A threshold value for a system with demographic fluctuations.

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Abstract. In this paper we study a threshold value for the survival in the predator-prey models. The result obtained can be applied to the case of systems with some perturbation due to toxic effects or also in the case of biological fight in greenhouse crops.
Key words: Threshold value. Predator-prey. Persistence.

1 Introduction

Searching solutions for the important economic influence of plagues on crops has, as a matter of fact, evolved throughout time, and this evolution has been particularly fast in the last decades, this has been like that as a consequence of problems deriving from the exclusive use of synthetic pesticides (e.g., higher incidence of pests, appearance of new species, endurance to pesticides, costs increase, toxical and environmental problems)(Braungärtner and Gutierrez, 1989; Norton and Munford, 1993; Cabello, 1998).

It is relevant the role of organic enemies within crops, since they will give rise to a death toll which will regulate pest populations, this is called *"natural control"*. Obviously, this control is not enough in order to keep pests under the threshold in which it might cause important economic damage on crops.

Whenever the action of natural enemies is intentionally manipulated, then we are dealing with biological fight against pests, which is an alternative

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technique for quimical pest-control (Debach, 1984; Samways, 1990).

Within the frame of this biological fight, the role of the so-called predator-prey models has an essential role; several mathematical studies have been made about this, such as those by Lotka (1925), Volterra (1926), Watt (1959), Curry and Demichele (1977). Nevertheless, the practical performance of these works considering their relevance upon biological control will become apparen later on (Hassell and Waage, 1984; Hassell, 1988; Mackauer et al. 1990).

What can be inferred from the facts above is that intend to study the predator-prey models in order to obtain a precise threshold value for the survival of pests, so completing the result from theorem 1 [6] and the checking it through a series of real data taken from a predator-prey action in greenhouse crops.

2 The main result

We have the Kolmogorov-type equation,

$$x' = xG(c(t), x), \quad t \in \mathbb{R}_+ = [0, \infty). \tag{2.1}$$

In (2.1), x is a measure of the population; $c = c(t)$ represents demographic parameters. It is assumed that:
1) $G(z, x) \in C^1[\mathbb{R} \times \mathbb{R}_+, \mathbb{R}], \quad G(0, 0) = 0$
2) $G(z, x)$ is strictly decreasing in z and x.
3) $c(t)$ denoted a bounded continuous function.

In the given a bounded continuous function $\alpha : [0, +\infty) \to \mathbb{R}$ we define,

$$\alpha^* = \lim_{t \to +\infty} \sup \alpha(t) \quad ; \quad \alpha_* = \lim_{t \to +\infty} \inf \alpha(t)$$

in this section we prove a result that complements theorem 1 in [6]

Theorem 2.1 *If $G(c_*, 0) = 0$ then $x^* = 0$ for any function solution x of (2.1).*

We define $\Gamma := c_*$, a threshold value for the persistence of the solution x. To prove the Theorem 2.1 we need two intermediates results.

Proposition 2.2 *Let be a positive solution y the equation*

$$y' = y[m - f(t, y)], \tag{2.2}$$

where $m > 0$ is constant and $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ continuous and increasing function, is such that

$$\lim_{y \to 0} f(t, y) = 0 \quad uniformly \;\; in \;\; y. \tag{2.3}$$

If y is defined and bounded on $[0, +\infty)$, then $y_* > 0$.

Proof. For contradiction propose, suppose that

$$y_* = 0, \tag{2.4}$$

we shall show that

$$y^* > 0. \tag{2.5}$$

For contradiction propose, assume that $y(t) \to 0 \;\; as \;\; t \to +\infty$. By (2.3), then exists $t_0 \geq 0$ such that,

$$f(t, y(t)) \leq \frac{1}{2} m \quad \forall t \geq t_0,$$

an hence,

$$y' \geq \frac{1}{2} my \quad \text{on} \;\; [t_0, +\infty).$$

Consequently, $y(t) \to +\infty$ as $t \to +\infty$, and this contradiction prove that (2.5) holds.

