# STOCK ESTIMATION, ENVIRONMENTAL MONITORING AND EQUILIBRIUM CONTROL OF A FISH POPULATION WITH RESERVE AREA

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**Abstract:**

For sustainable exploitation of renewable resources, the separation of a reserve area is a natural idea. In particular, in fishery management of such systems, dynamic modelling, monitoring and control has gained major attention in recent years. In this paper, based on the known dynamic model of a fish population with reserve area, the methodology of mathematical systems theory and optimal control is applied. In most cases, the control variable is fishing effort in the unreserved area. Working with illustrative data, first a deterministic stock estimation is proposed using an observer design method. A similar approach is also applied to the estimation of the effect of an unknown environmental change. Then it is shown how the system can be steered to equilibrium in given time, using fishing effort as an open-loop control. Furthermore, a corresponding optimal control problem is also solved, maximizing the harvested biomass while controlling the system into equilibrium. Finally, a closed-loop control model is applied to asymptotically control the system into a desired equilibrium, intervening this time in the reserve area.

***Keywords:*** stock estimation, fishery resource management, reserve area, observer system, ecosystem monitoring, ecosystem control

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## 1. Introduction

In ecological management, in particular, in management of fish resources optimal control methods play an important role. Since the publication of the basic monograph by Clark (1976), a large number of publications have been dedicated to the development of the optimal control methodology applied to the management of renewable resources, see e.g. Goh (1980). We also call the attention to Clark (2010), the third, extended edition of Clark (1976). Modifying Clark's model, Chaudhuri (1986, 1988) studied combined harvesting and considered the perspectives of bioeconomics and dynamic optimization of a two-species fishery, see also Kar and Chaudhuri (2004) concerning two prey one predator fishery.

Recently, also qualitative properties of population systems, such as controllability and observability have been studied, see e.g. Shamandy (2002,) López et al. (2004), (2007a, b). For an overview of the applications of mathematical systems theory in this context, we refer the reader to the review Varga (2008).

Over the last decades, the problem of sustainability of marine fisheries, the study the effects of a reserve area has played an important role in the management of fish resources. In fact, the protection of a portion of the fishery stocks agains future overfishing, can be realized in a reserve (or no-take) area where fishing is prohibited, see e.g. Agardy (1997), Pauly et al. (2002) and a recent overview of the ecological effects of marine reserves a Lester et al. (2009). For a survey of criticalv science gaps in the application of reserve areas we refere to Sale et al. (2005).

To our knowledge, Dubey et al. (2003) was the first paper was the first paper where the effect of a reserve area on the exploitation of a fishery resource has been modelled and analyzed in terms of a continuous-time logistic dynamics. The authors derive sufficient conditions for the existence of equilibrium in the dynamic model, and they also analyze its stability properties. In Bischi and Lamantia (2007), based on a single-species discrete time logistic model with reserve area, the game-theoretic conflict of several fishing agents is studied, where the harvested fish is sold on a Cournot-type oligopolistic market. Cartigny et al. (2008) analyze the problem of designing the access of small- and large-scale fishermen to a protected fishing reproductive area. A comparison of different dynamic fishery models with reserve area is discussed in Loisel and Cartigny (2009).

In our paper, the stock estimation and monitoring of the considered population system will be based on the observability theory of nonlinear systems of Lee and Markus (1971), and on the observer design methodology of Sundarapandian (2002). We call the attention to the fact that the observer system also provides a deterministic stock estimation method for the reserved area, as well; see Guiro et al. (2009). In the latter a *global* observer was constructed for the same model of Dubey et al. (2003), with a different methodology, and a compact survey of observer design methods was also given. Although our observer is only *local*, i.e. provides stock estimates only near the equilibrium state, it may be more efficient than the global one, as shown in Gámez et al. (2011). In the latter paper the monitoring problem in the fishing effort model with reserve area has been studied, in particular, observer has been also constructed for the system under the effect of a seasonal change in the abiotic environment. In the present work we complete this with the estimation of an unknown environmental parameter, using the same observer design methodology.

Finally, applying the usual discounted *infinite time horizon* optimal control model e.g. Clark, 1976 and Goth, 1980 discuss the optimal harvesting policy in terms of the fishing effort model. In the present paper we will use the same logistic dynamics with the fishing effort as control variable, but first we deal with a *finite time horizon* control model proving that a disturbed system can be controlled into equilibrium from nearby states, in given time, by an appropriate fishing effort strategy. In addition, for the finite time horizon model, using a toolbox developed in a MatLab environment in Banga et al. (2005) and Hirmayer et al. (2009), we obtain a time-dependent harvesting strategy which is optimal among those that steer the disturbed system back into the equilibrium. As for the infinite time horizon model, applying a theorem of Rafikov et al. (2008), we find a linear feedback control which, from the actual state calculates the corresponding fishing effort that asymptotically steers the system into the required equilibrium.

