Assessing the Reliability of the Single Circular-Array Method for Love-Wave Ambient-Noise Surveying

by Antonio García-Jerez, Francisco Luzón, Manuel Navarro, and Miguel A. Santoyo

Abstract  The single circular array (SCA) method is a spatial autocorrelation (SPAC)–like technique for ambient noise exploration. Its main feature is the possibility of calculation of Love-wave dispersion curves by using centerless circular arrays of 3-component seismometers, allowing independent processing of each circle. Situations in which Rayleigh-wave and Love-wave arrivals or waves coming from different azimuths are mutually correlated are also correctly dealt with in this method. An algorithm for practical calculation of the SCA coefficient $B$ is described. The algorithm includes averaging over a set of time windows and minimizes the number of spectral ratios to be computed for the purposes of stability. Numerical tests show that SCA coefficients estimated in this way have quite a robust behavior. Bias due to use of a finite number of sensors, as well as to effects of nonpropagating incoherent noise, has been theoretically studied in both the deterministic and the stationary random-field formulations. Using a finite number of stations is a cause of bias even under isotropic illumination conditions. Nevertheless, its effect can be neglected for wavelengths-to-radius ratios above a threshold that depends on the number of evenly distributed sensors. By contrast, uncorrelated noise may affect the whole frequency band and is behind the limitations of the method at low frequencies.

Finally, we present the first real data test of this method, consisting of a comparison between theoretical and experimental Love-wave dispersion curves for a site where the structure is known. In practice, the minimum wavelength for direct velocity retrieval for a pentagonal array with radius $r$ was approximately $\lambda_{min}^L \sim 3r$, although this value depends on the signal-to-noise ratio. Experiments demonstrate that the usable range can be extended, mainly toward shorter wavelengths, if the effects of noise and of the finite number of sensors are included in the analysis.

Introduction

Microtremor surveying methods have been widely shown as powerful tools for the determination of ground structures, either by themselves or together with other geophysical techniques (e.g., Noguchi and Nishida 2002; Sakai and Morikawa 2006). The simplicity and low cost of these techniques are their major merits. On the subject, the work of Aki (1957, 1965) was a major milestone in the development of methods for calculation of surface-wave dispersion curves from simultaneous ambient-noise records (spatial autocorrelation [SPAC]–like methods, Green’s function retrieval, etc.).

Provided that a large enough number of stations are available, some of these methods have the capability of retrieving dispersion curves whatever the wave-field composition (Rayleigh-to-Love power ratio) is and for both isotropic and anisotropic illumination conditions. In this article, we will restrict our attention to these methods and more specifically to SPAC-type techniques.

To date, most of the practical studies using microtremors take advantage of Rayleigh-wave dispersion curves obtained from vertical-component records. The original method (vertical SPAC, or v-SPAC, with azimuthal averaging; Aki, 1965) required a circular or semicircular array of vertical-component sensors surrounding another central device. The azimuthally averaged cross correlation between the central sensor and those on the circumference is directly related to the Rayleigh-wave velocity. Some improvements and generalizations to the v-SPAC, as well as some other SPAC-like methods for vertical components, can be found in specific literature (e.g., Henstridge 1979; Bettig et al. 2001; Morikawa et al. 2004; Tada et al. 2007; Cho et al. 2006a).

Analysis of horizontal components of microtremors is comparatively harder due to the mixing of Love and Rayleigh waves in an unknown proportion. Several authors have studied the composition of the horizontal component from the experimental point of view. Miyadera and Tokimatsu
(1992) found a slight dominance of the Love-wave spectral power (ranging from 50% to 70%). Köhler et al. (2007) estimated Rayleigh-wave contents between 10% and 35% in the 0.5–2-Hz spectral band from Pulheim, Germany. A recent study by Endrun and Ohrnerberger (2009), performed in 20 different sites in Europe, points to frequency-dependent Rayleigh-wave contributions ranging from 10% to 60% in the 1–15 Hz band and around 50% at larger and lower frequencies. Details for some of these sites can be found in Endrun (2010). After an exhaustive literature review, Bonnefoy-Claudet et al. (2006) concluded that Love waves are predominant in the noise wave field for frequencies above 1 Hz. Nevertheless, there is some variability in results (mainly for lower frequencies) and cases of slightly predominant Rayleigh waves can also be found in the literature (e.g., Cornou, 2002). In spite of such a complicated behavior, the development of methods for horizontal-component processing is a topic of great interest because inversion of both Rayleigh-wave and Love-wave dispersion curves together should lead to better constrained ground models, considering their sensitivities to different structural characteristics. Pei (2007) has performed several synthetic tests, showing that the additional Love-wave constraint results in a significant improvement of inverted models in terms of resolution of low-velocity zones and high-velocity contrasts. Another synthetic test conducted by Köhler et al. (2007) also shows that joint inversion provides better resolution of the interface depths compared to separate inversion runs. These improvements were more evident when the whole set of good models was considered instead of giving the best fitting one as the only result. These authors also suggest that starting with Love-wave inversion may allow a correct identification of higher modes due to the broader mode separation of these surface waves.

Several SPAC-like methods are already available for dealing with Love waves. Okada and Matsushima (1989) showed that the original SPAC method can be adapted to analyze horizontal components in general wave-field conditions. In later years, several new methods have been developed, too. Tada et al. (2006) and García-Jerez et al. (2006, 2008a) developed some methods to work with an array setup made up of two concentric centerless circular distributions of sensors. Recently, García-Jerez et al. (2008b) have proposed a new method (the single circular array method, hereafter shortened to SCA) suitable for being used with a simpler array setup consisting of a single circular array without a central station. Contrary to the SPAC method, this technique does not require any previous estimate of Rayleigh-wave velocities to calculate Love-wave dispersion curves, provided that the number of stations used is large enough. The SCA method remains valid for both random wave fields (as shown in the following sections) and deterministic wave fields (García-Jerez et al., 2008b). Some other methods with similar capabilities have been proposed by Tada et al. (2009). Even though all these techniques are promising, real data tests of these methods are still scarce, and theoretical studies on the performance and robustness are still needed.

In this article, we present a robust scheme for practical implementation of the SCA method, as well as a detailed study of the main causes of possible biases in the estimated phase velocities. The effects of finite number of stations and uncorrelated noise are accounted for in both analytical and numerical approaches. Previous research on this topic, applied to other array methods, has been carried out by Okada (2006), Cho et al. (2006b; analytical point of view), Asten (2003, 2006), and Asten et al. (2004; numerical tests for the performance of the v-SPAC method, amongst others).

Finally, the improvements achieved by partial compensation of these effects are discussed for a real dataset obtained in the delta of the Andarax River (Almería, Spain).

A Robust Implementation of the Single Circular Array Method

The SCA method was originally formulated by García-Jerez et al. (2008b) under a deterministic description of the wave field. This method consists of a way for working out the Love-wave phase velocity under incidence of any arbitrary set of Rayleigh and Love plane waves recorded on a centerless circular array of seismometers. Nevertheless, the optimum application to real ambient-noise records requires a careful implementation of the method, defining suitable quantities to be stacked and minimizing the number of spectral ratios to be evaluated in order to improve the stability of the solution. The algorithm should be simple enough to make possible further analytical studies such as the statistical analysis of uncorrelated noise effects. For that purpose, the main definitions and basic results of the formulation of the SCA method will be outlined first.

At the circular frequency $\omega = 2\pi f$, the complex quantities $A_{L,j}(\omega; \tau)$ and $A_{R,j}(\omega; \tau)$ represent the amplitudes of a plane Love wave and the horizontal component of a Rayleigh wave, respectively, generated by the $j$-th source and spreading towards the azimuth $\varphi_j$ (see Fig. 1). Symbol $\tau$ represents the length of the time window used for evaluation of Fourier transforms, by applying $\int_{-\tau/2}^{\tau/2} e^{-i\omega t} dt$. The corresponding complex amplitude for the vertical component is $iA_{R,j}(\omega; \tau) / \chi(\omega)$, where the real quantity $\chi(\omega)$ represents the Rayleigh wave ellipticity for the circular frequency $\omega$.

Then, the weighted sum of amplitudes $A_{R}^m(\omega; \tau)$ and $A_{L}^m(\omega; \tau)$ for a set of $M$ sources can be defined as

$$A_{R}^m(\omega; \tau) = \sum_{j=1}^{M} \exp(-im\varphi_j)A_{X,j}(\omega; \tau),$$

with $X = R$ or $L$.

Let us name the radial, tangential, and vertical components of the motion recorder in a circumference of radius $r$ as $R(r, \theta, \omega; \tau)$, $T(r, \theta, \omega; \tau)$, and $Z(r, \theta, \omega; \tau)$, respectively. If the records were available at all the points on the circumference, the Fourier-series coefficients of the expansion of these
functions on the azimuthal coordinate $\theta$ could be empirically evaluated. They are defined as
\[
X_m(r, \omega; \tau) = \int_{-\pi}^{\pi} e^{-im\theta} X(r, \omega; \tau) d\theta,
\]
where $X = Z, R, \text{or } T$. As in García-Jerez et al. (2008b), these coefficients can be related with the sums of weighted amplitudes $A_{Rm}^n(\omega; \tau)$ and $A_{tm}^n(\omega; \tau)$: the Rayleigh wave ellipticity $\chi(\omega)$; and the Rayleigh-wave and Love-wave velocities $c_R(\omega)$ and $c_L(\omega)$. For the sake of completeness, these relations are listed in Appendix A.