By (2.4)-(2.5) and the arguments in section 1 [13], then exists a sequence $\{t_n\} \to +\infty$, such that

$$y(t_n) \to 0 \quad and \quad y'(t_n) = 0. \tag{2.6}$$

From this, $m = f(t_n, y(t_n))$ for all $n \in \mathbb{N}$, and by since again, $m = 0$. This contradiction end the proof. ■

Lemma 2.3 *Let $[\rho_0, \rho_1]$ be a compact interval. If given $\epsilon > 0$ there exists $\delta > 0$ with the following property.*

$G(z, 0) - G(z, \mu) < \delta$ for some $(z, \mu) \in [\rho_0, \rho_1] \times (0, +\infty)$ then, $\mu < \epsilon$.

Proof. Suppose on the contrary the existence of $\epsilon > 0$ and sequence $\{z_n\}$,$\{\mu_n\}$ in $[\rho_0, \rho_1]$, $[\epsilon, +\infty)$ respectively such that,

$$G(z_n, 0) - G(z_n, \mu_n) \to 0 \quad as \quad n \to +\infty, \tag{2.7}$$

without loss of generality, we can assume that $z_n \to z_0$ for some $z_0 \in [\rho_0, \rho_1]$. On the other hand $G(z_n, \mu_n) \leq G(z_n, \epsilon)$ and by (2.7),

$$G(z_0, 0) \leq G(z_0, \epsilon),$$

$G(t, x)$ is strictly increasing in x, and this contradiction end the proof. ■

Proof of Theorem 2.1 Let us define,

$$\rho_0 = \inf(c), \quad \rho_1 = \sup(c) \quad \text{and fix } \ \epsilon > 0$$

Fix also $\delta > 0$ satisfying the assuming in Lemma 2.3. Since,

$$m_\alpha := G(c_* - \alpha, 0) \to 0 \quad \text{as} \quad \alpha \to 0;$$

There exists $\beta > 0$ such that $m_\beta < \delta$. On the other hand, there exists $t_0 \geq 0$ such that, $c(t) \geq c_* - \beta \quad \forall t \geq t_0$, and hence,

$$x' \leq x[m_\beta - f(t, x)] \quad on \quad [t_0, +\infty)$$

when $f(t, x) := G(c(t), 0) - G(c(t), x)$.

Let y be the solution of (2.2) determined by the initial condition $y(t_0) = x(t_0)$. By the monotony of function G we have $f(t, y) \leq m_\beta$, then $y \leq e^{m_b t} y(t_0)$ and this implies that y is defined and bounded on $[t_0, +\infty)$. and by comparison $y(t) \geq x(t)$ for all $t \geq t_0$.

On the other hand, it is easy to show that f satisfy the assumptions in Proposition 2.2 an hence $y_* > 0$.

By Lemma 2.2 of [13] there exists a sequence $t_n \to +\infty$, such that, $y'(t_n) \to 0$ and $y(t_n) \to y^* > 0$. And by (2.2),

$$m_\beta = \lim_{n \to +\infty} \left[G(c(t_n), 0) - G(c(t_n), y(t_n)) \right].$$

Since $c(t)$ is bounded, we can assume, without loos of generality, that $c(t_n) \to z_0 \in [\rho_0, \rho_1]$, and them

$$m_\beta = G(z_0, 0) - G(z_0, y^*) < \delta.$$

By Lemma 2.3, $y^* < \epsilon$ and the proof follows since $x^* \leq y^*$. ■

3 Application to the predator-prey model

When analyzing the predator-prey model, we can consider the predator activity as an external factor which acts upon the development of the prey population. According to this, we will try to obtain some conclusions based upon a predator-prey model.