The paper is organized as follows: In Section 2 first, from Dubey et al. (2003) we recall the dynamic model of a fish population with reserve area, and sufficient conditions for the stable coexistence of both subpopulations under the effect of a constant fishing effort. Then, based on a systems theoretical approach, a deterministic stock estimation method is proposed. Section 3 is devoted to the estimation of the effect of an unknown environmental change, applying the same observer design methodology of the previous section. In Section 4, we deal with the equilibrium control of the system in given time, using fishing effort as an open-loop control, and also solve a corresponding optimal control problem. Finally, in Section 5, a closed-loop control model is applied to asymptotically control the system into a desired equilibrium, and a Discussion section completes the paper. For the reader’s convenience, certain concepts and theorems of mathematical systems theory applied in the main body of the paper are shortly summarized in the Appendix.

## 2. Deterministic stock estimation by observation in the fishing area

First, from Dubey et al. (2003) we recall the dynamics of the fish population moving between two areas, an unreserved one (1) where fishing is allowed, and a reserved one (2) where fishing is prohibited. At time $t$, let $x_1(t)$ and $x_2(t)$ be the respective biomass densities of the same fish species inside the unreserved and reserved areas, respectively. Assume that the fish subpopulation of the unreserved area migrates into reserved area at a rate $m_{12}$, and there is also an inverse migration at rate $m_{21}$. Let $E$ be the fishing effort applied to harvesting in the unreserved area and let us assume that in each area the growth of the fish population follows a logistic model. The dynamics of the fish subpopulations in the unreserved and reserved areas are then assumed to be governed by the following system of differential equations (2.1)-(2.2):

$$\dot{x}_1 = r_1 x_1 \left(1 - \frac{x_1}{K_1}\right) - m_{12} x_1 + m_{21} x_2 - qEx_1 \tag{2.1}$$

$$\dot{x}_2 = r_2 x_2 \left(1 - \frac{x_2}{K_2}\right) + m_{12} x_1 - m_{21} x_2 , \tag{2.2}$$

where $r_1$ and $r_2$ are the intrinsic growth rates of the corresponding sub-populations, $K_1$ and $K_2$ are the carrying capacities for the fish species in the unreserved and reserved areas, respectively; $q$ is the catchability coefficient of in the unreserved area. All parameters $r_1$, $r_2$, $q$, $m_{12}$, $m_{21}$, $K_1$ and $K_2$ are positive constants.

In Dubey et al. (2003), it is checked that for a unique positive equilibrium $x^* = (x_1^*, x_2^*)$ of the dynamic model (2.1)-(2.2) the following set of inequalities are sufficient:

$$\frac{r_2(r_1 - m_{12} - qE)^2}{K_2 m_{21}} < \frac{(r_2 - m_{21})r_1}{K_1} \tag{2.3a}$$

$$(r_2 - m_{21})(r_1 - m_{12} - qE) < m_{12}m_{21} \tag{2.3b}$$

$$\frac{r_1 x_1^*}{K_1} > r_1 - m_{12} - qE. \tag{2.3c}$$

Furthermore, the Lyapunov function

$$V(x) := \left( x_1 - x_1^* - x_1^* \ln \frac{x_1}{x_1^*} \right) + \frac{x_2^* m_{21}}{x_1^* m_{12}} \left( x_2 - x_2^* - x_2^* \ln \frac{x_2}{x_2^*} \right)$$

also implies asymptotic stability of equilibrium $x^*$ for system (2.1)-(2.2), globally with respect to the positive orthant of $\mathbf{R}^2$. Throughout the paper we shall suppose conditions (2.3a)-(2.3c) to guarantee the stable coexistence of the system applying a constant reference fishing effort.