The SCA method provides a straightforward calculation of the Love-wave phase velocity from the subset of equations generated by evaluation of $Z_{11}(r, \omega; \tau), R_{11}(r, \omega; \tau),$ and $T_{11}(r, \omega; \tau)$.

The key equation of this method (see equations 15 and 16 in García-Jerez et al. 2008b) can be rewritten as
\[
\frac{\text{Im}[(T_{-1}Z_{+1} + T_{+1}Z_{-1})(R_{-1}Z_{+1} - R_{+1}Z_{-1})^*]}{|R_{-1}Z_{+1} - R_{+1}Z_{-1}|^2} = \frac{J_0(x_L)}{J_0(x_L) + J_2(x_L)},
\]
where $x_L = \omega r/c_L(\omega)$, $x_R = \omega r/c_R(\omega)$, the asterisk indicates complex conjugation, and $J_n(x)$ represents the $n$-th order Bessel function. Some changes in the nomenclature in comparison with García-Jerez et al. (2008b) have been listed in Table 1. The counterpart of the left side of equation (3) in the original formulation of the SCA method (equation 16 in García-Jerez et al. 2008b) is an equivalent fraction with denominator $R_{-1}Z_{+1} - R_{+1}Z_{-1}$. Because quantity $R_{-1}Z_{+1} - R_{+1}Z_{-1}$ preserves phase information from the records, it is not suitable for the time-windows averaging operation described subsequently. Thus, dealing with equation (3) is more suitable for the purposes of this article.

For convenience, we shall abbreviate the numerator and denominator on the left side of equation (3), after a normalization by $4\pi^2\tau^2$, as
\[
n_{Bij} = \frac{1}{4\pi^2\tau^2} \text{Im}[(T_{-1}Z_{+1} + T_{+1}Z_{-1})(R_{-1}Z_{+1} - R_{+1}Z_{-1})^*],
\]
and
\[
d_{Bij} = \frac{1}{4\pi^2\tau^2} |R_{-1}Z_{+1} - R_{+1}Z_{-1}|^2.
\]

An important aim is to determine how this method should be applied to a description of the microtremor wave field in terms of stochastic processes. If the wave field were stationary, the scheme should allow a simple link with the quantities usually defined for these processes (i.e., the frequency-direction power spectral densities of the signal and the power spectral density of the uncorrelated noise) with moderate analytical effort and should prove a robust behavior in simulations and real-data tests. The simplest attempt to define how the method is applied to a stochastic process (i.e., to a set of realizations of the wave field) is to compute $(n_{Bij}/d_{Bij})$, where notation $(\cdot)$ stands for averaging over a set of time windows of length $\tau$, corresponding to different realizations of the stochastic process. If microtremors are ergodic, $(\cdot)$ can be evaluated by using different time windows of a single microtremor record. Nevertheless, we verified that the robustness of this quantity is worse than in other possible implementations. Moreover, the presence of a quotient inside the expectation

### Table 1

<table>
<thead>
<tr>
<th>Description</th>
<th>In This Article</th>
<th>In García-Jerez et al. (2008b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radius of the array</td>
<td>$r$</td>
<td>$R$</td>
</tr>
<tr>
<td>Vertical-component records</td>
<td>$Z$</td>
<td>$W$</td>
</tr>
<tr>
<td>Radial-component records</td>
<td>$R$</td>
<td>$U^{mk}$</td>
</tr>
<tr>
<td>Tangential-component records</td>
<td>$T$</td>
<td>$U^{\text{tg}}$</td>
</tr>
<tr>
<td>Number of stations</td>
<td>$N$</td>
<td>$N$</td>
</tr>
<tr>
<td>Number of source azimuths</td>
<td>$M$</td>
<td>$N$</td>
</tr>
<tr>
<td>Complex amplitude of the $j$-th Rayleigh wave in horizontal component</td>
<td>$A_{Rj}^j$</td>
<td>$A_{Rj}^H$</td>
</tr>
<tr>
<td>$m$-th order weighted sum of complex amplitudes of Rayleigh wave in horizontal component</td>
<td>$A_n^m$</td>
<td>$A_R^m$</td>
</tr>
</tbody>
</table>
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Instead, we propose defining the quantity \( B_{II} \) as a suitable estimator for the left side of equation (3) as

\[
B_{II} = \frac{\langle n_{B_{ij}} \rangle}{d_{B_{ij}}}. \tag{6}
\]

Because \( d_{B_{ij}} \) is, by definition, a positive (or, rarely, zero) quantity, it can be evaluated for the set of windows and subsequently averaged without putting at risk the stability of the later calculation of the ratio. We have checked that definition (6) is more stable than \( \langle n_{B_{ij}}/d_{B_{ij}} \rangle \) by means of numerical and real-data tests (not shown here). In addition, the analytical study of the effects of uncorrelated noise and effects of using a finite number of stations, as well as its adaptation to the usual hypotheses made for stochastic microtremor wave fields, are simpler. These will be the subjects of the following sections.

Before taking the ratio in definition (6), the numerator and denominator of \( B_{II} \) are still dependent on all the wave-field parameters. In fact, by using the relations listed in Appendix A and some algebra, we conclude that their expressions for a given time window are

\[
n_{B_{ij}} = \frac{\pi^2}{\chi^2} \left| A_L^+ A_R^- \right|^2 f_L^2(x_R) J_0^2(x_L) - J_2^2(x_L), \tag{7}
\]

and

\[
d_{B_{ij}} = \frac{\pi^2}{\chi^2} \left| A_L^+ A_R^- \right|^2 + A_L^2 A_R^- J_1^2(x_R) + J_2^2(x_L). \tag{8}
\]

Some aspects of the significance of these functional forms are discussed in the next section.

Relation between Deterministic and Stationary Random Field Descriptions

Because the SCA method was formulated in a deterministic context, definitions of frequency–direction power spectral densities for Rayleigh and Love waves can be initially disregarded. Nevertheless, such a formulation in terms of stationary random fields allows an easier description of simple cases and makes possible further comparisons with other techniques. Thus, we briefly explore in this section a way of rewriting the main deterministic equations of the SCA method in a context in which (1) the microtremor is stationary in space and time and (2) Rayleigh and Love waves are mutually uncorrelated.

We will use the notations \( f^L(\omega, \varphi) \) and \( f^R(\omega, \varphi) \) for the frequency–direction power spectral densities of Love and horizontal components of Rayleigh waves, respectively. Once multiplied by \( d\varphi d\omega \), these real quantities represent the respective intensities of plane-wave components spreading to azimuths in the range \( \varphi \) to \( \varphi + d\varphi \) with frequencies from \( \omega \) to \( \omega + d\omega \). Both \( f^L(\omega, \varphi) \) and \( f^R(\omega, \varphi) \) are non-negative and even functions of \( \omega \). The \( m \)-order Fourier-series coefficients of the spectral densities, \( f^L_m(\omega) \) and \( f^R_m(\omega) \), are defined as

\[
f^L_m(\omega) = \int_{-\pi}^{\pi} e^{-im\varphi} f^L(\omega, \varphi) d\varphi, \tag{9}
\]

and

\[
f^R_m(\omega) = \int_{-\pi}^{\pi} e^{-im\varphi} f^R(\omega, \varphi) d\varphi, \tag{10}
\]

with \( X = R \) or \( L \). Here \( f^L_m(\omega) = f^L_m(\omega) \) and, in the case of an isotropic wavefield, \( f^L_0(\omega) \) is the only nonzero coefficient.

Under the hypotheses mentioned previously, the following correspondences between all the averaged products \( A^m_u A^m_v \) and \( A^m_w A^m_o A^m_r \), with \( X, Y, W \) and \( Z = R \) (Rayleigh) or \( L \) (Love), and the Fourier coefficients of the frequency–direction power spectral densities can be stated:

\[
\left\{ \frac{1}{2\pi} A^m_u A^m_v \right\} \to \delta_{XY} f^L_m-n, \tag{11}
\]

\[
\left\{ \frac{1}{4\pi^2} A^m_w A^m_o A^m_r \right\} \to \delta_{WY} \delta_{ZX} f^W_{j-m} f^L_{l-n} + \delta_{WZ} \delta_{XY} f^W_{j-m} f^L_{l-n}, \tag{12}
\]

where \( \delta \) is the Kronecker delta and \( j, l, m, \) and \( n \) are integer numbers. Symbol \( \to \) stands for limit as \( \tau \) tends to infinity and \( \Delta \varphi \) tends to zero, where \( \varphi \) has to be interpreted as the azimuthal coordinate of the Fourier–Stieltjes representation of the wave field (see Appendix B). Equation (10) plays an important role in most of the formulations of the SPAC-like methods (e.g., it is analogous to equation 53 in the paper by Cho et al. 2006b). Similarly, equation (11) is necessary for the translation of the SCA method into the stationary random-field framework, although that equation does not have a counterpart in the standard derivations. Further details on the derivation of equation (11) are given in Appendix B.