We considers,

$$x' = x(a - bx - dy) \tag{3.1}$$

$$y' = y(\alpha - \beta x - \gamma y) \tag{3.2}$$

which $a, b, d, \alpha, \beta, \gamma$ positive constants which determine a predator-prey model with friction.

We define $c(t) := a - dy(t)$, in this way we have that (3.1) is a equation of type (2.1) where the thresholds value $\Gamma = \dfrac{a}{d}$, in this way of Theorem 2.1 we have if $y(t) \to \frac{a}{d}$ as $t \to +\infty$, then we have the extinction of the prey.

We are trying to confront the result obtained above with unpublished data (Novartis S.A.), where the activity of the pest population *"frankliniella occidentalis"* (Thys: Thripidae) is studied and also that of its predator *"Orius"* (Hem: Anthocoridae) on pepper crops on greenhouse, located in La Mojonera Almeria (Spain) through the year 1997-98. According to the model (3.1)(3.2), the following graph is obtained in [5], which is represented in figure 1. However, the same result. i.e., extinguishment is obtained whenever y reaches the Γ threshold, as shown in figure 2. On the contrary, it can also be observed that whenever this threshold value is not reached by the y predator population, then the x prey population manages to persist, as shown in figure 3.

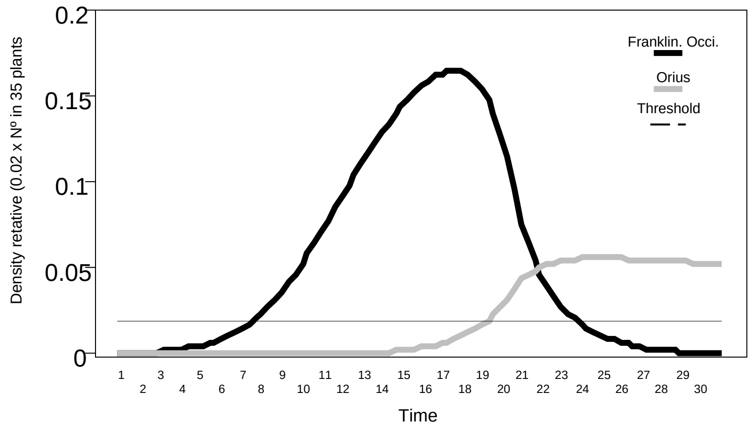

Figure 1: Adjustment Orius-Frankliniella in pepper in 1997/98. La Mojonera (Almería)

