Robust observer design by sign-stability for the monitoring of population systems

Inmaculada López[a*], María Luz Puertas[a], Carmelo Rodríguez[a],
Manuel Gámez[a] and Zoltán Varga[b]

[a] *Department of Mathematics, University of Almería, La Cañada de San Urbano, 04120 Almería, Spain,*
[b] *Department of Mathematics, Szent István University, Páter K. u. 1., H-2103 Godollo, Hungary*

Abstract

Monitoring problem in population ecology can be formalized as observer design for the population system in question: Supposing that we observe only certain species considered indicators, we want to recover or estimate the whole state process of the population system. In the present paper, for stably coexisting population systems, a new approach to the design of the corresponding observer system is proposed which, from the knowledge of the observed indicator(s), estimates the state process with exponential convergence. In the usual observer design, an auxiliary matrix, the so-called gain matrix must be found that guarantees the mentioned exponential convergence. The novelty is in that due to the required sign-stability (or qualitative stability) of the interaction pattern, the designed observer system (i.e. the gain matrix) is robust against quantitative changes in the inter- and intra-specific interactions. In other words, under sign-stability conditions, in the observer design the same gain matrix can be used even if, due to environmental changes, the intensities of certain interactions suffer a quantitative change in the meanwhile. The requirement of sign-stability of the interaction pattern can be considered rather natural, since in a stably coexisting population system, it means for example that a predator-prey relation does not change into a prey-predator interaction, and interactions neither appear nor disappear within the system. Our approach to robust observer design is illustrated on model population systems, such as trophic chains of type resource-producer-primary consumer-secondary consumer and Lotka-Volterra system with vertical structure.

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[*] Corresponding author. *E-mail addresses:* milopez@ual.es (I. López), mpuertas@ual.es (M.L. Puertas), crt@ual.es (C. Rodríguez), mgamez@ual.es (M. Gámez), Varga.Zoltan@gek.szie.hu (Z. Varga)

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1. Introduction

1.1. Systems-theoretical background

Mathematical Systems Theory (MST) was mostly developed and mainly used for engineering purpose. The first monograph, laying down the basic concepts and theorems of MST was Kalman et al. [1]. The theory and applications MST is based on the so-called state-space model, where the state of an object (in the context of engineering usually called a plant) is described by a time-dependent state vector influenced by a time-dependent input or *control function.* The different concepts of *controllability* are related to the requirement that the system, from a given initial state, can be steered to a required state, applying an appropriate control. Furthermore, given a state-space model, we usually cannot (or do not want to) observe the whole state vector, only the time-dependent vector of certain state variables, considered as output, or *observation function. Observability* of a system, roughly speaking means that from the knowledge of the observation function, in principle, the true state process can be uniquely recovered. The *monitoring problem* we consider in the present paper, is related to the observation side of the state-space model. In practice, it is not enough to know that to a given observation function there uniquely corresponds a state process, but we also want to effectively calculate (or at least estimate) the latter. This can be solved by the construction of a so-called *observer system* (shortly observer), the solution of which efficiently approximates the unknown true solution of the original system. We will consider a monitoring problem of population ecology. This means that in a system of several interacting populations living in the same habitat, we observe only certain species (called indicator species), and from these observations, by appropriate observer design, we estimate the whole state process.

The development of state-space modeling in population biology looks back to the last three decades. While in Kalman et al. [1] only linear systems were studied, most models of population biology are nonlinear. By proving basic theorems, controllability and observability in frequency-dependent systems of population genetics have been first studied e.g. in Varga [2,3]. For recent results in this research line, see Gámez et al. [4]. Applying the nonlinear observer design methodology of Sundarapandian [5], for the effective monitoring of different ecological systems, observer systems have been constructed e.g. in Gámez et al. [6,7]. In particular, for monitoring in Lotka-Volterra population systems, we refer the reader to López et al. [8]. A review paper summarizing the different branches of applications of MST in population biology is Varga [9], more recent surveys are Gámez [10] and Varga et al. [11].

1.2. Robustness and sign-stability

As a continuation of the research line concerning monitoring in nonlinear systems, in this paper the aspect of robustness will be studied. In the context of system modeling, "robustness is a property that allows a system to maintain its functions against internal and external perturbations", see e.g. Kitano [12]. In particular, robustness of an observer (or a state estimation) means insensitivity to system disturbance, model uncertainty and sensor noise, as considered in Marquez and Riaz [13]. In the latter paper, an overview of different approaches to robust observer design is given, and a new method of the authors for the robust observer design is also presented.

In the present paper we will focus on the state monitoring of stably coexisting population systems consisting of populations of different species. In technical terms, stable coexistence means that the dynamics of the population system has an asymptotically stable positive equilibrium (or alternatively a stable limit cycle, but the latter case will not be considered here). Asymptotical stability is usually guaranteed by the stability of the Jacobian A of the right-hand side system dynamics at the equilibrium. (A matrix is called stable or Hurwitz matrix, if all its eigenvalues have negative real parts.) A is also called the *community matrix* of the population system, since its ij-entry characterizes the effect of species j on species i. The sign pattern $\mathrm{sgn} a_{ij}$ indicates the quality of the interactions. A stable matrix A will be called *sign-stable* (or qualitatively stable), if it remains stable with any change of its entries which does not violate the sign pattern of A. A basic reference on the role of sign-stability in the study of the dynamic behavior of population system is Svirezhev and Logofet [14], see also Logofet [15].

In the present paper, for stably coexisting population systems, a new approach to the design of the corresponding observer system is proposed which, from the knowledge of the observed indicator(s), estimates the state process with exponential convergence. In the usual observer design, an auxiliary matrix, the so-called gain matrix must be found that guarantees the mentioned exponential convergence. The novelty is in that, due to the required sign-stability (or qualitative stability) of the interaction pattern, in the observer design the gain matrix is robust against quantitative changes in the inter- and intra-specific interactions. In other words, in the observer design the same gain matrix can be used even if, due to environmental changes, the intensities of certain interactions suffer a quantitative change in the meanwhile.

The paper is organized as follows: In Section 2, based on Jeffries et al. [16], we summarize how sign-stability can be checked in terms of the signed directed graph associated with the interaction matrix. In Section 3 the basics of observer design are recalled and, based on the sign-stability of matrices, a robust observer design method is proposed. Section 4 is devoted to the robust observer design for trophic chains of type 'resource - producer - primary consumer - secondary consumer'. In Section 5 similar observer design for Lotka-Volterra systems with vertical structure is considered. A discussion Section 6 closes our study. In the Appendix some calculations supporting the main body of the paper are summarized.