Counterparts of \( n_{B_{ij}} \) and \( d_{B_{ij}} \) in a Stationary Random Wave Field

On the basis of equation (11), the expressions of \( n_{B_{ij}} \) and \( d_{B_{ij}} \) given by equations (7) and (8) can be rewritten as

\[
\langle n_{B_{ij}} \rangle = \frac{8\pi^4}{\chi^2} \left\{ f^R_0 f^R_0 + \text{Re}[f^R_2 f^R_2] \right\} J_0^2(x_R) [J_0^2(x_L) - J_2^2(x_L)] \tag{12}
\]

and

\[
\langle d_{B_{ij}} \rangle = \frac{8\pi^4}{\chi^2} \left\{ f^R_0 f^R_0 + \text{Re}[f^R_2 f^R_2] \right\} J_1^2(x_R) [J_0^2(x_L) + J_2^2(x_L)]. \tag{13}
\]

Unlike the centerless circular array method for Love waves (CCA-L) developed by Tada et al. (2009), our numerator and denominator depend on \( f^R_0 \) and \( f^R_2 \) instead of only on higher-order Fourier coefficients of the spectral densities. It implies that \( B_{II} \) does remain stable under isotropic illumination. The superiority of the SCA method in such wave-field conditions
will be illustrated subsequently in a numerical test. On the other hand, loss of accuracy is still possible at the zero crossing of $J_1(x_R)$ or due to near-zero spectral powers of Love waves ($f^L(\omega, \varphi) \approx 0$) or Rayleigh waves in the vertical component ($f^R(\omega, \varphi) / \chi^2(\omega) \approx 0$). The latter restriction also holds for the v-SPAC method.

**Biases in the Estimates of $B_{II}$**

The finite number of sensors in any real array setup may represent an important source of bias in both the estimate of $B_{II}$ and the derived Love-wave phase velocity. This effect is expressed as a dependence on the wave-field directionality, which becomes more significant as the radius-to-wavelength ratio increases. In the case of methods for Love-wave analysis, a scarce number of stations also implies an undesired dependence on the Rayleigh-to-Love power ratio (RLR) and on the Rayleigh-wave velocity. This fact was noticed by Aki (1957), who pointed out that cross correlation between radial (or tangential) components recorded at two stations had different expressions, depending on the type of dominant surface waves. García-Jerez et al. (2008b) evaluated this source of bias, also called directional aliasing or finite-N effects, for the SCA method by means of some simple numerical tests. Illumination consisted of a single monochromatic plane wave (with Rayleigh and Love components), and four incidence azimuths were checked.

We shall now study finite-N effects on the estimates of $B_{II}$ in an analytical and more general way, closely following the methodologies developed by Cho et al. (2006b) and Okada (2006). On the basis of equation (11), it will be possible to evaluate aliasing effects for wave fields described as a stationary random process. These effects will be mapped for isotropic wave fields and for several array setups.

**Effects of a Finite Number of Sensors**

Let us consider a centerless circular array made up of $N$ evenly distributed sensors. The first step in the application of this method requires evaluation of $n_{B_{II}}$ and $d_{B_{II}}$ for the set of time windows following their definitions (equations 4 and 5). Because records are not available everywhere on the circumference, the $m$-order Fourier-series coefficients of the components of motion (defined in equation 2), must be estimated by replacing the integration with a sum over the set of sensors:

$$\tilde{X}_n(r, \omega; \tau) \equiv \frac{2\pi}{N} \sum_{j=1}^{N} e^{-imj\Delta\theta}X(r, j\Delta\theta, \omega; \tau), \quad (14)$$

where $X$ represents $R$, $T$, $Z$; $\Delta\theta = 2\pi/N$; and the $^*$ symbol identifies the biased quantities obtained from the $N$-sensor array. If quantities $n_{B_{II}}$ and $d_{B_{II}}$ are evaluated from biased Fourier-series coefficients, they will be renamed to $\tilde{n}_{B_{II}}$ and $\tilde{d}_{B_{II}}$. Then, the biased counterpart of $B_{II}$ is defined as

$$\tilde{B}_{II} \equiv (\tilde{n}_{B_{II}})/(\tilde{d}_{B_{II}}). \quad (15)$$

We are now interested in obtaining a way to relate $\tilde{B}_{II}$ to the array and wave-field characteristics. For that purpose, we note that the biased quantity $X_{BII}(r, \omega; \tau)$ can be expressed as the sum of a set of unbiased Fourier-Series coefficients $X_{\omega, j; n}(r, \omega; \tau)$, with $j = 0, \pm 1, \pm 2, \ldots$ (see Cho et al. 2006b). Thus, quantities $\tilde{n}_{B_{II}}$ and $\tilde{d}_{B_{II}}$ can be expressed as

$$\tilde{n}_{B_{II}} = \frac{1}{4\pi^2} \sum_{j,l,m,n=-\infty}^{+\infty} \text{Im}[(T_{1+j} Z_{1+j} \chi_{1+j} + R_{1+j} Z_{1+j} \chi_{1+j})], \quad (16)$$

and

$$\tilde{d}_{B_{II}} = \frac{1}{4\pi^2} \sum_{j,l,m,n=-\infty}^{+\infty} (R_{1+j} Z_{1+j} \chi_{1+j} - R_{1+j} Z_{1+j} \chi_{1+j}) \times (R_{1+j} Z_{1+j} \chi_{1+j} - R_{1+j} Z_{1+j} \chi_{1+j}). \quad (17)$$

Here, equations (A1–A3) in Appendix A provide the relations to express equations (16) and (17) in terms of $c_L(\omega)$, $c_R(\omega)$, $\chi(\omega)$, and the products of four sums of weighted amplitudes ($A_w B_x A_y A_z$), where $j$, $l$, $m$, and $n$ are integers and $W, X, Y,$ and $Z$ take the values $R$ (Rayleigh) or $L$, (Love). Moreover, equations (16) and (17) can also be used if a microtremor is described as a stationary random field. In such a case, equation (11) provides the necessary link between $A_w B_x A_y A_z$ and the Fourier-series coefficients of the frequency–direction power spectral densities. The replacements of equations (A1–A3) and (11) in equations (16) and (17) lead to two somewhat lengthy expressions, which are not explicitly given here. Nevertheless, these calculations can be programmed without great effort. In the particular case of an isotropic wave field, $\tilde{B}_{II}$ can be written as a function of $x_L$, $x_R$, and $RLR = f^R_0 / f^L_0$. Alternatively, the dependence of $\tilde{B}_{II}$ on the RLR can be expressed as dependence on the relative power of Rayleigh waves in the total horizontal component, that is $f^R_0 / (f^L_0 + f^R_0)$ or $RLR/(1 + RLR)$.

Figure 2 quantifies the effects of a finite number of stations for an isotropic wave field in different array setups made up of $N$ evenly distributed sensors. Left panels show $\tilde{B}_{II}$ for different values of $N$, RLR, $x_L$, and $x_R$. Changes of sign are indicated with solid black lines. Right panels show the difference between the actual value of $\tilde{B}_{II}$ for a given RLR and the corresponding result for $RLR = 0$ (wave field free of Rayleigh waves). This difference is also plotted versus $x_L$ and $x_R$. Note that, in the reference case $RLR = 0$, $B_{II}$ becomes independent of $x_R$. Each subplot corresponds with the value of the RLR between 0.1 (91% Love waves and 9% Rayleigh waves) and 1.3 (43% Love waves and 57% Rayleigh waves). This range has been chosen according to the predominance of Love waves in microtremors found in most of the empirical studies (see Introduction). In the pentagonal case ($N = 5$), we observe a very slight dependence of $\tilde{B}_{II}$ on
the $RLR$ up to $x_L \cong 2.6$, assuming that $x_R \leq x_L$ (i.e., for $c_R \geq c_L$ or below the white diagonal segments marked in the right panels of Figure 2). The situation indeed gets better as $c_R / c_L$ increases. For example, if $x_R / x_L = 0.75$, $\hat{B}_{II}$ shows variations within $\pm 0.1$ up to $x_L = 3.65$ when $RLR$ varies in the whole range $0.1 < RLR < 1.3$. This example is also marked in the figure with solid white segments. Tests with 7, 9, and 11 stations show a gradual improvement of the capabilities for rejection of Rayleigh waves (lower bands in the right panels of Figure 2 increasing in size or, equivalently, dependence on $x_R$ diminishing in all panels). Again, the zone $x_R < x_L$ shows faster improvements as $N$ increases.