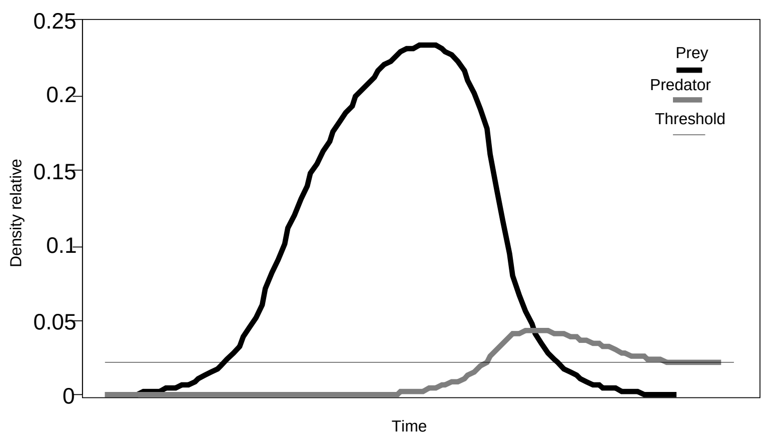

Figure 2: Predator equal to Threshold value Γ

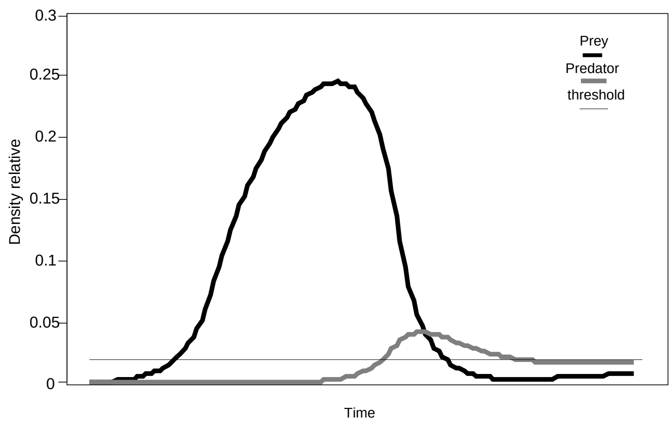

Figure 3: Predator under Threshold value Γ

4 Discusion.

Many studies have been concerned with biological fight in the last few years, since it could become a relevant instrument, either alternate or complementary, for pest-controlling in vegetable cultive. Thus, one of the main problems would be to establish the extent to which the predator should be used in order to extinguish or keep the pest alive, in a harmless level for crops. The results obtained in this work could be extremely useful in order to determine precisely how the process above can be carried out, since it may be relevant even from an economic point of view.

It can also be mentioned the relevance of these results within the field of biological fight control, an issue under observation in plenty of studies nowadays.

References

[1] Baungärtner, J. ;Gutierrez, A.P. Simulation techniques applied to crops and pest models. En: Cavalloro, R.; DeLucchi, V. (Ed.).Parasitis 88. Proceeding of a Scientific Congress. Barcelona 25-28 de octubre 1988. *Boletin de Sanidad Vegetal, Plagas*; Fuera de serie, 17, (1989), 175-214.

[2] Cabello T. *Racionalización del empleo de fitosanitarios: eficacia, técnicas de aplicación y respeto a la fauna auxiliar*. Encuentro ambiental almeriense. Documentación de la comisión de trabajo "Agricultura intensiva", Almería , (1998), 26-52.

[3] Curry, G.; DeMichele, D.W. Stochastic analysis for the description and synthesis of predator-prey systems. it Canad. Entomol., 109, (1977), 1167-1174.

[4] DeBach, P. *Control biológico de las plagas de insectos y malas hierbas*. CECSA. México, (1984), 949 pp.

[5] Gamez, M. Modelo depredador.presa. Aplicaciones al control biológico. *Tesis doctoral. Universidad de Almería* (1999).

[6] Hallam, T.G. and Zhien, M. Persistence in population models with demographic fluctuations. *Jour. Math. Biology*, 24, (1986) 327-339.

[7] Hassell, M.P. *Dinámica de la competencia y la depredación*. Oikos-tau Barcelona, (1988), 102 pp.

[8] Hassell, M.P. ;Waage, J.K. Host-parasitoid population interactions. *Ann. Rev. Entomol.*, 29, (1984), 89-114.

[9] Lotka, A.J. *Elements of physical biology*. Williams & Wilkins, Baltimore, Maryland, USA, (1925).

[10] Mackauer, M. et al. *Critical issues in biological control*. Intercept. Andover, (1990), 330 pp.

[11] Norton, G.A.; Mumford, J.D. *Decision tools for pest management*. C.A.B. International. Wallingford, (1993), 279 pp.

[12] Samways, M.J. *Control biológico de plagas y malas hierbas*. Oikos-tau. Barcelona, (1990), 84 pp.

[13] Tineo, A. Asymptotic behaviour of positive solution of the nonautonomous Lotka-Volterra competition equation.. *Differ. and integral equation*, 6 n2, (1993) 449-457.

[14] Volterra, V. Variazioni e fluttuazioni del numero d'individui in specie animali conviventi. *Mem. Roy. Acad. Naz. Dei Lincei*, 2 (1926), 31-113.

[15] Watt, K.E.F. A mathematical model of effect of densities of attacked and attacking species on the number attacked. *Canad. Entomol.*, 91 (1959) 129-144.