2. Sign-stability and digraphs

Definition 2.1. Two $n \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ are *sign-equivalent* if $\operatorname{sgn} a_{ij} = \operatorname{sgn} b_{ij}$ $(\operatorname{sgn} = +, -, 0)$. A matrix A is sign-stable if every matrix B sign-equivalent to A is stable.

Definition 2.2. Let $A = (a_{ij})$ be a $n \times n$ matrix

1. The *signed digraph* $SD(A)$ associated with A is the digraph (directed graph) with n vertices $\{v_1, v_2, \cdots, v_n\}$ and such that there is an arc from vertex v_j to vertex v_i if and only if $a_{ij} \neq 0$. Moreover, such arc is signed with $\operatorname{sgn} a_{ij}$.
2. The *graph* $G(A)$ associated with A is the undirected graph with n vertices $\{u_1, u_2, \cdots, u_n\}$ and such that there is an edge between v_i and v_j if and only if $i \neq j$ $a_{ij} \neq 0$ and $a_{ji} \neq 0$.

Sign-stability of matrix A can be studied by means of some properties of both $SD(A)$ and $G(A)$. On the one hand, properties of $SD(A)$ are related to its cycles while properties of $G(A)$ can be described in terms of partitions of its vertex set.

Definition 2.3. An *m-cycle* in $SD(A)$ is a sequence of m arcs $v_{i_1} \to v_{i_2} \to \cdots \to v_{i_m} \to v_{i_1}$, such that $v_{i_j} \neq v_{i_k}$. In particular, a 1-cycle, also called a *loop*, is an arc from a vertex to itself.

Definition 2.4. A *δ-coloring* in $G(A)$ is a partition of its vertex set $\{u_1, u_2, \cdots, u_n\}$ into two sets, black and white, satisfying:

i. if $a_{ii} \neq 0$, then u_i is a black vertex;

ii. no black vertex has exactly one white neighbor;

iii. each white vertex has at least a white neighbor.

Definition 2.5. An ε*-coloring* in $G(A)$ is a partition of its vertex set $\{u_1, u_2, \cdots, u_n\}$ into two sets, black and white, satisfying:

i. if $a_{ii} \neq 0$, then u_i is a black vertex;
ii. no black vertex has exactly one white neighbor;
iii. no white vertex has a white neighbor.

Note that conditions i. and ii. are equal in both definitions, while condition iii. is different.

The *trivial partition*, where all vertices are black, is both a δ-coloring and an ε-coloring. From Jeffries et al. [16], we recall the following characterization of sign stability:

Theorem 2.1. A square matrix A is sign-stable if and only if the following conditions hold:

1. each 1-cycle in $SD(A)$ is signed "$-$";
2. each 2-cycle in $SD(A)$ is signed "$- +$";
3. there is no m-cycle in $SD(A)$ with $m \geq 3$;
4. each δ-colouring in $G(A)$ is trivial;
5. each ε-colouring in $G(A)$ is trivial.

Remark 2.1. If matrix $A = (a_{ij})$ is such that $a_{ii} \neq 0$ for every index $i \in \{1, 2, \cdots, n\}$, then it is clear that A satisfies conditions 4 and 5. Therefore, in this case just conditions about cycles of digraph $SD(A)$ should be checked in order to study sign-stability.

3. Robust observer design by sign-stability

First, from Sundarapandian [5], we recall the basics of the standard observer design for nonlinear observation systems. Given positive integers *m, n*, let

$$f : \mathbf{R}^n \to \mathbf{R}^n, \quad h : \mathbf{R}^n \to \mathbf{R}^m,$$

consider the following observation system

$$\dot{x} = f(x) \tag{3.1}$$

$$y = h(x) \ , \tag{3.2}$$

where f and h are continuously differentiable functions, and $x^* \in \mathbf{R}^n$ is an equilibrium of dynamics (3.1) (i.e. $f(x^*) = 0$) and to the equilibrium there corresponds the zero observation: $h(x^*) = 0$. y is called the *observed function*.

Definition 3.1. Observation system (3.1)-(3.2) is called *locally observable near equilibrium* x^*, *on a given time interval* $[0,T]$, if there exists $\varepsilon > 0$, such that for any two different solutions x and $\bar{x}$ of system (3.1) with $| x(t) - x^* | < \varepsilon$ and $| \bar{x}(t) - x^* | < \varepsilon$ $(t \in [0,T])$, the observed functions $h(x(t))$ and $h(\bar{x}(t))$ are different. For the sake of brevity, the reference to $[0,T]$ is usually suppressed).

For a sufficient condition for local observability near the equilibrium, we need the linearization of the observation system (3.1)-(3.2) by the calculation of the corresponding Jacobian matrices:

$$A := f'(x^*) \quad \text{and} \quad C := h'(x^*).$$

Theorem 3.1. (Lee and Markus [17]). Suppose that

$$rank\left(C \,|\, CA \,|\, CA^2 \,|\, ... \,|\, CA^{n-1}\right)^T = n. \tag{3.3}$$

Then observation system (3.1)-(3.2) is locally observable near the equilibrium x^*.

Now, the observer design will be based on Sundarapandian [5]. Let us consider observation system (3.1)-(3.2).

Definition 3.2. Given a continuously differentiable function $g : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$, system

$$\dot{z} = g(z, y) \tag{3.4}$$

is called a *local asymptotic (respectively, exponential) observer for observation system* (3.1)-(3.2), if for the composite system (3.1)-(3.2), (3.4) satisfies the following requirements:

i) If $x(0) = z(0)$, then $x(t) = z(t)$, for all $t \geq 0$.

ii) There exists a neighborhood V of the equilibrium x^* of $\mathbf{R}^n$ such that for all $x(0), z(0) \in V$, the estimation error $z(t) - x(t)$ decays asymptotically (respectively, exponentially) to zero.

Theorem 3.2. (Sundarapandian, [5]). Suppose that observation system (3.1)-(3.2) is Lyapunov stable at equilibrium x^*, and that there exists a matrix K, the so-called *gain matrix*, such that matrix $A - KC$ is stable, where $A = f'(x^*)$ and $C = h'(x^*)$. Then dynamics defined by

$$\dot{z} = f(z) + K[y - h(z)] \tag{3.5}$$

is a local exponential observer for observation system (3.1)-(3.2).

Remark 3.1. We note that rank condition (3.3), together with the stability condition for equilibrium x^* already imply the existence of a gain matrix *K* providing the required observer system.