Tests conducted by García-Jerez et al. (2008b) using a single plane wave showed a significant instability at $(x_L, x_R) = (1.6, 3.2)$ for $N = 5$, $RLR = 1$, and $\varphi = 12^\circ$ (as well as for other equivalent incidence azimuths). We have verified that the behavior of $\hat{B}_{II}$ at this point becomes rapidly smoother as the azimuthal range from which the waves are coming increases in width, but some distortion can still be distinguished under isotropic illumination. That point has been marked with stars in Figure 2 (in panels corresponding to $RLR = 0.9$), confirming that it lies in an area of important aliasing effects. Although situations in which Rayleigh waves are much slower than Love waves, as that described previously ($c_R = c_L / 2$), may represent a difficulty for applying this methodology with a limited number of stations, they are probably irrelevant in practice.

Effects of Nonpropagating Incoherent Noise

The microtremor records are always contaminated with incoherent noise due to either electronic processes of the

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**Figure 2.** Variation in $\hat{B}_{II}$ as a function of $x_L$ and $x_R$. Wave field is assumed to be isotropic. Each panel corresponds to a number of sensors in the array ($N$) and a Rayleigh-to-Love power ratio $RLR$. Left panels show $B_{II}$, whereas right panels show its variation from the result for $RLR = 0$ (keeping $N$ unchanged). Thin black lines in left panels show the changes of sign of $B_{II}$ (zero crossings and divergences). Horizontal white lines show the positions of the zeros of $J_1(x_L)$. The positions of the divergences of $\hat{B}_{II}$ for $N \to \infty$ (divergences of the right side of equation 3) are marked with vertical white lines. Solid and dotted gray lines in those panels corresponding to $N = 5$ show the $x_R$ vs. $x_L$ relationships for both the numerical and the 50-m real-data tests performed, respectively. The white star shows a particular couple $(x_L, x_R)$ commented on in the text. The color version of this figure is available only in the electronic edition.

(Continued)
recording system or some external phenomena violating the plane-wave postulation (wind, near sources, etc.). These effects have been already studied for several SPAC-like methods (Cho et al. 2006a, 2006b) and are often evidenced by decreased amplitudes in the (oscillating) correlation coefficients. We follow a similar scheme, based on three assumptions: (1) uncorrelated noise can be modeled as a stationary random process uncorrelated with the signal; (2) noise waveforms appearing in records of different stations or in different components of a given station are mutually uncorrelated; (3) power spectral densities of noise are the same for all horizontal sensors and for all vertical sensors. These power spectral densities will be named $P_\text{n}/(n)$, respectively, where the superscript $n$ stands for noise. In our framework, effects of uncorrelated noise are revealed as additive terms in the estimators of $\hat{h}_\text{BII}^{n}$ and $\hat{d}_\text{BII}^{n}$. Provided that their noisy counterparts (i.e., those including signal and noise) are noted as $\hat{h}_\text{BII}^{(s+n)}$ and $\hat{d}_\text{BII}^{(s+n)}$, respectively, the contribution of noise is given by the following relations (see Appendix C):

$$\hat{h}_\text{BII}^{n} \approx \hat{h}_\text{BII}^{s} + \frac{2\pi}{N_T} p_\text{V}(n) \text{Re}[i(\hat{T}_{-1} \hat{R}_{-1}^{n} - \hat{T}_{1} \hat{R}_{1}^{n})].$$  \hspace{1cm} (18)

and

$$\hat{d}_\text{BII}^{n} \approx \hat{d}_\text{BII}^{s} + \frac{32\pi^4}{N_T^2} p_\text{H}(n) p_\text{V}(n) + \frac{2\pi}{N_T} p_\text{H}(n) (\hat{Z}_{-1} \hat{Z}_{-1}^{n} + \hat{Z}_{1} \hat{Z}_{1}^{n}) + \frac{2\pi}{N_T} p_\text{V}(n) (\hat{R}_{-1} \hat{R}_{-1}^{n} + \hat{R}_{1} \hat{R}_{1}^{n}).$$  \hspace{1cm} (19)
The equalities hold for $\tau \to \infty$. Note that all Fourier-series coefficients on the right sides of these equations are evaluated in the corresponding free of noise situation. Finally, the noisy estimator of $B_{II}$ is directly obtained as $\hat{B}_{II}^{(s+n)} = (\hat{h}_{RI})^{(s+n)}/(\hat{d}_{RI})^{(s+n)}$. Once more, terms in equations (18) and (19) involving azimuthally averaged weighted records can be rewritten as functions of the wave-field characteristics $c_L(\omega), c_R(\omega); \chi(\omega)$; and products of two sums of weighted amplitudes $A_L^k(\omega; \tau)A_R^l(\omega; \tau)$ (where $j$ and $l$ are integer numbers and $X$ and $Y$ stand for Rayleigh or Love). First, quantities $\hat{X}_m(r, \omega; \tau)$ are replaced with $\sum_{j=-\infty}^{+\infty} X_{m+jN}(r, \omega; \tau)$, where $X$ stands for $R$, $T$, or $Z$; then, expressions (A1–A3) in Appendix A are replaced. In the particular case of a wave field described as a stationary random field, such calculations yield (after some algebra)

$$\frac{i}{2\pi} \hat{B}_{II} = \frac{2\pi}{2}\sum_{\xi=-\infty}^{+\infty} i^{-\xi N} \{\text{Re}[f_{\xi N}\hat{h}_{\xi N}^{+1}(x_R)] + \text{Re}[f_{\xi N}\hat{h}_{\xi N}^{+1}(x_L)]\},$$

(20)

and

$$\frac{i}{2\pi} \hat{B}_{II} = \frac{2\pi}{2}\sum_{\xi=-\infty}^{+\infty} i^{-\xi N} \{\text{Re}[f_{\xi N}\hat{h}_{\xi N}^{+1}(x_L)] + \text{Re}[f_{\xi N}\hat{h}_{\xi N}^{+1}(x_R)]\},$$

(21)

where functions $g_{\xi N}(x)$ and $h_{\xi N}^{+1}(x)$ have been defined as

$$g_{\xi N}(x) = \sum_{l=-\infty}^{+\infty} J_{(\xi+l)N+1}(x)J_{IN+1}(x),$$

(23)

and

$$h_{\xi N}^{+1}(x) = \sum_{l=-\infty}^{+\infty} (J_{(\xi+l)N}(x) + s_1 J_{(\xi+l)N+2}(x)(J_{IN}(x)$$

$$+ s_2 J_{IN+2}(x))),$$

(24)

with $s_1$ and $s_2$ taking values $-1$ or $+1$. This later step has been carried out by using relation (10) in order to introduce the Fourier-series coefficients of the frequency–azimuthal spectral densities. Note that imaginary terms in equations (21) and (22) cancel out due to the properties $g_{\xi N}(x) = g_{-\xi N}(x)$ and $h_{\xi N}^{+1}(x) = h_{-\xi N}^{+1}(x)$. In particular, for an isotropic wave field, $\hat{B}_{II}^{(s+n)}$ can be rewritten as a function on three meaningful quantities only, apart from $N$, $x_L$, and $x_R$. Such quantities are the noise-to-signal power ratios for the horizontal ($\text{NSRH}$) and vertical ($\text{NSRV}$) components, defined as $\text{NSRH} = f_{\xi}^R/(f_0^R + f_0^L)$ and $\text{NSRV} = f_{\xi}^V/ f_0^R$, respectively, as well as the Rayleigh-to-Love power ratio $\text{RLR} = f_0^R/f_0^L$.

Finally, it should be mentioned that a different way of assembling averaging was proposed for the SCA method by García-Jerez et al. (2008b, section 4.2). The robustness of that algorithm, which is not equivalent to the calculation of $B_{II}$ defined previously, was checked under uncorrelated noise by means of a simple numerical test by using a single plane wave and introducing random time shifts into the records of the different stations. Although the performance of both the algorithms may be comparable (both of them generate stable estimators), the complexity of that technique prevents us for obtaining suitable analytical expressions for uncorrelated noise and finite-$N$ effects.

A Numerical Example for a Shallow Survey Using a Five-Sensor Array

Effects of a finite number of stations are exemplified in Figures 3 and 4 for a one-layer over half-space ground structure listed in Table 2. Figure 3a shows the reference Rayleigh-wave and Love-wave fundamental-mode phase velocities, as well as the Rayleigh-wave ellipticity. Fundamental modes will be considered dominant in the following calculations.

