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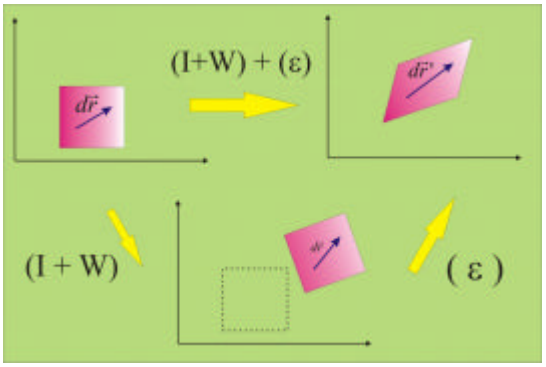
formulario 3: Deformaciones

GIRO Y DEFORMACIÓN

1. Movimiento general

$$d\vec{r}\,' = (I + W)d\vec{r} + e d\vec{r}$$



2. Matriz de giro

$$W = \begin{pmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial u_2} - \frac{\partial u_2}{\partial u_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial u_3} - \frac{\partial u_3}{\partial u_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial u_1} - \frac{\partial u_1}{\partial u_2}\right) & 0 & \frac{1}{2}\left(\frac{\partial u_2}{\partial u_3} - \frac{\partial u_3}{\partial u_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial u_1} - \frac{\partial u_1}{\partial u_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial u_2} - \frac{\partial u_2}{\partial u_3}\right) & 0 \end{pmatrix}$$

3. Matriz de deformación

$$e = \begin{pmatrix} \frac{\partial u_1}{\partial u_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial u_2} + \frac{\partial u_2}{\partial u_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial u_3} + \frac{\partial u_3}{\partial u_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial u_1} + \frac{\partial u_1}{\partial u_2}\right) & \frac{\partial u_2}{\partial u_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial u_3} + \frac{\partial u_3}{\partial u_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial u_1} + \frac{\partial u_1}{\partial u_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial u_2} + \frac{\partial u_2}{\partial u_3}\right) & \frac{\partial u_3}{\partial u_3} \end{pmatrix}$$

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial u_j} + \frac{\partial u_j}{\partial u_i}\right) = \frac{1}{2}(u_{i,j} + u_{j,i})$$

$$e = \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix} = \begin{pmatrix} e_{xx} & e_{xy} & e_{xz} \\ e_{yx} & e_{yy} & e_{yz} \\ e_{zx} & e_{zy} & e_{zz} \end{pmatrix} = \begin{pmatrix} e_{x} & e_{xy} & e_{xz} \\ e_{yx} & e_{y} & e_{yz} \\ e_{zx} & e_{zy} & e_{z} \end{pmatrix}$$

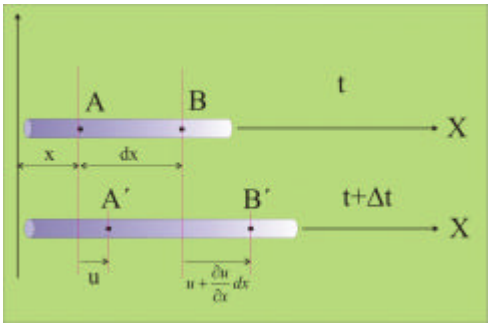
4. Componentes ingenieriles de la deformación

$$g_{ij} \equiv \left(\frac{\partial u_i}{\partial u_j} + \frac{\partial u_j}{\partial u_i} \right) = (u_{i,j} + u_{j,i}) \quad \Rightarrow \quad e = \begin{pmatrix} e_x & \frac{1}{2} g_{xy} & \frac{1}{2} g_{xz} \\ \frac{1}{2} g_{yx} & e_y & \frac{1}{2} g_{yz} \\ \frac{1}{2} g_{zx} & \frac{1}{2} g_{zy} & e_z \end{pmatrix}$$

INTERPRETACIÓN GEOMÉTRICA DE LOS ELEMENTOS DE e

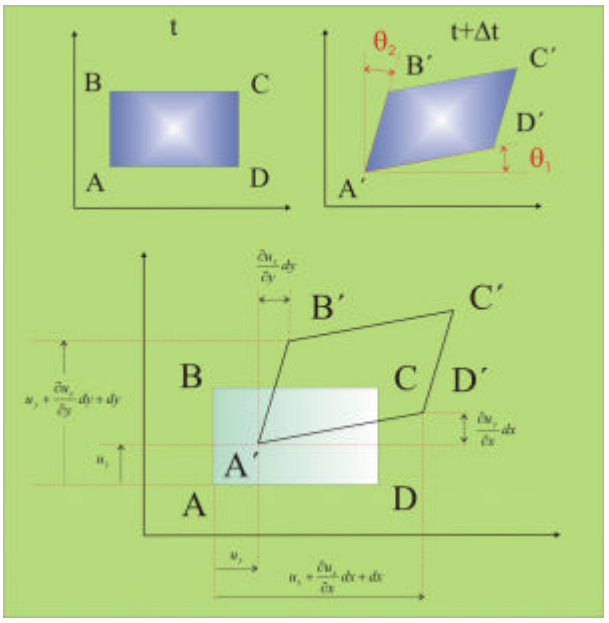
1. Elementos diagonales

$$e_x = \frac{\partial u}{\partial x} = \frac{A'B' - AB}{AB} \quad \Rightarrow \quad e_n = \frac{d\vec{r}_n^{\,'} - d\vec{r}_n}{d\vec{r}_n}$$