From this theorem it can be seen that the gain matrix plays a key role in observer design. In fact, since it guarantees the stabilization of the zero equilibrium of the error dynamics for: $e=z-x$:

$$\dot{e} = g(x+e, h(x)) - f(x)$$

In the present paper, we propose a new approach to *observer design* such that, under sign-stability conditions, the obtained observer system will be robust against quantitative changes in the inter- and intra-specific interactions. In other words, in the observer design the same gain matrix can be used even if, e.g. due to environmental changes, the intensities of certain interactions suffer a quantitative change in the meanwhile. Now, the observer design will be carried out as follows: Let us suppose that A is sign stable. Now, if we choose a gain matrix K such that adding $-KC$ to A does not change the sign pattern of the latter, then matrix $A-KC$ will also be stable. Assume now that there is a quantitative change in the model parameters such that the new Jacobian A' remains sign-equivalent to A, and $A'-KC$ is also sign-equivalent to A'. Then, for the changed system, the same gain matrix K will provide a local exponential observer for the changed system, too.

In the following sections the above observer design scheme will be illustrated by the monitoring of model population systems, such as trophic chains of type resource-producer-primary consumer-secondary consumer and Lotka-Volterra systems with vertical structure.

4. Monitoring of a trophic chain by robust observer design

4.1. The dynamic model of a trophic chain

In this section we show how our observer design approach can be applied to the monitoring of a four-level trophic chain of the type *resource - producer - primary consumer - secondary consumer.* (The monitoring of a simpler trophic chain, with the classical observer design was considered in Varga et al. [18]) The present model would describe how a resource moves through a trophic chain, consisting of the following components and interactions:

resource, the *0th trophic level* (solar energy or inorganic nutrient), which is incorporated by

a plant population, the *1st trophic level (producer)*, which transfers it to

a herbivorous animal population, the *2nd trophic level (primary consumer)*, which is consumed by

a carnivorous animal population, the *3rd trophic level (secondary consumer).*

Bearing in mind the possible types of 0th level (energy or nutrient), we present several types of trophic chains: *open chains* (without recycling), *closed chains* (with recycling) and *partially closed chains.*

Let us denote by x_0 the time-varying quantity (density) of "free" resource present in the system, x_1, x_2 and x_3, in function of time, the biomass (or density) of the producer (species 1), the primary consumer (species 2) and the secondary consumer (species 3), respectively. Let Q be the resource supply considered constant in the model; γ_{01} and γ_{10} the consumption rate from the resource by species 1, and the conversion rate from resource to biomass of species 1, respectively; γ_{ii} the intra-specific rate (or *autolimitation*) for species $i = 1,2,3$; γ_{12} and γ_{23} the consumption rate from species 1 by species 2, and from species 2 by species 3, respectively; γ_{21} and γ_{32} the corresponding conversion rates. Both the plant and the animal species, in the absence of the resource and the other species, are supposed to decrease, and exponentially at low densities, with respective rates of decrease (Malthus parameters) $\varepsilon_1, \varepsilon_2$ and ε_3.

In a *closed system* the dead individuals of species 1, 2 and 3 are recycled into free nutrient at respective rates $0 < \beta_1 < 1$, $0 < \beta_2 < 1$ and $0 < \beta_3 < 1$, while for an *open system* (where there is no natural recycling) $\beta_1 = 0, \beta_2 = 0, \beta_3 = 0$ holds. In particular we shall study the case where $\beta_1 > 0, \beta_2 = 0, \beta_3 = 0$, that is, a *partially closed system.*

Then with model parameters

$$Q, \varepsilon_1, \varepsilon_2, \varepsilon_3, \gamma_{01}, \gamma_{10}, \gamma_{11}, \gamma_{22}, \gamma_{33}, \gamma_{12}, \gamma_{21}, \gamma_{23}, \gamma_{32} > 0; \quad \beta_1, \beta_2, \beta_3 \in [0,1[, \tag{4.1}$$

the dynamic model for the trophic chain can be set up as follows:

$$\dot{x}_0 = Q - \gamma_{01}x_0x_1 + \beta_1\varepsilon_1x_1 + \beta_2\varepsilon_2x_2 + \beta_3\varepsilon_3x_3 \tag{4.2}$$
$$\dot{x}_1 = x_1[-\varepsilon_1 + \gamma_{10}x_0 - \gamma_{11}x_1 - \gamma_{12}x_2] \tag{4.3}$$
$$\dot{x}_2 = x_2[-\varepsilon_2 + \gamma_{21}x_1 - \gamma_{22}x_2 - \gamma_{23}x_3] \tag{4.4}$$
$$\dot{x}_3 = x_3[-\varepsilon_3 + \gamma_{32}x_2 - \gamma_{33}x_3] \tag{4.5}$$

Now, in order to consider a general interaction pattern, we will study a *partially closed system*, with $\beta_1 > 0, \beta_2 = 0, \beta_3 = 0$.

Let us start with checking that there exists a positive equilibrium x^* for system (4.2)-(4.5):

Theorem 4.1. Suppose that: 1. The resource supply Q is high enough, and 2. The consumption rate from the resource by species 1, γ_{01}, is small enough. Then, for the partially closed system (4.2)-(4.5) with $\beta_1 > 0, \beta_2 = 0, \beta_3 = 0$, there exists a positive equilibrium x^*.

Proof. See Appendix.

From now on, throughout Section 4, we assume that the conditions of Theorem 4.1 are satisfied.

4.2. Stable coexistence of the trophic chain

Let us study the stability of the equilibrium from the Jacobian of system (4.2)-(4.5) at the positive equilibrium x^*. The Jacobian of the right-hand side of (4.2)-(4.5) at the positive equilibrium x^* is

$$A = \begin{pmatrix} -\gamma_{01}x_1^* & -\gamma_{01}x_0^* + \beta_1\varepsilon_1 & 0 & 0 \\ \gamma_{10}x_1^* & -\gamma_{11}x_1^* & -\gamma_{12}x_1^* & 0 \\ 0 & \gamma_{21}x_2^* & -\gamma_{22}x_2^* & -\gamma_{23}x_2^* \\ 0 & 0 & \gamma_{32}x_3^* & -\gamma_{33}x_3^* \end{pmatrix}. \tag{4.6}$$

We shall suppose that

$$\beta_1\varepsilon_1 < \gamma_{01}x_0^* \tag{4.7}$$

to guarantee sign-stability of matrix A. Let us check it. Since $x^* > 0$,and every γ_{ij} figuring in the model is positive, we have that $a_{ii} \neq 0$, in particular $a_{ii} < 0$, for $i \in \{1,2,3,4\}$. The signed digraph $SD(A)$ associated with the Jacobian matrix is shown in Figure 1.

Figure 1. Signed digraph associated with A

It is clear that $SD(A)$ satisfies conditions 1, 2 and 3 of the Theorem 2.1 (see also Remark 2.1), hence A is sign-stable.