![Figure 3](image-url)

**Figure 3.** (a) Fundamental-mode dispersion curves and ellipticity ($\chi$) for the ground model listed in Table 2. (b) SCA coefficient $\hat{B}_{II}$ calculated from the curves in panel (a) for a virtual array consisting of $N$ evenly distributed stations on a circumference of 250-m radius. Isotropic wave field and an RLR of 0.2 have been assumed.
Figure 4. Panels (a–d): modeled $\hat{B}_{11}$ vs. frequency relation for the ground structure shown in Table 2 and a wave field made up of a finite sum of plane waves with random phases. Four illumination azimuthal widths ($\Delta$) have been considered: $2^\circ$, $30^\circ$, $60^\circ$, and $90^\circ$. Different symbols correspond to respective central azimuths of the triangle-shaped incident wave field: $\varphi = 0^\circ$, $9^\circ$, $18^\circ$, $27^\circ$, and $36^\circ$, provided that one of the five stations is placed at $0^\circ$ azimuth. The thick gray line shows the isotropic wave-field case. The virtual array radius is 250 m. Predominance of the fundamental modes and $RLR = 0.2$ are assumed. Panels (e–h): respective theoretical solutions for continuous triangle-shaped frequency–direction power spectral densities.
Figure 3b shows the theoretical shapes of $\hat{B}_{II}$ for the incidence of an isotropic wave field in an $N$-sensor array, with $N = 3, 4, 5, 6$ or 7. A virtual array radius of 250 m and an RLR of 0.2 have been used. Contrary to the SPAC coefficients, $\hat{B}_{II}$ does not converge to the $N = \infty$ case (right side of equation 3 and solid gray line in Fig. 3b) as the wave field tends to become more isotropic but only as the number of stations grows. As shown, shapes of functions $\hat{B}_{II}$ corresponding to $N > 4$ remain similar almost up to the first trough ($x_L = 3.83$), whereas different smooth versions of the $N = \infty$ case hold for higher frequencies. Therefore, at least a pentagonal array is required to get negligible finite-$N$ effects in a considerable wavelength range. Note that the hexagonal array gives a wider range of fit to the $N = \infty$ curve than does a pentagonal and a triangular array, contrary to the three-component SPAC (3c-SPAC) method developed by Okada and Matsushima (1989). When that method is formulated under our hypotheses for stationary random wave fields, the correlations between the central station and two stations at diametrically opposite azimuths provide redundant information (see Appendix C in the paper by Cho et al. 2006b). This is the reason why a regular hexagonal array shows the same aliasing effects as a triangular array in the SPAC theory. Because the SCA method involves the correlations between all pairs of stations in the centerless array, these symmetrically placed sensors cannot be ignored. In addition, correlations between parallel equidistant pairs are not equivalent for the SCA method due to the azimuthal weighting. Consequently, we expect that finite-$N$ effects will always tend to decrease as the number of sensors increases.

Figure 4, shows the effects of a finite number of stations for an array made up of five evenly distributed sensors under anisotropic illumination with a predominant propagation direction. We used $f^R(\omega, \phi)$ and $f^L(\omega, \phi)$ functions with a triangle-shaped dependence on the azimuth (Fig. 1); thus, expressions of $f^R_m(\omega)$ and $f^M_m(\omega)$ can be evaluated analytically. After that, the theoretical results of $\langle \hat{n}_{BII} \rangle$, $\langle \hat{d}_{BII} \rangle$, and $\hat{B}_{II}$ can be obtained from equations (11) and (15)–(17). Array size and the RLR were the same as in Figure 3b. The series in equations (16) and (17) have been truncated at $j, l, m$, and $n = \pm 5$. We checked that higher order terms have little influence within the frequency range shown. Thin lines in Figure 4e–h show these analytical results for wave beam widths of $2^\circ, 30^\circ, 60^\circ$, and $90^\circ$ and different maximum power directions, whereas gray lines show the isotropic wave-field case. In addition, each situation was modeled by using a set of plane waves with azimuth-dependent amplitudes obeying $f^L(\omega, \phi)^{1/2}$. The azimuthal increment was chosen as the minimum value between $1^\circ$ and $1/30$ times the wave beam width. The wave-field records were synthesized at the virtual stations’ positions and averaged by following equation (14) in order to estimate $n_{BII}$ and $d_{BII}$ from their definitions (equations 4 and 5). Then, these two quantities were averaged over a total of 500 realizations with all Rayleigh and Love phases randomly varied from one to another. Finally, $\hat{B}_{II}$ was calculated as the ratio between the resulting mean values $\langle \hat{n}_{BII} \rangle$ and $\langle \hat{d}_{BII} \rangle$ and shown in Figure 4a–d by using different symbols, depending on the central azimuth of the wave beam. Comparison between the analytical (Figure 4e–h) and the numerical (Figure 4a–d) results shows that the azimuthal density of sources and the number of stacked time windows in the numerical test is large enough. For a beam width of $30^\circ$, $\hat{B}_{II}$ remains stable under variations of the wave-field directions in the range from 0 to 1.25 Hz only ($x_L$ up to 2.62). The situation gets rapidly better as the source azimuthal width grows, in fact, $\hat{B}_{II}$ becomes almost nonsensitive to the spreading direction up to 2.3 Hz ($x_L = 6.32$) for a $60^\circ$-width wave beam.

Figure 5 shows $\hat{B}_{II} (i + n)$-versus-$f$ functions calculated for the dispersion curves and the isotropic wave field used in Figure 3, in the case of a pentagonal array. Noise-to-signal power ratios (NSR) are assumed to be the same for horizontal and vertical components. The rest of the array and wave-field characteristics considered here are: $N = 5, r = 250$ m, and RLR = 0.2.

Table 2

<table>
<thead>
<tr>
<th>Layer</th>
<th>$V_s$ (m/s)</th>
<th>$V_p$ (m/s)</th>
<th>$H$ (m)</th>
<th>$\rho$ (g/cm$^3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>500</td>
<td>935</td>
<td>100</td>
<td>2.1</td>
</tr>
<tr>
<td>2</td>
<td>1000</td>
<td>1870</td>
<td>$\infty$</td>
<td>2.1</td>
</tr>
</tbody>
</table>

Figure 5. Uncorrelated noise effects on $\hat{B}_{II}$ calculated for the ground model listed in Table 2 under isotropic wave field (dominance of fundamental modes is also assumed). Noise-to-signal power ratios (NSR) are assumed to be the same for horizontal and vertical components. The rest of the array and wave-field characteristics considered here are: $N = 5, r = 250$ m, and RLR = 0.2.
favorable effect does not hold for the SCA method, though variations on the zero-crossing frequencies are hardly appreciated in Figure 5.

Figure 6 shows the combined effects of uncorrelated noise and finite-N for the cases studied in Figure 4. In this case, the noise-to-signal power ratios have been fixed to $NSRH = 0.3$ and $NSRV = 0.2$. 

Figure 6. Same as Figure 4 except for the presence of uncorrelated noise with intensities given by $NSRH = 0.3$ and $NSRV = 0.2$. 

[Graphs showing combined effects for different angles and noise scenarios]
NSRH = 0.3 and NSRV = 0.2 in the entire frequency range. Uncorrelated noise has been simulated by adding terms with amplitude $\sqrt{P_H^{(n)}}$ or $\sqrt{P_V^{(n)}}$ and arbitrary phase angles to the Fourier-transformed records, for each virtual station, component, and time window (realization). The great likeness between theoretical results (Fig. 6e–h) and their simulated counterparts (Fig. 6a–d) confirms the rapid convergence of the series in equations (20)–(22), in the studied frequency range. These series have been truncated at orders $\pm 7$ in $\xi$.

An additional test comparing the SCA with the CCA-L method (Tada et al., 2009) is shown in Figure 7. We used an almost isotropic wave field (left panels) as well as a wave field with a 90°-width range of dominant azimuths (right panels). Both the tests were performed with and without uncorrelated noise. Although further studies about the biases in the CCA-L coefficient are required, the following preliminary conclusions may be inferred from this figure: (1) the SCA method shows stable sensitivity to uncorrelated noise, whatever the source distribution is; (2) the CCA-L method will be almost insensitive to uncorrelated noise if the wave field has a predominant direction; (3) the CCA-L method shows important finite-$N$ effects (even at long wavelengths) for isotropic illumination, which are not observed for the SCA method; and (4) the CCA-L method is unable to give any estimation of the phase velocities in the presence of uncorrelated noise and/or stochastic errors in the case of isotropic wave fields. The latter point seems to agree with the limitations advanced by Tada et al. (2009).

