{X_j}$ vector lattice, and $q_3 \geq q_1 \circ q_2$ on $\bar{\mathbb{R}}^{X_3}$.

Now, using the properties of the integral considered and having in mind the corollary 2.1., we obtain a Fubini's theorem for q-integrable functions.

THEOREM 3.1. *If* $f \in B_3^{q_3}$ *then*

i) $q_2 f, (q_2)_* f \in B_1^{q_1}$.

ii) There exist $A_k \subset X_1$, $k \in \mathbb{N}$, q_1*-null sets, such that* $f_x \in B_2^{q_2}$ *for all* $x \in X_1 - \bigcup_{k=1}^{\infty} A_k$.

iii) There exists $g \in B_1^{q_1}$ *defined by* $I_2(f_x)$ *if* $f_x \in B_2^{q_2}$ *and such that* $I_3(f) = I_1(g)$.

Proof i) For $f \in B_3^{q_3}$, by (2), (3), (5) and lemma 2.1, we have

$$q_3(f) \geq (q_1 \circ q_2)(f) = q_1(q_2 f) \geq \left\{ \begin{matrix} q_1[(q_2)_* f] \\ (q_1)_*[q_2 f] \end{matrix} \right\}$$

$$\geq (q_1)_*[(q_2)_* f] \geq (q_3)_*(f),$$

so that, $q_1(q_2 f) = (q_1)_*(q_2 f) \in \mathbb{R}$, and by lemma 2.2., $q_2 f \in B_1^{q_1}$.

Similarly, $(q_2)_* f \in B_1^{q_1}$.

ii) For $x \in X_1$, set $h(x) := q_2(f_x) - (q_2)_*(f_x)$. One has $0 \leq h \in B_1^{q_1}$ and $q_1(h) = 0$.

Now, let $A_k := \left\{ x \in X_1; h(x) \geq \frac{1}{k} \right\}$, $k \in \mathbb{N}$. Since $q_1(\chi_{A_k}) \leq k q_1(h) = 0$, A_k are q_1-null sets, and by (6) and lemma 2.2., $f_x \in B_2^{q_2}$ for all $x \in X_1 - \bigcup_{k=1}^{\infty} A_k$.

iii) Finally, if $g \in \bar{\mathbb{R}}^{X_1}$ such that $I_2(f_x) \leq g(x) \leq (I_2)_*(f_x)$ for all $x \in X_1 - \bigcup_{k=1}^{\infty} A_k$, then by lemma 2.2., we obtain $g \in B_1^{q_1}$ and $I_1(g) = I_1(I_2 f) = I_3(f)$. ■

THEOREM 3.1. *contains properly that of Elsner [7] p. 269, for which we obtain a simplified proof. Adeed, example 2 below shows that there exist functions which theorem 3.1. is applicable, but not the corresponding results of [7] or [11].*

Remarks 2.

2.1. In the Bourbaki situation, where the nonnegative *linear functional* $I : B \to \mathbb{R}$ is τ-continuous, i.e. $I(h_n) \to 0$ if net $(h_n) \subset B$ decreases pointwise to 0, the space $L^\tau = L^\tau(B, I)$ of Bourbaki-I-integrable functions and the corresponding integral extension $I^\tau : L^\tau \to \mathbb{R}$ are well defined (see for example Pfeffer [13] p. 44), with Daniell-$L^1(B, I) \subset L^\tau$.

Here $I^\tau \geq I_1^\tau \circ I_2^\tau$ and there is an analogue to theorem 2.1. (see [13], p. 186).

Special cases: X =open sets $\subset \mathbb{R}^n$, Ω = {intervals}, μ =Lebesgue measure μ_L^n on Ω. Also, $C_0(X, \mathbb{R})$ with arbitrary Hausdorff space X and any nonnegative linar I on $C_0(X, \mathbb{R})$, which is automatically τ-continuous.