Remark 4.1. Since sign-equivalence in Definition 2.1. is reflexive, A is a stable matrix, implying asymptotic stability of equilibrium x^*. In biological terms this means stable coexistence of the considered trophic chain.

4.3. Observation of certain components of the trophic chain

Suppose that for technical or economic reasons it is convenient to observe only one of the components of the trophic chain, and we want to recover the densities of the rest of the components. Suppose we observe either the resource or the carnivore.

Observation of the resource. For technical reason we suppose that the deviation of resource density from its equilibrium value is observed, that is, we have

$$y = h(x) = x_0 - x_0^*,$$

with

$$C = h'(x^*) = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}.$$

Then

$$\det\left(C \middle| CA \middle| CA^2 \middle| CA^3\right)^T = -\gamma_{12}^2 \gamma_{23} (\beta_1 \varepsilon_1 - \gamma_{01} x_0^*)^3 \left(x_1^*\right)^2 x_2^*,$$

which is different from zero if and only if $\beta_1 \varepsilon_1 \neq \gamma_{01} x_0^*$. Hence, by condition (4.7) we have $rank \left(C \middle| CA \middle| CA^2 \middle| CA^3\right)^T = 4$, implying the local observability of the trophic chain near the positive equilibrium x^*, observing the resource.

Remark 4.2. Inequality $\beta_1 \varepsilon_1 \neq \gamma_{01} \mathrm{x}_0^*$ always holds in our model, because otherwise by the equilibrium condition in equation (4.2) we would have

$$Q - \gamma_{01} x_0^* x_1^* + \beta_1 \varepsilon_1 x_1^* = Q + x_1^* (-\gamma_{01} x_0^* + \beta_1 \varepsilon_1) = Q = 0,$$

but the resource supply Q in our model is positive.

Observation of the carnivorous (predator) species. Let us consider the following observation function

$$y = h(x) = x_3 - x_3^*,$$

with

$$C = h'(x^*) = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then

$$\det\left(C \middle| CA \middle| CA^2 \middle| CA^3\right)^T = \gamma_{10} \gamma_{21}^2 \gamma_{32}^3 x_1^* \left(x_2^*\right)^2 \left(x_3^*\right)^3 > 0,$$

and therefore we have that the system is locally observable near the positive equilibrium observing the carnivorous species.

Robust observer design, observing either the resource or the carnivore. We can proceed according to the methodology of Section 3, applying first Theorem 3.2. According to Remark 4.1, the positive equilibrium x^* is asymptotically stable, for both considered observation situations we only need to find an appropriate gain matrix *K* such that matrix *A-KC* is sign-stable, where $C = h'(x^*)$ and *h* is the corresponding observation function.

Considering the *observation of the resource*, we have $C = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$. Choosing $K = col\begin{pmatrix} k_1 & 0 & 0 & 0 \end{pmatrix}$ with some $k_1 > 0$, let us check that *A-KC* is sign-stable. Indeed, matrix

$$A - KC = \begin{pmatrix} -\gamma_{01}x_1^* - k_1 & -\gamma_{01}x_0^* + \beta_1\varepsilon_1 & 0 & 0 \\ \gamma_{10}x_1^* & -\gamma_{11}x_1^* & -\gamma_{12}x_1^* & 0 \\ 0 & \gamma_{21}x_2^* & -\gamma_{22}x_2^* & -\gamma_{23}x_2^* \\ 0 & 0 & \gamma_{32}x_3^* & -\gamma_{33}x_3^* \end{pmatrix}$$

keeps the same sign structure of *A* because *A-KC* is the equal to matrix *A* except the 11-entry, which is negative in both matrices. Therefore *A-KC* is sign-stable, and applying Theorem 3.2 we obtain a local exponential observer for system (4.2)-(4.5). Now, if due to a sign-preserving change of the interaction parameters, matrix *A* changes to *A*' having the same sign pattern, then matrix *A*'*-KC* will also be sign-equivalent to *A*', and therefore stable. Hence we can conclude that *K* remains a valid gain matrix for the changed system, too. In other words we obtained a *robust observer* for the original system where the resource is observed.

As for the *robust observer design with observation of the carnivore*, we can proceed in an analogous way. Indeed, observing even any species *i* putting

$$y = h(x) := x_i - x_i^* \quad i \in \{1,2,3\},$$

again for a gain matrix we can choose a column matrix *K* with the only non-zero entry $k_{i+1} > 0$.

This construction, similarly to the case of the observation of the resource, also results in a robust observer.

Example 4.1. Let us consider the model trophic chain (4.2)-(4.5) with illustrative parameter values

$Q = 10;\ \varepsilon_1 = 0.1, \varepsilon_2 = 0.4, \varepsilon_3 = 0.2;\ \beta_1 = 0.2$

$\gamma_{01} = 0.3, \gamma_{10} = 0.15, \gamma_{11} = 0.1, \gamma_{22} = 0.2, \gamma_{33} = 0.1, \gamma_{12} = 0.1, \gamma_{21} = 0.5, \gamma_{23} = 0.15, \gamma_{32} = 0.1$

Let us shortly denote by

$$\dot{x} = f(x) \tag{4.8}$$

the dynamic model of the trophic chain (4.2)-(4.5), with the above, numerically given concrete parameters. A straightforward checking shows that these parameters satisfy conditions 1 and 2 of Theorem 4.1, therefore we have a positive equilibrium for the corresponding system (4.8), namely $x^* = (6.58\ , 5.28\ , 3.3\ , 7.9)$, which is asymptotically stable. Condition (4.7) also holds, implying that the corresponding Jacobian (4.6) is sign-stable. Let us observe the resource. Then we have

$$y = h(x) := x_0 - x_0^* \tag{4.9}$$

It is easy to check that e.g. $K = col(2.5\ ,0\ ,0\ ,0)$ is an appropriate gain matrix. *A-KC* is sign-stable, and hence stable. Applying Theorem 3.2, with gain matrix *K* we define the observer system

$$\dot{z} = f(z) + K[y - h(z)]\,. \tag{4.10}$$

Let us see how, in the knowledge of the observed function *y*(*t*)=*h*(*x*(*t*)) the unknown solution *x*(*t*) of system (4.8) can be estimated (exponentially approximated) by a solution *z*(*t*) of the observer (4.10). Figure 2 shows how the solution of (4.9) with initial value $x(0) = (5{,}8{,}7{,}2)$, considered unknown, can be approximated by the solution of the observer system with initial value $z(0) = (6{,}9{,}6{,}3)$