Tests with Real Data

A practical test has been conducted in the delta of the Andarax River (Almería City, Spain) by using five pentagon-shaped arrays with radii of 12, 18, 25, 50, and 94 m. Several geophysical studies have been previously performed at this site (see for example, Instituto Geológico y Minero de España–Instituto de Reforma y Desarrollo Agrario, 1977; Navarro et al. 1997; Pulido-Bosch et al. 2004). Amongst them, there are 19 close boreholes directed to the construction of a desalination plant. The main landform in the area is formed by recent alluvial deposits (down to around 20 m), composed of silt, sand, and gravel, overlying a layer of coarse gravels and sands over a bed of lutites. Pliocene-aged materials arise around 60-m depth as alternating lutites and sandstone strata overlying Pliocene gravel and sand and finally, a thick layer of Miocene marls. Triassic rocks (dolomitic limestone and phyllites) arise around a depth of 600 m,

![Figure 7.](image-url)

**Figure 7.** Panel (a): modeled $\hat{B}_{ij}$ (circles) and $\hat{B}^{(s+a)}_{ij}$ (squares) for the ground structure shown in Table 2 and a pentagonal array with $r = 250$ m. Propagation of fundamental modes and isotropic illumination are assumed. Solid lines show the theoretical functions derived from the presented formula. Dashed line is the theoretical result for a dense array. (b) Same for a triangle-shaped illumination (Fig. 1) with the characteristics indicated in the title. (c) Noisy (squares) and nonnoisy (circles) CCA-L coefficients (Tada et al., 2009) for the isotropic wave-field case. (d) Same for the anisotropic illumination used in panel (b). The rest of the wave-field and noise characteristics (RLR, NSRH, NSRV, and number of realizations) are the same as in Figure 6.
constituting the stiff basement. A suitable model for the subsurface structure is shown in Figure 8. The thicknesses of layers are obtained from the boreholes and electrical surveys, whereas the fine tuning of the elastic parameters of each layer was obtained by joint inversion of v-SPAC survey data together with the horizontal-to-vertical spectral ratio (HVSR). The arrays were made up of five three-component broadband seismometers. We used CMG-3ESP and CMG-6TD broadband recorders manufactured by Guralp with reliable response bands from 120 s to 50 Hz and from 30 s to 100 Hz, respectively, and an independent Global Positioning System (GPS)–based timing system. The recording time was at least of 1 hour for each radius, and the sampling rate was 100 samples/s. Traces were band-pass-filtered between 0.1 and 30 Hz and divided in sets of overlapping windows of 20 s. This length can be considered suitable for the analysis of frequencies down to 0.5–1 Hz. Record portions were tapered 5% of their lengths and Fourier transformed. Then, the ±1-order Fourier-series coefficients of the vertical, radial, and tangential components were calculated following equation (14). Finally, $n_{B_{II}}$ and $d_{B_{II}}$ were obtained from equations (4) and (5). These two quantities were averaged over the set of time windows, avoiding those containing clearly nonstationary signals. We indicate the ratio between them as $B_{II}^{exp}$ (with the superscript standing for experimental value), which corresponds to the empirical estimation of $B_{II}$ (or $B_{II}^{(1+n)}$ provided that uncorrelated noise is considered in the subsequent analysis). These quantities are shown with black lines in Figure 9a–e for each array radius.

A direct estimate of the Love-wave phase-dispersion curve can be obtained by considering that $B_{II}^{exp}$ approximately equals the right side of equation (3). As shown previously, this approach is suitable for small values of $x_L$ within the first decreasing part of $B_{II}^{exp}$, mainly up to the first zero crossing ($x_L < 1.84$), and assumes that uncorrelated noise effects and finite $N$ effects can be neglected. In this case, $x_L$ has been calculated from $B_{II}^{exp}$ by using a suitable power series (García-Jerez et al. 2008b). Results are shown in Figure 10a, together with the actual fundamental-mode Love-wave and Rayleigh-wave dispersion curves calculated from the ground model. In order to improve clarity, such velocities are not shown in ranges where they are clearly

![Figure 8](image-url)
unreliable. Points at which $x_L = 1.84$ are indicated for each radius with an arrow. The minimum frequencies at which phase velocities can be obtained in this manner depend on the signal-to-noise power ratios, which seem to be poor for frequencies below the thresholds marked in Figure 9a. In spite of the involved assumptions, a remarkable resemblance is found between experimental and theoretical dispersion curves for the fundamental mode.

Alternatively, Figure 10b shows Love phase velocities obtained by following a three-step procedure that takes into account finite-$N$ and uncorrelated noise effects. If the SCA method is used in this way and the wave field is assumed to be isotropic, quantities $cR(f)$, $RLR(f)$, $NSRH(f)$, and $NSRV(f)$ must be either fitted or estimated by some other method to take into account the consequent perturbations in the forward calculation of $B_{II}(f)$ or $B_{II}(f)^{n+n}(f)$. If the wave field is not isotropic but $f^2(\omega, \varphi)/f^2(\omega, \varphi)$ is still independent of $\varphi$, the shape of the relation $f^2(\omega, \varphi)$ versus $\varphi$ will also be needed.

First, a unique smooth-dispersion curve with the form $c_L(f) = A f^{-e^{2b}}$ (e.g., Saccorotti et al., 2001), where $A$, $a$, and $b$ are real parameters, was fitted to the whole dataset. In this step, $B_{II}$ curves are compared with the forward calculations of $B_{II}$ for $RLR = 0$, assuming isotropic wave field in all the calculations. All fits were carried out by using the simplex downhill method (Nelder and Mead, 1965) and misfit functions defined by $\sum_{i=1}^{N_i} (B_{II}^{\exp}(f_i) - B_{II}(f_i))^2$ or $\sum_{i=1}^{N_i} (B_{II}^{\exp}(f_i) - B_{II}^{n+n}(f_i))^2$, where $f_1$, $f_2$, ..., $f_{N_i}$ are the frequencies in which the estimate of $B_{II}^{\exp}$ shows a good quality (bands enclosed between vertical dashed lines in Fig. 9a-e). Misfits corresponding with different radii were normalized by the respective values of $N_{i}$ and subsequently added in order to define a global misfit. The smooth dispersion curve obtained is given by $A = 0.937$, $a = 0.575$, and $b = 0.0214$. After this, we fitted a global constant $RLR$ for the whole dataset and a linear $NSR$-versus-$f$ relation for each radius. Such a linear behavior seems to work well with our dataset in some frequency bands (those marked in Fig. 9a-e), but this cannot be considered a general rule. Parameters defining $c_L(f)$ were kept constant, and the relation $NSRV = NSRH = NSR$ was assumed here. Because the Rayleigh-wave dispersion curve is necessary in this second step, it was obtained from SPAC analysis of the vertical components. The value obtained for the $RLR$ was 0.36 (i.e., 26.5% of Rayleigh-wave power), whereas the $NSR$-versus-$f$ relations have been shown in Figure 9f. Finally, refined smooth Love-wave dispersion curves were fitted for each radius, using the global dispersion curve obtained previously as the initial

![Figure 9](image-url)

**Figure 9.** (a–e): Black lines show $B_{II}^{\exp}$ obtained by using several pentagonal arrays deployed at the test site. The title above each panel specifies the radius of the array. Solid gray lines show the respective calculation of $B_{II}^{n+n}$ from previously fitted smooth dispersion curves, $NSR$ and $RLR$ (see text). Dashed gray lines show the range where the data have been used. Dashed and dotted lines show $B_{II}$ functions calculated from the theoretical fundamental mode and first higher-mode Love-wave dispersion curves, respectively (obtained from the structure), and assuming $NSR = 0$, $RLR = 0$, $N = 5$, and an isotropic wave field. (f) Linear $NSR$ vs. $f$ relations fitted for each array.
model and keeping the noise-to-signal power ratios and the
RLR unvaried. These curves are shown in Figure 10b,
whereas their mean value is shown in Figure 10c. Gray lines
in Figure 9a–e represent forward calculations of $\hat{B}_{II}^{(s+n)}$ for
each radius obtained from the final fitting. By comparison
with the experimental data, we suspect that most of the NSRs
in Figure 9f are probably overestimated at high frequencies
due to the oversimplified assumption of linear behavior. In
such a case, amplitudes of maxima and minima of $\hat{B}_{II}^{(s+n)}$
cannot be properly fitted (e.g., around 11 Hz in Fig. 9c). By
contrast, noise seems to be underestimated at frequencies
below the frequency ranges fitted in Figure 9a–e. Dashed
and dotted black lines show the theoretical shapes of the
$\hat{B}_{II}$ functions, assuming $NSR = 0$, $RLR = 0$, $N = 5$, and an
isotropic wave field for both the fundamental mode and the
first higher-mode Love-wave dispersion curves calculated
directly from the ground structure. Even though further
research on the effects of higher modes is necessary, their
contributions can be suspected in some frequency ranges
(e.g., at the bump around 10 Hz for $r = 12$ m and at the
unexpected trough around 13 Hz for $r = 18$ m).

We have checked that our fittings have a limited sensitiv-
ity to the RLR. Note that the method becomes completely
insensitive to the RLR as $N$ tends to infinity. For example, if

![Figure 10.](image)

(Continued)
we replace $RLR$ with 1 before the final step (i.e., before fitting the five smooth segments of dispersion curves), the resulting mean dispersion curve will vary 3.3\% in average from the final result shown in Figure 10c. The maximum variation (lower than 8\%) holds in the range 20.6–22.5 Hz. It suggests that an approximate estimation of the $RLR$ may suffice in many ordinary wave-field conditions.

The overall results indicate that this scheme allows a suitable consideration of directional aliasing and correlated noise effects, extending the usable frequency range of the data beyond the first minimum of $B^{\exp}_{II}$. Further refinements of the numerical algorithm could be developed in future work on the basis of the theory developed here.