Note that $I_1 \circ I_2$ is a Daniell or Bourbaki integral according to whenever I_1 and I_2 were Daniell or Bourbaki integrals, respectively.

We recall that, if $|f| \in B_3$ whenever $f \in B_3 := B_1 \otimes B_2$ (= "tensor product space"), then B_3 is a product system, so the above results can be applied to some classical measure product spaces (see [13] & 15).

2.2. As in remark 1.1., for $f \in \mathbb{R}^{X_j}$, $j = 1, 2, 3$, we define the Riemann upper integral $q_j(f) = I_j^-(f)$. Then the set of all the proper Riemann-I_j-integrable functions $R_{prop}(B_j, I_j)$ is the closure of B_j with respect to the integral-seminorm $I_j^-(|\cdot|)$. If $f \in R_{prop}(B_j, I_j)$, $I_j(f) := I_j^-(f) = (I_j^-)_*(f) \in \mathbb{R}$.

Since, $inf\{I_1(I_2 f);\ f \leq h \in B_3\} \geq inf\{I_1(I_2 h);\ I_2^- f \leq I_2 h,$

$h \in B_3\} \geq inf\{I_1(g);\ I_2^- f \leq g \in B_1\}$, one has $I_3^-(f) \geq (I_1^- \circ I_2^-)(f)$ for all $f \in \bar{\mathbb{R}}^{X_3}$, and all the above is applicable ((3), (7) hold), so that, theorem 2.1. gives the corresponding Fubini theorem for the proper Riemann-I-integrable functions $R_{prop}(B_3, I_3)$.

We retain the basic assumptions as formulated in definition 2.3., and (7).

(8) For any $f \in \bar{\mathbb{R}}^{X_j}$, $j = 1, 2, 3$, $q_{j,\ell}(f) := \sup\{q_j(f \wedge h); 0 \leq h \in B_j\}$.

For $f \in \mathbb{R}^{X_3}$, we define $(q_{2,\ell} f)(x) := q_{2,\ell}(f_x)$ for each $x \in X_1$.

$B_j^{q_{j,\ell}}$ denotes the set of all the $q_{j,\ell}$-integrable functions and $I^{q_{j,\ell}}$ the unique $q_{j,\ell}$ $(|\cdot|)$-continuous extension of I_j to $B_j^{q_{j,\ell}}$.

In all that follows we add two other basic assumptions:

(9) Given $0 \leq h \in B_1$, $0 \leq g \in B_2$ there exists $k \in B_3$ such that $g(y) \leq k(x, y)$ if $h(x) > 0$.

(10) If $0 \leq h \in B_1$ then $h \wedge 1 \in B_1$ (Stone's condition) and $q_1(h \wedge \varepsilon) \to 0$, as $\varepsilon \to 0$.

Observe that the above assumptions are fullfilled in most applications, for example for step functions or continuous functions with compact support (see remark 1.3.).

LEMMA 3.1. *If (7), (9) and (10) hold, and* $f \in \bar{\mathbb{R}}_+^{X_3}$ *such that the following condition holds*

(*) *To* f *there exists* $g \in B_2^{q_{2,\ell}}$ *such that* $f_x \leq g$ *for each* $x \in X_1$.

Then, $q_{3,\ell}(f) \geq (q_{1,\ell} \circ q_{2,\ell})(f)$.

Proof. For $g \in B_2^{q_{2,\ell}}$ and $\varepsilon > 0$, there exists $t_\varepsilon \in B_2$ such that $q_{2,\ell}(|g - t_\varepsilon|) < \varepsilon$.

Now, with $0 \leq g \leq |g - t_\varepsilon| + |t_\varepsilon|$ and (*), we have

$$f_x = f_x \wedge g \leq |g - t_\varepsilon| \wedge f_x + |t_\varepsilon| \wedge f_x \leq |g - t_\varepsilon| + |t_\varepsilon| \wedge f_x.$$

By definition, $(q_{1,\ell} \circ q_{2,\ell})(f) = q_{1,\ell}(q_{2,\ell} f) := sup\{q_1[(q_{2,\ell} f) \wedge h];\ \leq h \in B_1\}$.