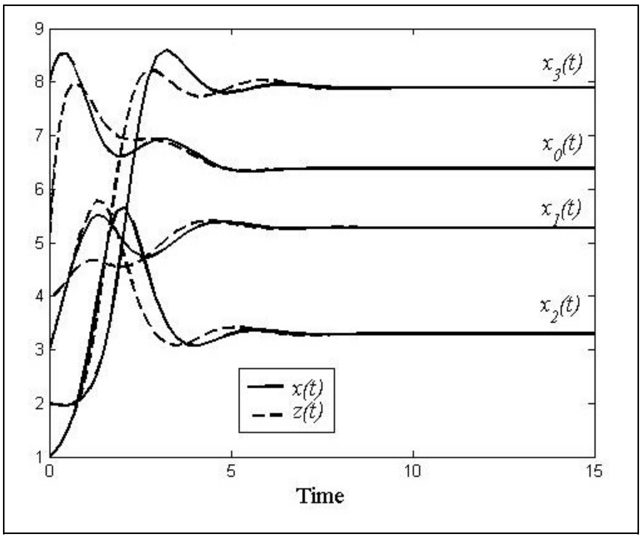

Figure 2. Solution of the observer system, approximating an "unknown" solution of the original system

Now we show the *robustness of the constructed observer* (4.10). Let us assume that, for example due to some environmental change, there is a quantitative change in the model parameters, such that the new Jacobian *A'* remains sign-equivalent to *A*, and *A'–KC* is also sign-equivalent to *A'*. Then, for the changed system, the same gain matrix *K* will provide a local exponential observer for the changed system, too. More concretely, let us suppose that, e.g. due to environmental changes the grazing activity of the herbivore and the predatory activity of the predator, i.e. γ_{12}, γ_{23} are reduced, say, by 0.05. This also changes the equilibrium to $x^* = (5.89\ ,5.72\ ,4.24\ ,10.74)$, nevertheless, the sign pattern of matrix *A* remains the same, and by the robustness of the observer, the previous gain matrix *K*=col(2.5,0,0,0) remains valid. Calculating with the same initial values $x(0) = (5,8,7,2)$, $z(0) = (6,9,6,3)$, the corresponding observer really works, as shown in Figure 3.

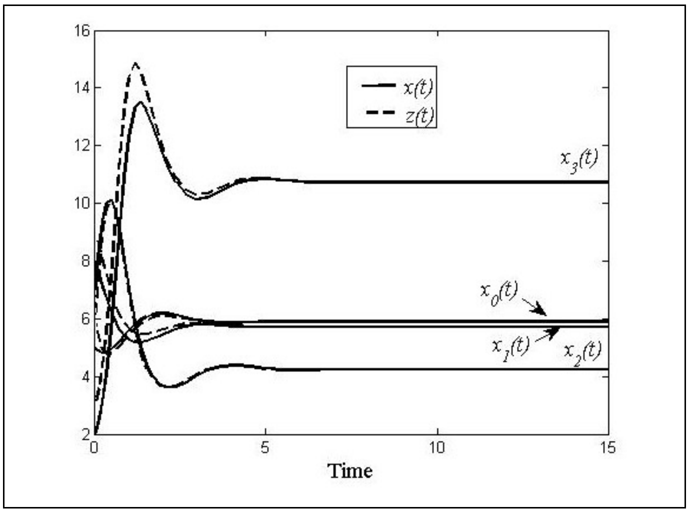

Figure 3. Robust observer also approximates the “unknown” solution of the changed system

5. Robust observer design for a Lotka-Volterra system with vertical structure

Let us consider the following two predator-two prey Lotka-Volterra system where species 1 and 3 are preys and species 2 and 4 are predators; $\varepsilon_i > 0 \quad (i \in \{1,2,3,4\})$ is the Malthus parameters of species *i*, γ_{ij} the interactions parameter expressing the effect of species *j* on species *i* and, in particular, and $\gamma_{ii} > 0 \quad (i \in \{1,2,3,4\})$ is the rate of autolimitation in species *i*, for each species. By convention, $\gamma_{12}, \gamma_{21}, \gamma_{23}, \gamma_{32}, \gamma_{34}, \gamma_{43} > 0$. We will consider the Lotka-Volterra model (5.1)

$$\begin{aligned} \dot{x}_1 &= x_1[\varepsilon_1 - \gamma_{11}x_1 - \gamma_{12}x_2] \\ \dot{x}_2 &= x_2[-\varepsilon_2 + \gamma_{21}x_1 - \gamma_{22}x_2 + \gamma_{23}x_3] \\ \dot{x}_3 &= x_3[\varepsilon_3 - \gamma_{32}x_2 - \gamma_{33}x_3 - \gamma_{34}x_4] \\ \dot{x}_4 &= x_4[-\varepsilon_4 + \gamma_{43}x_3 - \gamma_{44}x_4] \end{aligned} . \tag{5.1}$$

In order to write this system in the compact form

$$\dot{x} = \mathrm{Diag}x \cdot [\varepsilon - \Gamma x],$$

in terms of the we define the interaction matrix Γ as follows:

$$\Gamma = \begin{bmatrix} \gamma_{11} & \gamma_{12} & 0 & 0 \\ -\gamma_{21} & \gamma_{22} & -\gamma_{23} & 0 \\ 0 & \gamma_{32} & \gamma_{33} & \gamma_{34} \\ 0 & 0 & -\gamma_{43} & \gamma_{44} \end{bmatrix}$$

and Malthus parameter vector

$$\varepsilon := \begin{pmatrix} \varepsilon_1 & -\varepsilon_2 & \varepsilon_3 & -\varepsilon_4 \end{pmatrix}^T$$

we can express system (5.1) in the following compact form

$$\dot{x} = \mathrm{Diag}x \cdot [\varepsilon - \Gamma x],$$

where $\mathrm{Diag}x$ is the diagonal matrix formed by vector $x = (x_1, x_2, x_3, x_4)$.

5.1. Equilibrium and stability

Positive equilibrium and local asymptotic stability

It is very easy to check that matrix Γ is invertible, so dynamics (5.1) has a unique equilibrium

$$x^* = \Gamma^{-1}\varepsilon .$$

In our model we are interested in a positive equilibrium.

Theorem 5.1. If the following condition holds

$$\varepsilon_1 > \varepsilon_2 \frac{\gamma_{11}}{\gamma_{21}}; \quad \varepsilon_2 > \max\left\{ \varepsilon_3 \frac{\gamma_{23}}{\gamma_{33}}, \varepsilon_4 \frac{\gamma_{23}}{\gamma_{43}} \right\}; \quad \varepsilon_3 > \varepsilon_1 \frac{\gamma_{32}}{\gamma_{12}} + \varepsilon_4 \frac{\gamma_{33}}{\gamma_{43}};$$

that is, if some Malthus parameters are high enough, then there exists a positive equilibrium x^* for system (5.1).