Conclusions

Different authors have shown that inclusion of Love-wave velocities in surface wave–based inversion algorithms results in better constrained ground models. The reliability of the SCA method, a suitable technique for the calculation of Love-wave dispersion curves from microtremor measurements, has been explored in this article. This method can be used for the analysis of almost any sort of surface wave fields: with or without correlation between Love and Rayleigh waves or among waves coming from different azimuths and for isotropic and anisotropic source distributions. Thus the SCA method is, in principle, more versatile than some other techniques (e.g., the CCA-L method). An implementation of the method, based on the $B_{II}$ estimator defined here, has shown quite robust behavior for synthetic and real datasets, being still simple enough to allow an analytical study of its behavior under several causes of experimental limitations. A set of equations has been derived and implemented to account for biases due to directional aliasing, including a description in terms of the Fourier-series coefficients of the Rayleigh-wave and Love-wave frequency–direction power spectral densities. Suitable formulae have been also given to account for the effects of incoherent noise on $B_{II}$ in both deterministic and random fields. These relations provide useful information on the reliability of the method and have been successfully tested by using forward numerical simulations of $B_{II}$ and $B^{(i+n)}_{II}$ for a simple ground model under incidence of plane surface waves. Our numerical tests confirmed that the stability of $B_{II}$ under variations in the source azimuths increases as the width of the wave beam increases. In our numerical test, a pentagonal array provided stable results up to $x_L = 6.32$ for illumination coming from an azimuthal width of 60°. Nevertheless, further research is necessary to check these results in more general ground models.

We have also applied the SCA method to calculate dispersion curves from a real dataset recorded with pentagonal arrays at a site where the ground structure was known. This experiment confirms that the equations derived here can significantly extend the usable wavelength range of the SCA method. In fact, experimental shapes of $B_{II}$ have been successfully interpreted up to $x_L = 11.7$ (in average), assuming isotropic illumination, although this limit may strongly depend on the wave-field characteristics. By contrast, if effects of directional aliasing and incoherent noise were neglected, reliable velocities could be obtained up to around $x_L = 1.84$ only.

Tests performed in this article consisted of forward comparisons where the ground structure is known. Thus, some additional research is necessary in order to find the optimum operational procedure, including accurate estimation or fitting of signal-to-noise power ratios and the optimum calculation of dispersion curves and evaluation of the associated uncertainties.
Data and Resources

The microtremor waveforms used in this article belong to the Group on Applied Geophysics of the University of Almería (Spain). Records are not currently available for public use. Some of the calculations were done using the parallel computing equipment WaveCom II at the University of Almería.

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References


Appendix A

Fourier-Series Coefficients of the Vertical, Radial, and Tangential Components versus the Azimuth

Let us consider a wave field made up of an arbitrary sum of Rayleigh and Love plane waves recorded in an infinite dense circular array of radius \( r \). Assuming that there is a dominant mode for each frequency and type of wave (Rayleigh and Love), the \( m \)-order weighted azimuthal average of the vertical (\( Z \)), radial (\( R \)), and tangential (\( T \)) components of motion, defined in equation (2), can be expressed as

\[
Z_m(r, \omega, \tau) = 2\pi(-i)^{m-1} \frac{A_m^R}{\chi} J_m(x_R),
\]

\[
R_m(r, \omega, \tau) = (-i)^{m-1} \pi A_m^R [J_{m-1}(x_R) - J_{m+1}(x_R)]
+ (-i)^m \pi A_L^m [J_{m-1}(x_L) + J_{m+1}(x_L)],
\]

\[
T_m(r, \omega, \tau) = (-i)^{m-1} \pi A_L^m [J_{m-1}(x_R) + J_{m+1}(x_R)]
+ (-i)^m \pi A_R^m [J_{m-1}(x_L) - J_{m+1}(x_L)],
\]

respectively (García-Jerez et al. 2008b), where \( A_R^m(\omega, \tau) \) and \( A_L^m(\omega, \tau) \) have been defined in equation (1), \( \chi(\omega) \) is the ellipticity of the predominant mode of Rayleigh waves (imaginary part) defined previously, \( x_L(\omega) = r_\omega/c_L(\omega) \), and \( x_R(\omega) = r_\omega/c_R(\omega) \).

Appendix B

Derivation of Equation (11)

The product of sums of weighted complex amplitudes \( A_R^m A_Y^p A_Z^q \), where \( W, X, Y, \) and \( Z \) stand for Rayleigh (\( R \)) or Love (\( L \)) and \( m, n, p, \) and \( q \) are integers, has been defined for any deterministic wave field, in accordance with equation (1). Nevertheless, a suitable expression for its mean value in a stationary random wave field can also be easily derived.

Let us consider firstly the vertical motion at the array center (\( r = 0 \)) for such a stationary random field. The expression of that time history can be taken, for example, from Cho et al. (2006b) (equation 18 in their paper). Using our nomenclature and assuming dominance of a single mode, it yields

\[
z(0, t) = \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} e^{i\omega t} i\chi(\omega) d\zeta(\omega, \varphi).
\]

Integrals in this relation have to be interpreted in the Stieltjes sense. \( d\zeta(\omega, \varphi) \) is called the random spectral measure of the process. By definition, if this integral exists, it must coincide with the limit

\[
z(0, t) = \lim_{\tau \to 0} \sum \sum e^{i\Delta\omega t} \frac{i}{\chi(i\Delta\omega)} \times \Delta\zeta_R(\Delta\omega, j\Delta\varphi; \tau)
\]

(Priestley, 1981, pp. 247–248), where \( \Delta\omega = \frac{2\pi}{\tau} \) and \( \Delta\varphi = \frac{2\pi}{M} \).

On the other hand, under the approach used in the present article for a deterministic wave field consisting of a superposition of \( M \) plane Rayleigh waves, the vertical record at \( r = 0 \) for \( 0 < t < \tau \), is

\[
z(0, t) = \frac{1}{2\pi} \sum \sum e^{i\Delta\omega t} \frac{i}{\chi(i\Delta\omega)} A_{R,y}(\Delta\omega; \tau) \Delta\omega
\]

(see example equation 4.11.8 in Priestley, 1981 for the dependence on \( \omega \)).

By comparison between equations (B2) and (B3), we observe that the stochastic case in equation (B2) can be formally thought as the limit, \( M \to \infty \) of the deterministic case in equation (B3) for a wave field consisting of a superposition of \( M \) plane Rayleigh waves coming from evenly distributed azimuths in the whole range \([0, 2\pi]\) and complex amplitudes given by \( A_{R,y}(\Delta\omega; \tau) \). In a similar manner, stochastic representations involving Love waves match those formulated in the deterministic context if \( \Delta\zeta_L(\Delta\omega, j\Delta\varphi; \tau) \) is replaced with \( A_{L,y}(\Delta\omega; \tau) / \tau \).

We will assume statistical independence among waves coming from different azimuths and between different types of waves (Rayleigh or Love). The following properties of the complex amplitudes also can be assumed by analogy with properties of \( \Delta\zeta_R \) and \( \Delta\zeta_L \) in a stationary stochastic wave field:

\[
\langle A_{X,y}(\Delta\omega; \tau) \rangle = 0,
\]

\[
\left\langle \frac{1}{2\pi \Delta\varphi} A_{X,y}(\Delta\omega; \tau) A_{L,k}^*(m\Delta\omega; \tau) \right\rangle
\]

\[
\to \delta_{XY} \delta_{jk} \delta_{lm} f^X(\omega, \varphi),
\]

and

\[
\langle A_{X,y}(\Delta\omega; \tau) A_{X,y}(\Delta\omega; \tau) \rangle = 0, \quad l \neq 0,
\]

where \( j, k, l, \) and \( m \) are integers and \( X \) stands for \( R \) (Rayleigh) or \( L \) (Love). Equation (B4) indicates that the microtremor wave field is a zero-mean stationary stochastic.
process. Equation (B5) is the orthogonality relation. The limit $\tau \to \infty$, $\Delta \phi \to 0$, $l \Delta \omega \to \omega$, and $j \Delta \phi \to \phi$ is understood in equation (B5). Equation (B6) can be demonstrated from the orthogonality relation for $\Delta \chi(0, \omega, j \Delta \phi; \tau)$, taking into account that $\Delta \chi(0, \omega, j \Delta \phi; \tau) = \Delta \chi(-0, \omega, j \Delta \phi; \tau)$. Because $\Delta \chi(\omega, \phi; \tau)$, $O\left(\sqrt{\Delta \omega} / O(\sqrt{\Delta \phi}) \right)$ as $\tau \to \infty$, $\Delta \phi \to 0$ (e.g., Priestley, 1981, p. 247 for the dependence on $\Delta \omega$; Okada, 2006), we can infer that $A_{X,j}(\omega; \tau) \sim O(\sqrt{\tau}) O(\sqrt{\Delta \phi})$.