To $0 \leq h \in B_1$ and $|t_\varepsilon| \in B_2$, by (9), there exists $l \in B_3$ such that $|t_\varepsilon| \leq l_x$ on X_1, if $hx) > 0$.

Thus, one gets $q_{2,\ell}(f_x) \leq q_{2,\ell}(|g - t_\varepsilon|) + q_{2,\ell}(|t_\varepsilon| \wedge f_x) < \varepsilon + q_{2,\ell}(|t_\varepsilon| \wedge f_x)$, so that, $q_1[(q_{2,\ell} f) \wedge h] \leq q_1(\varepsilon \wedge h) + q_1[q_{2,\ell}(|t_\varepsilon| \wedge f_x) \wedge h] = q_1(q_1[q_2(|t_\varepsilon| \wedge f_x) \wedge h]$,

Finally, since $f_x \wedge |t_\varepsilon| \leq f_x \wedge l$ if $h(x) > 0$, one has $q_1[(q_{2,\ell} f) \wedge h] \leq q_1(h \wedge \varepsilon) + q_1[q_2(f_x \wedge l_x) \leq q_1(h \wedge \varepsilon) + q_3(f \wedge I)$, and with $\varepsilon \to 0$ we conclude the result. ∎

In lemma 3.1. the boundedness of the section functions is necessary by example 1 below.

We shall now apply the inequality established in lemma 1.3. to give the following generalization of theorem p. 141 of Hoffmann [11] (see also Elsner [7], with (7), and [5] theorem 2).

THEOREM 3.1. (Fubini theorem for q_ℓ-integrable functions). *If (7), (9) and (10) hold, and $f \in B_3^{q_{3,\ell}}$ such that $|f_x| \leq g \in B_2^{q_{2,\ell}}$ for each $x \in X_1$, the the following assertions hold:*

i) There exist $A_k \in X_1$, $k \in \mathbb{N}$, $q_{1,\ell}$-null sets such that $f_x \in B_2^{q_{2,\ell}}$ for each $x \in X_1 - \bigcup_{k=1}^{\infty} A_k$.

ii) There exists $k \in B_1^{q_{1,\ell}}$ defined by $q_{2,\ell}(f)$ if $f_x \in B_2^{q_{2,\ell}}$, such that $q_{1,\ell}(k) = q_{3,\ell}(f)$, i.e. $I^{q_{3,\ell}}(f) = (I^{q_{1,\ell}} \circ I^{q_{2,\ell}})(f)$.

Proof The proof is similar to the one of theorem p. 141 of [11] by application of lemma 3.1. We will denote only the main steps.

i) For $f \in B_3^{q_{3,\ell}}$ and $\varepsilon > 0$, there exists $t \in B_3$ such that $q_{3,\ell}(|f - t|) < \varepsilon$.

For each $x \in X_1$ set $\phi(x) := inf\{q_{2,\ell}(|f_x - h|)$, for each $h \in B_2\}$ and set $A_k := \left\{x \in X_1; \phi(x) \geq \frac{1}{K}\right\}$, $k \in \mathbb{N}$. With lemma 1.3. the sets A_k, $k \in \mathbb{N}$, are $q_{1,\ell}$-null, and i) follows immediatelly.

ii) It is suffices to see that there is $(q_{2,\ell} k_n) \subset B$ such that $q_{1,\ell}(|(q_{2,\ell} k_n) - k|) \leq 3\ (q_{1,\ell} \circ q_{2,\ell})(|k_n - f|) \leq 3\ q_{3,\ell}(|k_n - f|)$, where $(k_n) \subset B_3$ and $q_{3,\ell}(|k_n - f|) \to 0$, as $n \to \infty$. So that, $(q_{1,\ell} \circ q_{2,\ell})(k_n) \to q_{1,\ell}(k)$, as $n \to \infty$, and $q_{1,\ell}(k) = q_{3,\ell}(f)$. ■

Remarks 3.