Proof. See Appendix.

Now, we are going to see the (local) asymptotic stability of equilibrium checking the sign-stability of the Jacobian matrix of system (5.1)

$$A = -(Diag\ x^*) \cdot \Gamma = \begin{bmatrix} -\gamma_{11}x_1^* & -\gamma_{12}x_1^* & 0 & 0 \\ \gamma_{21}x_2^* & -\gamma_{22}x_2^* & \gamma_{23}x_2^* & 0 \\ 0 & -\gamma_{32}x_3^* & -\gamma_{33}x_3^* & -\gamma_{34}x_3^* \\ 0 & 0 & \gamma_{43}x_4^* & -\gamma_{44}x_4^* \end{bmatrix}.$$

The signed-digraph $SD(A)$ associated to the Jacobian matrix is shown in Figure 4.

Figure 4. Signed digraph associated with A

The digraph of A satisfies conditions 1, 2 and 3 of the Theorem 2.1, so A is sign-stable, in particular stable. Therefore equilibrium x^* is asymptotically stable for system (5.1).

Dissipativity and global asymptotic stability of the equilibrium

First from Logofet [15] we recall the classical definition of dissipativity of a Lotka-Volterra population system: Let

$$\dot{x} = \mathrm{Diag}x \cdot [\varepsilon - \Gamma x] \tag{5.2}$$

a general four-species Lotka-Volterra system, and $D = -\Gamma$.

Theorem 5.2. System (5.2) is dissipative, then equilibrium x^* is asymptotically stable, and its basin of attraction is R_+^n.

Proof. System (5.1) has the following form

$$\dot{x} = \mathrm{Diag}x \cdot [\varepsilon - \Gamma x].$$

By the definition of dissipative system (see e.g. Logofet [15]), we need to find constants $\alpha_1, \ldots, \alpha_4 > 0$ such that

$$V(x) = \sum_{i,j=1}^{4} \alpha_i d_{ij} x_i x_j < 0 \quad (x \in R^4, x \neq 0),$$

where, to keep the usual form of the definition, we put $D = (d_{ij}) = -\Gamma$.

$$\begin{aligned} V(x) = \alpha_1[-\gamma_{11}x_1^2 - \gamma_{12}x_1x_2] + \alpha_2[\gamma_{21}x_2x_1 - \gamma_{22}x_2^2 + \gamma_{23}x_2x_3] + \\ \alpha_3[-\gamma_{32}x_3x_2 - \gamma_{33}x_3^2 - \gamma_{34}x_3x_4] + \alpha_4[\gamma_{43}x_4x_3 - \gamma_{44}x_4^2] \end{aligned}$$

Now fix $\alpha_1 = 1$ and set $\alpha_2 = \dfrac{\gamma_{12}}{\gamma_{21}}$. Then $\alpha_1\gamma_{12} = \alpha_2\gamma_{21}$ and the x_1x_2 terms are cancelled.

Similarly, in order to cancel the x_2x_3 terms, we have to find an α_3 such that

$\alpha_2\gamma_{23} = \alpha_3\gamma_{32}$, hence $\alpha_3 = \dfrac{\gamma_{12}\gamma_{23}}{\gamma_{21}\gamma_{32}}$. We can similarly find $\alpha_4 = \dfrac{\gamma_{12}\gamma_{23}\gamma_{34}}{\gamma_{21}\gamma_{32}\gamma_{43}}$. Since each γ_{ii} is positive, we obtain $V(x) = -\sum_{i=1}^{4}\alpha_i\gamma_{ii}x_i^2 < 0$. Therefore system (5.2) is dissipative.

Furthermore, from the dissipativity of system (5.2) it follows that

$$W(x) = \sum_{i=1}^{4}\alpha_i x_i^*\left(\frac{x_i}{x_i^*} - \ln\frac{x_i}{x_i^*} - 1\right) \quad (x \in \mathbf{R}^4, x > 0.)$$

is a Lyapunov function implying asymptotic stability of equilibrium x^* and its basin of attraction is R_+^n.

Example 5.1. Now we illustrate the global asymptotic stability of the positive equilibrium in the positive orthant of $\mathbf{R}^4$. To this end we take the following parameters: $\varepsilon_1 = 6.5$; $\varepsilon_2 = 5$; $\varepsilon_3 = 6$; $\varepsilon_4 = 1$; $\gamma_{11} = 1.1$; $\gamma_{12} = 0.1$; $\gamma_{21} = 1$; $\gamma_{22} = 0.2$; $\gamma_{23} = 0.5$; $\gamma_{32} = 0.1$; $\gamma_{33} = 0.7$; $\gamma_{34} = 0.8$; $\gamma_{43} = 0.9$; $\gamma_{44} = 0.3$.

For these parameters the conditions of Theorem 5.1 are satisfied, and we have a positive equilibrium x^* = (5.22 , 7.51 , 2.55 , 4.33). In Figure 5, decomposing the four dimensions, we show the solutions of system (5.1) tending to the equilibrium from different initial points.

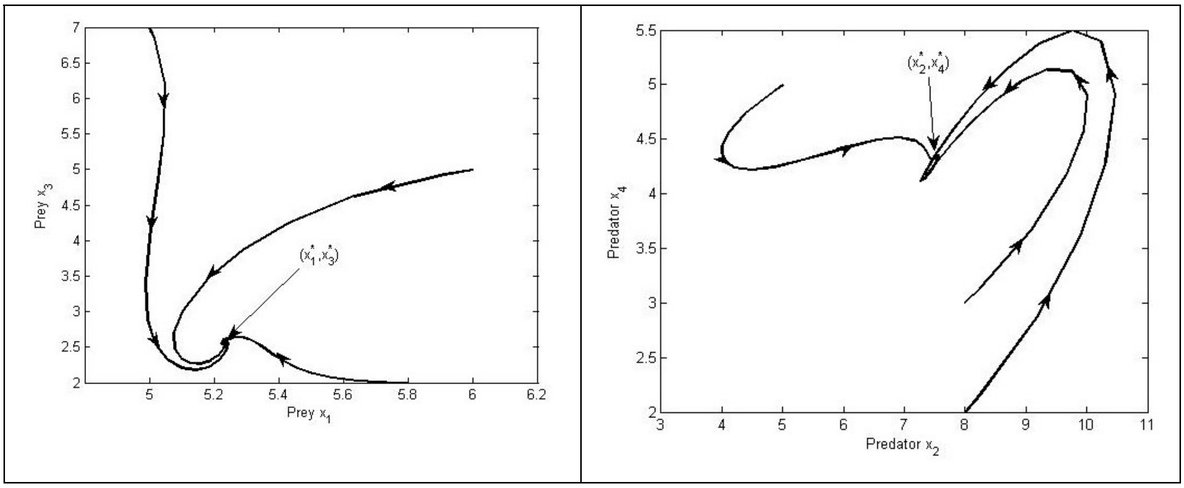

Figure 5. Global asymptotic stability of the positive equilibrium of system (5.1)