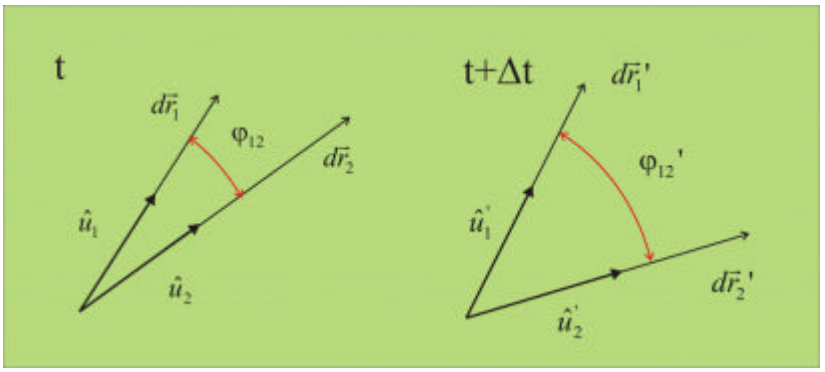
2. Elementos no diagonales

$$q_1 + q_2 = g_{xy} = 2e_{xy}$$



3. Deformación de ángulos

$$\Gamma_{12} \equiv j_{12} - j'_{12} = \frac{2\,\hat{u}_1^t\; e\; \hat{u}_2 - \left(e_{n1} + e_{n2}\right)\cos j_{12}}{sen\, j_{12}}$$



TENSOR DE DEFORMACIÓN

1. Tensor de deformación de Lagrange

$$e_{ij}^L \equiv \frac{1}{2}\left(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}\right) \quad \text{en coordenadas lagrangianas}$$

$$\left(dr'\right)^2 - \left(dr\right)^2 = 2\, e_{ij}^L\; dx_i\; dx_j$$

2. Tensor de deformación de Euler

$$e_{ij}^{E} \equiv \frac{1}{2}\left(u_{i,j} + u_{j,i} - u_{k,i}u_{k,j}\right) \quad \text{en coordenadas eulerianas}$$

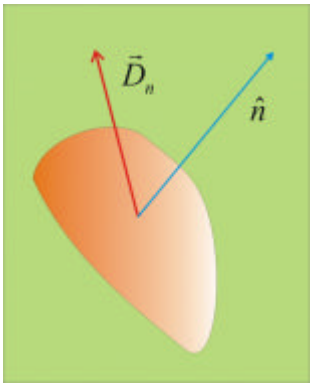
$$(dr')^2 - (dr)^2 = 2\, e_{ij}^{E}\; dX_i\; dX_j$$

3. Tensor de deformación de Cauchy

$$e_{ij} \equiv \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

ESTADO DE DEFORMACIÓN

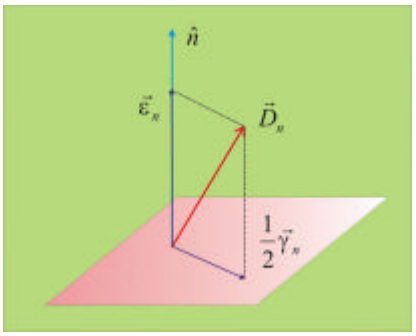
1. Ecuación de Cauchy



$$\vec{D} = e^t \hat{n} \quad ó \quad D_i = e_{ji} n_j$$

$$e = \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix}$$

2. Componentes intrínsecas del vector deformación



$$\vec{D}_n = \vec{e}_n + \frac{1}{2}\vec{g}_n \quad \Rightarrow \quad D_n^2 = e_n^2 + \left(\frac{1}{2} g_n\right)^2$$

$$e_n = \vec{D}_n \cdot \hat{n}$$

$$\left(\frac{1}{2} g_n\right)^2 = D_n^2 - e_n^2$$

DEFORMACIONES Y DIRECCIONES PRINCIPALES

1. Representación del tensor de deformaciones en las direcciones principales

$$e_{ij} = \begin{pmatrix} e_I & 0 & 0 \\ 0 & e_{II} & 0 \\ 0 & 0 & e_{III} \end{pmatrix}$$

2. Invariantes I del tensor de deformaciones

$$I_1(e) = e_{11} + e_{22} + e_{33} = e_I + e_{II} + e_{III}$$

$$I_2(e) = e_{11}e_{22} + e_{22}e_{33} + e_{33}e_{11} - e_{12}e_{21} - e_{13}e_{31} - e_{23}e_{32} =$$
$$= e_I e_{II} + e_{II} e_{III} + e_{III} e_I$$

$$I_3(e) = \begin{vmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{vmatrix} = e_I e_{II} e_{III}$$