3.1. With $q_i = I_1^-$, $i = 1, 2, 3$, of remark 1.1., according to remark 2. 2. (7) holds, and theorem 3.1. gives a Fubini type theorem for the abstract Riemann-I-integrable functions (see remarks 1.2, 1.3.).

3.2. With remark 1.3. for the $\lambda \times \mu$-finitely additive situation, (9) and (10) hold. Here, the Elsner-condition that $|f|$ be bounded and there exists $P \in$ ring generated by Ω_2 such that $supp(f) \subset X_1 \times P$ implies that there exists $g \in R_1(B_{\Omega_2}, l_{\mu_2})$ such that $|f_x| \leq g$ for each $x \in X_1$. So that, theorem 3.2. is applicable and contains Satz 10 of Elsner [7], so, we generalize in this way results analogous to the classical case.

3.3. Let us finally remark that our results can be reformulated for Banach space valued functions, using $f \cap g := \|f\|^{-1}(\|f\| \wedge g)f$, with $f : X \to E =$ Banach space, $g \in \bar{\mathbb{R}}_+$, of [9] p. 327 or [7] p. 266.

EXAMPLE 1. (see [7] p. 270, [11] p. 141).

Let $X_1 = \mathbb{R}$, $\Omega_1 = \{]a, b[;\ a, b \in \mathbb{R},\ a \leq b\}$, $\mu_L(]a, b[) = b - a$, $X_2 = \mathbb{N}$, $\Omega_2 = \{\mathbb{N} - A;\ A$ finite set $\subset \mathbb{N}\}$, $\lambda(A) = 0$, $\lambda(\mathbb{N}) = 1$ and $X_3 = \mathbb{R} \times \mathbb{N}$.

Given $g = \sum_{n=1}^{\infty} n\chi_{\{n\}\times]n,n+1[}$ one has $I_{3,\ell}^-(g) = 0 < \infty = (l_{1,\ell}^- \circ I_{2,\ell}^-)(g)$.

EXAMPLE 2. Let $X_1 = X_2 = \mathbb{N}$, $\Omega_1 = \Omega_2 = \{\mathbb{N} - A;\ A$ finete set $\subset \mathbb{N}\}$, $\mu_1 = \mu_2 = \mu$ additive measure $\mu(A) = 0$, $\mu(\mathbb{N}) = 1$. Let $X_3 = \mathbb{N} \times \mathbb{N}$, $\Omega = \{X_3 - A;\ A$ finite set $\subset X_3\}$, $v : \Omega \to \mathbb{R}$, $v(A) = 0$, $v(X_3) = 1$.

Since $I_v^- \geq I_{\mu\times\mu}^- \geq I_\mu^- \circ I_\mu^- =: I_3^-$, we have $R_{prop}(v, \mathbb{R}) \subset R_{prop}(\mu \times \mu, \mathbb{R}) \subset B_{\Omega_1\times\Omega_2}^{I_3^-}$.

With $f(m,n) = \dfrac{1}{m \wedge n}$, one has $f \in R_{prop}(\mu \times \mu, \mathbb{R})$ but $f \notin R_{prop}(v, \mathbb{R})$, and $f(m,n) = 1 \wedge \dfrac{m}{n}$ gives $f \in B^{I_3^-}_{\Omega_1 \times e\Omega_2}$ but $f \notin R_{prop}(\mu \times \mu, \mathbb{R})$.

Also, for $f(m,n) = \dfrac{1}{m}$ if n is even and $:= 0$ if n is odd, one has $f \in R_{prop}(\mu \times \mu, \mathbb{R})$ with $I^-_{\mu \times \mu}(f) = 0$, but for no $m \in X_1$ is $f_m \in R_{prop}(\mu, \mathbb{R})$, the exceptional set is all of X_1.

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