5.2. Observation of an indicator species

Observability

Let us consider the observation of the predator 4, that is,

$$y = h(x) = x_4 - x_4^*,$$

with

$$C = h'(x^*) = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then

$$\det\left(C\middle|CA\middle|CA^2\middle|CA^3\right)^T = \gamma_{21}\gamma_{32}^2\gamma_{43}^3 x_2^*\left(x_3^*\right)^2\left(x_4^*\right)^3 \neq 0$$

due to $x^* > 0$ and $\gamma_{21}, \gamma_{32}\gamma_{43} > 0$. Therefore, $rank\left(C\middle|CA\middle|CA^2\middle|CA^3\right)^T = 4$ and we obtain the local observability of system near the positive equilibrium x^*, observing this predator 4.

Construction of a robust observer system

We have that the positive equilibrium of system (5.1) is globally asymptotically stable, so proceeding similarly to the case of the non-Lotka-Volterra type trophic chain. We continue with the observation of the predator 4, that is,

$$y = h(x) := x_4 - x_4^*,$$

so $C = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$. Choosing gain matrix $K = col\begin{pmatrix} 0 & 0 & 0 & k_4 \end{pmatrix}$ with any fixed $k_4 > 0$. Then matrix

$$A - KC = \begin{bmatrix} -\gamma_{11}x_1^* & -\gamma_{12}x_1^* & 0 & 0 \\ \gamma_{21}x_2^* & -\gamma_{22}x_2^* & \gamma_{23}x_2^* & 0 \\ 0 & -\gamma_{32}x_3^* & -\gamma_{33}x_3^* & -\gamma_{34}x_3^* \\ 0 & 0 & \gamma_{43}x_4^* & -\gamma_{44}x_4^* - k_4 \end{bmatrix}$$

is sign-stable, and hence also stable. Thus, by Theorem 3.2 we obtain a local exponential observer for system (5.1). Then, in similar way as in Section 4, a robust observer can be designed.

Example 5.2. Let us numerically illustrate now the robust observer design for the Lotka-Volterra system (5.1). Working with the parameters of Example 5.1, let us observe predator 4. As we can easily check, gain matrix *K*=col(0,0,0,20). Now the solution of system (5.1) with initial value

$x(0)=(8,3,1,2)$, considered unknown, and the solution of the observer system with initial value $z(0)=(5,4,3,1)$ are plotted in Figure 6.

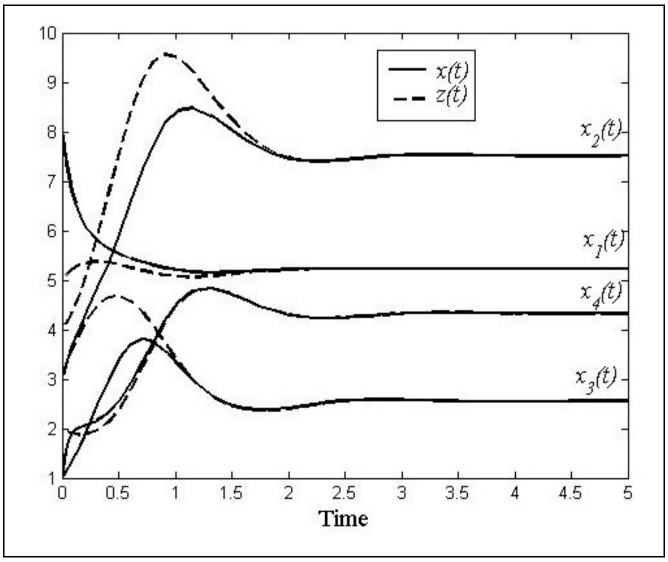

Figure 6. Solution of the observer system approaching an “unknown” solution of system (5.1)

Now let us suppose that by an environmental change the carrying capacity of each species decreases. In our parametrization this is reflected in the increase of the autolimitation coefficients. Let us calculate with an increase of each γ_{ii} by 0.2. Then checking the corresponding new Jacobian A' we obtain that gain matrix K=col(0,0,0,20) remains valid for the changed model, too. Figure 7 shows how this gain matrix provides the required approximation of the state process of the changed Lotka-Volterra system.

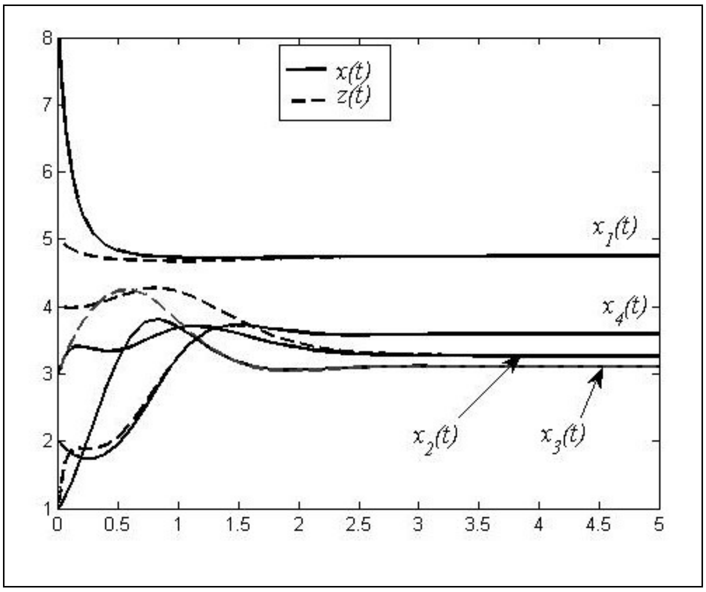

Figure 7. For the changed model, the same gain matrix provides the required approximation of the state process, with $x(0)=(8,3,1,2)$ and $z(0)=(5,4,3,1)$.

6. Discussion

The monitoring problem concerning population systems (also called biological communities, see e.g. Svirezhev and Logofet [14]) is closely related to the stability properties of the system. In the paper stably coexisting population systems are considered not only because of their ecological relevance, but also because of the technical importance of stability for the mathematical tools of monitoring.