3. Invariantes J del tensor de deformaciones

$$J_1(e) = I_1 = e_{ii}$$

$$J_2(e) = \frac{1}{2}\left(I_1^2 + 2I_2\right) = \frac{1}{2} e_{ij} e_{ji}$$

$$J_3(e) = \frac{1}{3}\left(I_1^3 + 3I_1 I_2 + 3I_3\right) = \frac{1}{3} e_{ij} e_{jk} e_{ki}$$

DEFORMACIONES MÁXIMAS

1. Deformación normal máxima

$$e_n^{máx} = D_n$$

2. Deformaciones tangenciales máximas

$$\frac{1}{2} g_n^{máx} = \pm \frac{e_I - e_{II}}{2} \qquad \frac{1}{2} g_n^{máx} = \pm \frac{e_{II} - e_{III}}{2} \qquad \frac{1}{2} g_n^{máx} = \pm \frac{e_I - e_{III}}{2}$$

DEFORMACIONES OCTAÉDRICAS

1. Deformación normal octaédrica

$$e_{no} = \frac{1}{3}\left(e_I + e_{II} + e_{III}\right)$$

2. Deformación tangencial octaédrica

$$\frac{1}{2}g_{no} = \frac{1}{3}\sqrt{(e_I - e_{II})^2 + (e_{II} - e_{III})^2 + (e_I - e_{III})^2}$$

REPRESENTACIÓN OCTAÉDRICO-DESVIADOR

1. Descomposición

$$e = e_{oct} + e_{des} \quad \begin{cases} e_{oct} = \begin{pmatrix} e_{no} & 0 & 0 \\ 0 & e_{no} & 0 \\ 0 & 0 & e_{no} \end{pmatrix} \\ e_{des} = \begin{pmatrix} e_{11} - e_{no} & e_{12} & e_{13} \\ e_{21} & e_{22} - e_{no} & e_{23} \\ e_{31} & e_{32} & e_{33} - e_{no} \end{pmatrix} \end{cases} \Rightarrow \quad e_{ij} = e_{no} d_{ij} + e_{des\,ij}$$

2. Interpretación geométrica

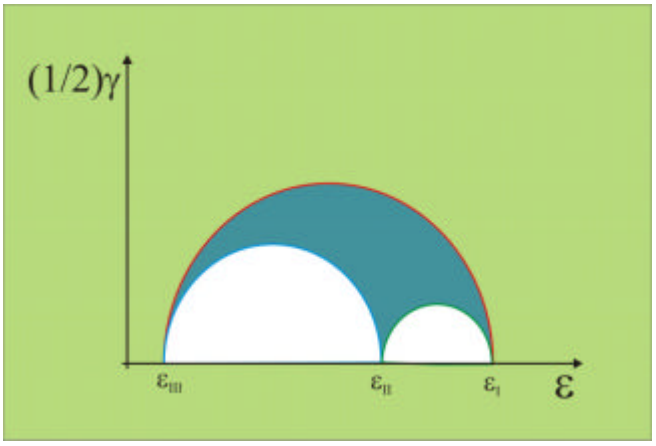
$$e = \frac{dV' - dV}{dV} = e_I + e_{II} + e_{III} = 3e_{no} = 3I_1(e)$$

REPRESENTACIÓN DE LAMÉ

$$\frac{x^2}{e_I^2} + \frac{y^2}{e_{II}^2} + \frac{z^2}{e_{III}^2} = 1$$

REPRESENTACIÓN DE MOHR

1. Círculos de Mohr 3D



2. Círculos de Mohr 2D

$$e_n = \frac{e_x + e_y}{2} + \frac{e_x - e_y}{2}\cos(2a) + \frac{1}{2}g_{yx}sen(2a)$$

$$\frac{1}{2}g_n = \frac{e_y - e_x}{2}sen(2a) + \frac{1}{2}g_{yx}\cos(2a)$$

ECUACIONES DE COMPATIBILIDAD

$$2\frac{\partial^2 e_{xy}}{\partial x \partial y} = \frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2}$$

$$2\frac{\partial^2 e_{xz}}{\partial x \partial z} = \frac{\partial^2 e_{xx}}{\partial z^2} + \frac{\partial^2 e_{zz}}{\partial x^2}$$

$$2\frac{\partial^2 e_{yz}}{\partial y \partial z} = \frac{\partial^2 e_{yy}}{\partial z^2} + \frac{\partial^2 e_{zz}}{\partial y^2}$$

$$\frac{\partial^2 e_{xx}}{\partial y \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial e_{xy}}{\partial z} + \frac{\partial e_{xz}}{\partial y} - \frac{\partial e_{yz}}{\partial x}\right)$$

$$\frac{\partial^2 e_{yy}}{\partial x \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial e_{xy}}{\partial z} - \frac{\partial e_{xz}}{\partial y} + \frac{\partial e_{yz}}{\partial x}\right)$$

$$\frac{\partial^2 e_{zz}}{\partial x \partial y} = \frac{\partial}{\partial z}\left(-\frac{\partial e_{xy}}{\partial z} + \frac{\partial e_{xz}}{\partial y} + \frac{\partial e_{yz}}{\partial x}\right)$$