From their definitions, and using the linearity of the operator $\langle \cdot \rangle$, the ensemble average of $A_{W}^{m} A_{X}^{n} A_{Y}^{p} A_{Z}^{q}$ can be written as

$$\langle A_{W}^{m}(\omega; \tau) A_{X}^{n}(\omega; \tau) A_{Y}^{p}(\omega; \tau) A_{Z}^{q}(\omega; \tau) \rangle = M \sum_{j_{1}=1}^{M} \sum_{j_{2}=1}^{M} \sum_{j_{3}=1}^{M} \sum_{j_{4}=1}^{M} e^{-i(mj_{1}+nj_{2}+pj_{3}+qj_{4})\Delta \phi}$$

$$\times \langle A_{W,j_{1}}(\omega; \tau) A_{X,j_{2}}(\omega; \tau) A_{Y,j_{3}}(\omega; \tau) A_{Z,j_{4}}(\omega; \tau) \rangle$$

(B7)

for any $\omega_i = l \Delta \omega$. Addsends in equation (B7), having at least one nonrepeated amplitude in the product $A_{W,j_{1}} A_{X,j_{2}} A_{Y,j_{3}} A_{Z,j_{4}}$, vanish because the expectation of statistically independent random variables factorizes, and they are zero mean (equation B4). Because squared complex amplitudes also have zero mean (equation B6), the only contributing terms are those satisfying either $W = Y$, $j_{1} = j_{3}$, $X = Z$, $j_{2} = j_{4}$, or $W = Z$, $j_{1} = j_{4}$, $X = Y$, $j_{2} = j_{3}$, or both of them ($W = X = Y$, $j_{1} = j_{2} = j_{3} = j_{4}$). It yields

$$\langle A_{W}^{m}(\omega; \tau) A_{X}^{n}(\omega; \tau) A_{Y}^{p}(\omega; \tau) A_{Z}^{q}(\omega; \tau) \rangle = \delta_{WY} \delta_{XZ} M \sum_{j_{1}=1}^{M} \sum_{j_{2}=1}^{M} e^{-i(mj_{1}+nj_{2}+pj_{3}+qj_{4})\Delta \phi}$$

$$\times \langle A_{W,j_{1}}(\omega; \tau) \rangle \langle A_{X,j_{2}}(\omega; \tau) \rangle \langle A_{Y,j_{3}}(\omega; \tau) \rangle \langle A_{Z,j_{4}}(\omega; \tau) \rangle$$

(B8)

The latter sum fixes the incorrect calculation of the expectation for terms with $j_{1} = j_{2}$ in the case $W = X = Y = Z$. In that case, the modulus squared complex amplitudes $|A_{W,j_{1}}(\omega; \tau)|^2$ and $|A_{X,j_{2}}(\omega; \tau)|^2$ are identical and have to be treated as fully correlated variables (e.g., Howard, 2002).

After dividing by $4\pi^2\tau^2$ and taking limits $\Delta \phi \to 0$ (i.e., the number of evenly distributed far-point sources, $M$, tending to infinity) and $\tau \to \infty$ (with $\omega \to 0$, $j \Delta \phi \to \phi$, symbols $\Delta \phi \sum_{j=1}^{M}$ can be formally replaced with $\int_{-\infty}^{\infty} d\phi$, and the following relation can be simplified by those terms of equation (B8) proportional to $\delta_{WY} \delta_{XZ}$ and $\delta_{WZ} \delta_{XY}$:

$$\Delta \phi \sum_{j=1}^{M} e^{-i(mn-pq)\Delta \phi} \left( \frac{1}{2\pi \Delta \phi} |A_{W,j}(\omega; \tau)|^2 \right)$$

$$\to \int_{-\infty}^{\infty} e^{-i(mn-pq)\Delta \phi} f_{\omega}(\omega, \varphi) d\varphi = f^{W}_{m-p}(\omega).$$

(B9)

Definition (9) and equation (B5) have been taken into account. By contrast, the term proportional to $\delta_{WY} \delta_{XZ} f_{\omega}^{W}$ vanishes because the integrand is $O(\Delta \phi)$ as $\Delta \phi \to 0$:

$$e^{-i(mn-pq)\Delta \phi} \left\{ \left( \frac{1}{\tau} |A_{W,j}(\omega; \tau)|^2 \right)^2 \right\} \sim O(\Delta \phi).$$

(B10)

Finally, equations (B8)–(B10) yield

$$\left( 1 - \frac{1}{4\pi^2\tau^2} A_{W}^{m} A_{X}^{n} A_{Y}^{p} A_{Z}^{q} \right)$$

$$\to (\delta_{WY} \delta_{XZ} f_{m-p}^{W} f_{n-q}^{W} + \delta_{WZ} \delta_{XY} f_{m-q}^{W} f_{n-p}^{W}).$$

(B11)

Appendix C

Derivation of Equations (18) and (19)

In order to demonstrate equations (18) and (19), we note that, from their definitions,

$$\langle \hat{\beta}_{B_{0}}^{(s+n)} \rangle = -\text{Im} \left[ \int_{-1}^{1} T_{s+n}^{(s)} Z_{s+n+1}^{(s+n)} \right.$$

$$\left. + \hat{T}_{s+n}^{(s+n)} Z_{s+n-1}^{(s)} \hat{R}_{s+n}^{(s+n)} Z_{s+n}^{(s)} \hat{R}_{s+n-1}^{(s+n)} \right],$$

(C1)

and

$$\langle \hat{\beta}_{B_{0}}^{(n)} \rangle = \left( 1 - \frac{1}{4\pi^2\tau^2} \right) \left[ \hat{R}_{s+n}^{(s+n)} Z_{s+n+1}^{(s+n)} - \hat{R}_{s+n}^{(s+n)} Z_{s+n}^{(s+n)} \right],$$

(C2)

Any record can be expressed as the sum of its signal and noise parts. Hereafter, quantities with superscript $(s+n)$ include both the signal and the noise component of the record, whereas quantities with superscript $(n)$ refer to noise only. Those with no superscript include signal only. Thus,

$$\hat{X}_{m}^{(s+n)}(r, \omega; \tau) = \Delta \theta \sum_{j=1}^{N} e^{-imj\Delta \omega} [X(r, j\Delta \omega, \omega; \tau)$$

$$+ X^{(n)}(r, j\Delta \omega, \omega; \tau)]$$

$$= \hat{X}_{m}(r, \omega, \tau) + \hat{X}_{m}^{(n)}(r, \omega; \tau),$$

(C3)

where $X = Z$, R, or T.
For evaluation of the right sides of equations (C1) and (C2), it suffices to obtain the ensemble mean of the products:

\[ \frac{1}{4\pi^2} \mathbf{R}_m(r, \omega; \tau) \mathbf{R}_n(r, \omega; \tau^*) \]
\[ \frac{1}{4\pi^2} \mathbf{R}_m(r, \omega; \tau) \mathbf{R}_n(r, \omega; \tau^*) \]
\[ \frac{1}{4\pi^2} \mathbf{R}_m(r, \omega; \tau) \mathbf{R}_n(r, \omega; \tau^*) \]
\[ \frac{1}{4\pi^2} \mathbf{R}_m(r, \omega; \tau) \mathbf{R}_n(r, \omega; \tau^*) \]

and

\[ \frac{1}{4\pi^2} \mathbf{R}_m(r, \omega; \tau) \mathbf{R}_n(r, \omega; \tau^*) \]
\[ \frac{1}{4\pi^2} \mathbf{R}_m(r, \omega; \tau) \mathbf{R}_n(r, \omega; \tau^*) \]

with \( n, m = \pm 1 \).

Evaluation of the first one gives

\[ \frac{1}{4\pi^2} \mathbf{R}_m(r, \omega; \tau) \mathbf{R}_n(r, \omega; \tau^*) \]
\[ \frac{1}{4\pi^2} \mathbf{R}_m(r, \omega; \tau) \mathbf{R}_n(r, \omega; \tau^*) \]

where \( N > 2 \) and the set of considered values for \( m \) and \( n \) (i.e., \( -1, +1 \)) have been taken into account in the last step. The equality holds for \( \tau \to \infty \). A similar analysis can be carried out for the other three products listed previously. The respective results are

\[ \frac{1}{4\pi^2} \mathbf{R}_m(r, \omega; \tau) \mathbf{R}_n(r, \omega; \tau^*) \]
\[ \frac{1}{4\pi^2} \mathbf{R}_m(r, \omega; \tau) \mathbf{R}_n(r, \omega; \tau^*) \]

Zero mean of the random processes describing uncorrelated noise, as well as the statistical independence between signal and noise and between noise records in different components have been taken into account in the first step. The second step results from the statistical independence between noise waveforms of different stations and from the definition of the spectral densities of noise; for example,

\[ \frac{1}{2\pi^2} \mathbf{R}_m(r, \omega; \tau) \mathbf{R}_n(r, \omega; \tau^*) \]
\[ \frac{1}{2\pi^2} \mathbf{R}_m(r, \omega; \tau) \mathbf{R}_n(r, \omega; \tau^*) \]

with \( N > 2 \) and \(|m| = |n| = 1 \). Finally, equations (18) and (19) are obtained by replacing the analyzed products in equations (C1) and (C2) with the expressions (C4), (C6), (C7), and (C8).

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