Now, let us consider the problem of stock estimation in the reserve area on the basis of the biomass harvested in the free area. (For technical reason, its difference from the equilibrium value is supposed to be observed.) To this end, in addition to dynamics (2.1)-(2.2) we introduce an observation equation

$$y = h(x) := qE(x_1 - x_1^*), \tag{2.4}$$

representing the observation of the biomass harvested in the free fishing area. Then linearizing observation system (2.1)-(2.2)-(2.4) near the equilibrium, we get the Jacobian of the right-hand side of (2.1)-(2.2)

$$A := \begin{bmatrix} r_1 - 2r_1 \dfrac{x_1^*}{K_1} - m_{12} - qE & m_{21} \\ m_{12} & r_2 - 2r_2 \dfrac{x_2^*}{K_2} - m_{21} \end{bmatrix},$$

and the observation matrix

$$C := h'(x^*) = (qE \quad 0).$$

Now, for the linearized system we obviously have $\operatorname{rank}[C \mid CA]^T = 2$. Hence Theorem A.2 of Appendix implies local observability of the system near the equilibrium in the sense of Definition A.1 of Appendix. In other words, in principle, the whole system state (in particular the stock of the species in the reserve area) as function of time can be uniquely recovered, observing the biomass harvested per unit time. In the following illustrative example we will see how the state of the system (and hence the total stock) can be effectively calculated from the catch realized in the fishing area, applying the methodology of Sundarapandian (2002), see Appendix.

**Example 2.1.** For a possible comparison, in this numerical example we use the same parameters as Guiro et al. (2009): $r_1$=0.7, $r_2$=0.5, $K_1$=10, $K_2$=2.2, $m_{12}$=0.2, $m_{21}$=0.1, $q$=0.25 and $E$=0.9,

$$\begin{aligned} \dot{x}_1 &= 0.7x_1\left(1-\frac{x_1}{10}\right) - 0.2x_1 + 0.1x_2 - 0.25\cdot 0.9x_1 \\ \dot{x}_2 &= 0.5x_2\left(1-\frac{x_2}{2.2}\right) + 0.2x_1 - 0.1x_2. \end{aligned} \tag{2.5}$$

Now the positive equilibrium is $x^* = (4.85, 3.12)$ and with

$$K := \begin{pmatrix} 0 \\ 10 \end{pmatrix},$$

matrix $A - KC$ is Hurwitz; therefore by Theorem A.4 of Appendix we have the following observer system

$$\begin{aligned} \dot{z}_1 &= 0.7z_1\left(1-\frac{z_1}{10}\right) - 0.2z_1 + 0.1z_2 - 0.25\cdot 0.9z_1 \\ \dot{z}_2 &= 0.5z_2\left(1-\frac{z_2}{2.2}\right) + 0.2z_1 - 0.1z_2 + 10[y - 0.25\cdot 0.9(z_1 - x_1^*)]. \end{aligned} \tag{2.6}$$

If we take an initial condition $x^0 := (30, 120)$ for system (2.5), and similarly, we consider another nearby initial condition $z^0 := (35, 100)$ for the observer system (2.6), then the corresponding solution $z$ of the observer approaches the solution $x$ of the original system, as shown in Figure 1. We note that in this particular case the convergence is much faster than that of the global observer constructed in Guiro et al. (2009).