The Jacobian matrix A of the right-hand side of the population system model, calculated at an equilibrium plays a key role in the characterization of a population system. For stably coexisting population systems, a new approach to the design of the corresponding observer system is proposed which combines the well-known nonlinear observer design methodology with graph-theoretical characterization of the sign-stability of matrix A. A further novelty of the proposed method is that, due to the required sign-stability (or qualitative stability) of the interaction pattern described by matrix A, in the observer design the gain matrix is robust against quantitative changes in the inter- and intra-specific interactions. These changes are often due to environmental changes, often to abiotic effects caused by human activities.

As for the presented illustrative applications of the proposed monitoring methodology, we note that the presented robust observer design can be easily extended to longer trophic chains and similarly structured Lotka-Volterra models with $2n$ predators and $2n$ preys. A “degenerate” case of this model is when, instead of certain predatory interactions, commensalism takes place. We note that the latter also belongs to the class of verticum-type systems which also admit observer design by decomposition into subsystems, see Molnar et al. [19] We also mention that equilibrium control of the considered trophic chain was studied in Gámez et al. [20]. Finally, we call the attention to Boutat and Saif [21] which, unlike the mostly used local observers, proposes globally convergent observers for certain predator-prey models. The globality of an observe is an attractive property, but it needs a special transformation of the system to a so-called observer normal form, the construction of which, unlike the local observer design, may be rather complicated. Furthermore, especially for ecological purpose, it is worth it to develop the methodology of local observer design for the monitoring of ecological systems (or population systems). In fact, if a population system is far from an equilibrium, then the main issue of conservation ecology is to control the system into equilibrium, which needs a control-theoretical approach, see e.g. Varga [9], Gámez [10] and Gámez et al. [20].

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Appendix

Positive equilibrium for the trophic chain

Proof of Theorem 4.1

From equations (4.5), (4.4), (4.3) we obviously obtain the first three coordinates of the equilibrium, as function of the last one:

$$x_2^* = \frac{\gamma_{33}x_3^* + \varepsilon_3}{\gamma_{32}}, \quad x_1^* = \frac{\gamma_{22}x_2^* + \gamma_{23}x_3^* + \varepsilon_2}{\gamma_{21}}, \quad x_0^* = \frac{\gamma_{11}x_1^* + \gamma_{12}x_2^* + \varepsilon_1}{\gamma_{10}}. \tag{A.1}$$

Hence, for the existence of a positive equilibrium $x^* > 0$, it will be sufficient to prove that there exists an equilibrium with $x_3^* > 0$. From equation (4.2) we have

$$Q - \gamma_{01}x_0^*x_1^* + \beta_1\varepsilon_1x_1^* = 0.$$

From this equality, substituting x_0^* and x_1^*, as function of x_3^* according to (A.1), we obtain the following quadratic equation for x_3^* of the form:

$$s(x_3^*) := A(x_3^*)^2 + Bx_3^* + C = 0 \tag{A.2}$$

where A, B and C are functions of parameters (4.1) obtained using the "Mathematica" software. The quadratic function s obviously has a negative leading coefficient, $A < 0$. To prove the existence of at least one positive solution of (A.2) for x_3^*, a sufficient condition is that the vertex of the parabola corresponding to (A.2) has positive coordinates, that is, $\frac{-B}{2A}>0$ and $s\left(\frac{-B}{2A}\right)>0$. Since $A<0$, it is enough to search conditions to guarantee $B>0$ because that would imply $\frac{-B}{2A}>0$. Since according to the symbolic calculations we have $B=-\gamma_{01}D+E$ where $D, E>0$, if the rate γ_{01} of resource consumption by the producer is small enough, we have $B>0$. Analogously we get $s\left(\frac{-B}{2A}\right)=F+GQ$, where F and G depend on the parameters (4.1), and $G>0$. Therefore, considering a resource supply Q high enough we have $s\left(\frac{-B}{2A}\right)>0$, obtaining at least one solution $x_3^*>0$ of (A.2). Finally, from (A.1) we obviously get the existence of an equilibrium $x^*>0$.

Positive equilibrium for the Lotka-Volterra model with vertical structure

Proof of Theorem 5.1.

To proof the positivity of the equilibrium we shall calculate it and after that it will be easy to check that its positivity holds under condition (5.2). Firstly, we show the determinant of matrix Γ, that obviously is positive,

$$d=\det\Gamma=(\gamma_{12}\gamma_{21}+\gamma_{11}\gamma_{22})(\gamma_{34}\gamma_{43}+\gamma_{33}\gamma_{44})+\gamma_{11}\gamma_{23}\gamma_{32}\gamma_{44}>0.$$

Then, using the *Mathematica* software we obtain the equilibrium $x^*=\Gamma^{-1}\varepsilon$, which after some algebraic transformations takes the following form

$$x_1^*=\frac{\gamma_{12}\gamma_{34}(\varepsilon_2\gamma_{43}-\varepsilon_4\gamma_{23})+\gamma_{12}\gamma_{44}(\varepsilon_2\gamma_{33}-\varepsilon_3\gamma_{23})+\varepsilon_1\gamma_{22}(\gamma_{34}\gamma_{43}+\gamma_{33}\gamma_{44})+\varepsilon_1\gamma_{23}\gamma_{32}\gamma_{44}}{d}>0$$

$$x_2^*=\frac{\gamma_{11}\gamma_{23}(\varepsilon_4\gamma_{34}+\varepsilon_3\gamma_{44})+(\varepsilon_1\gamma_{21}-\varepsilon_2\gamma_{11})(\gamma_{34}\gamma_{43}+\gamma_{33}\gamma_{44})}{d}>0$$

$$x_3^*=\frac{\varepsilon_4\gamma_{34}(\gamma_{12}\gamma_{21}+\gamma_{11}\gamma_{22})\gamma_{34}+\gamma_{21}\gamma_{44}(\varepsilon_3\gamma_{12}-\varepsilon_1\gamma_{32})+\gamma_{11}\gamma_{44}(\varepsilon_3\gamma_{22}+\varepsilon_2\gamma_{32})}{d}>0$$

$$x_4^*=\frac{\gamma_{11}\gamma_{32}(\varepsilon_2\gamma_{43}-\varepsilon_4\gamma_{23})+\gamma_{11}\gamma_{22}(\varepsilon_3\gamma_{43}-\varepsilon_4\gamma_{33})+\gamma_{21}(\varepsilon_3\gamma_{12}\gamma_{43}-\varepsilon_4\gamma_{12}\gamma_{33}-\varepsilon_1\gamma_{32}\gamma_{43})}{d}>0$$

Hence the sufficient conditions in Theorem can be easily obtained.