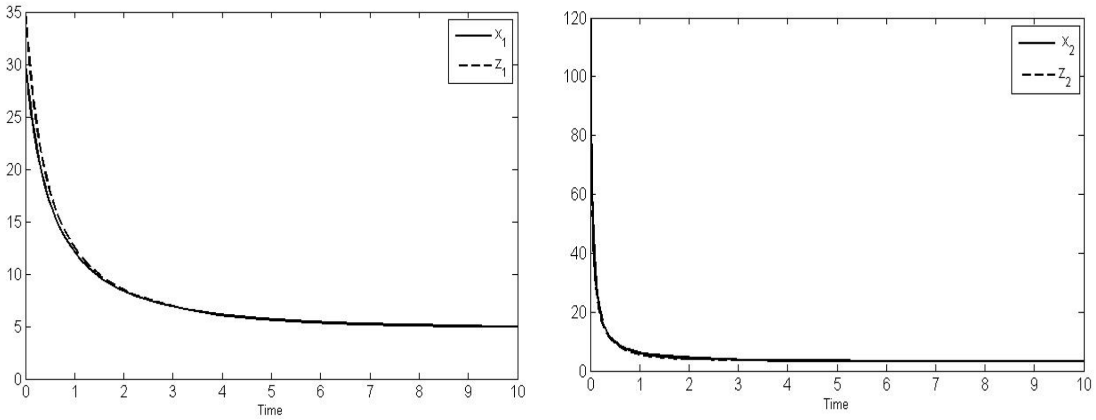



**Figure 1.** Solution of observer (2.6), approaching the solution of system (2.5)

## 3. Estimation of the effect of an unknown environmental change

Assume that the considered ecosystem consists, on the one hand, of a system of several interacting populations living in the given habitat, and the abiotic environment on the other. The latter may also be exposed to climatic (e.g. seasonal) changes and/or human intervention, such as e.g. pollution, described by certain abiotic parameters (e.g. temperature or concentration). In this section, considering the model (2.1)-(2.2), we suppose that the reference values of certain abiotic parameters change to unknown constant values. The effect of this change will be described by a small additive term (disturbance) $w \in \mathbf{R}$ in certain model parameters. In our illustrative numerical example it will be shown how we can recover the whole state process of the population system and estimate the unknown disturbance at the same time, by constructing and solving the corresponding observer system. Let us suppose, for example, that a disturbance takes place in the migration rates. Let us consider first the corresponding fishery system, completed with a trivial equation for the unknown constant parameter *w*,

$$\dot{x}_1 = r_1 x_1\left(1-\frac{x_1}{K_1}\right)-(m_{12}+c_1 w)x_1+(m_{21}+c_2 w)x_2-qEx_1 \quad (3.1)$$

$$\dot{x}_2 = r_2 x_2\left(1-\frac{x_2}{K_2}\right)+(m_{12}+c_1 w)x_1-(m_{21}+c_2 w)x_2 \quad (3.2)$$

$$\dot{w} = 0, \quad (3.3)$$

where $c_1$ and $c_2$ are positive constants. Since equilibrium $x^* = (x_1^*, x_2^*)$ is asymptotically stable for system (2.1)-(2.2), it is not hard to prove that equilibrium $(x_1^*, x_2^*, 0)$ is

Lyapunov stable for system (3.1)-(3.3). Therefore Theorem A.5 of Appendix can be applied for the observer design.

Let us suppose that, similarly to Section 2, the biomass harvested in unit time is observed:

$$y = h(x,w) := qE(x_1 - x_1^*). \tag{3.4}$$

Now the linearization of observation system (3.1)-(3.4) gives

$$A = \begin{pmatrix} r_1 - 2r_1 \dfrac{x_1^*}{K_1} - m_{12} - qE & m_{21} & -c_1x_1^* + c_2x_2^* \\ m_{12} & r_2 - 2r_2 \dfrac{x_2^*}{K_2} - m_{21} & c_1x_1^* - c_2x_2^* \\ 0 & 0 & 0 \end{pmatrix}; C := h'(x^*) = \begin{pmatrix} qE & 0 & 0 \end{pmatrix}. \tag{3.5}$$

Hence we easily obtain that $\operatorname{rank}[C \mid CA \mid CA^2]^T = 3$, if $K_2 \neq 2x_2^*$ and $c_1x_1^* \neq c_2x_2^*$. Therefore, by Theorem A.2 (see Appendix), the system is locally observable near the equilibrium, and applying the method of Sundarapandian (2002) we can construct a corresponding observer system, as shown in the following

**Example 3.1.** Using the same system parameters as in Example 2.1, with the presence of an unknown environmental disturbance $w$ and coefficients $c_1$=0.1, $c_2$=0.3, we have

$$\begin{aligned} \dot{x}_1 &= 0.7x_1\left(1 - \frac{x_1}{10}\right) - (0.2 + 0.1w)x_1 + (0.1 + 0.3w)x_2 - 0.25 \cdot 0.9x_1 \\ \dot{x}_2 &= 0.5x_2\left(1 - \frac{x_2}{2.2}\right) + (0.2 + 0.1w)x_1 - (0.1 + 0.3w)x_2 \\ \dot{w} &= 0 \end{aligned} \tag{3.6}$$

System (3.6) has a nonnegative equilibrium $\bar{x} = (4.85, 3.12, 0)$, and with

$$K := \begin{pmatrix} 0 \\ 0 \\ 100 \end{pmatrix}$$

matrix $A - KC$ is Hurwitz, therefore by Theorem A.5 of Appendix we can construct the following observer system:

$$
\begin{aligned}
\dot{z}_1 &= 0.7z_1\left(1-\frac{z_1}{10}\right)-(0.2+0.1z_3)x_1+(0.1+0.3z_3)z_2-0.25\cdot 0.9z_1 \\
\dot{z}_2 &= 0.5x_2\left(1-\frac{x_2}{2.2}\right)+(0.2+0.1z_3)z_1-(0.1+0.3z_3)z_2 \\
\dot{z}_3 &= 100[y-0.25\cdot 0.9(z_1-4.85)].
\end{aligned}
\tag{3.7}
$$

If we suppose that environmental perturbation corresponds to the value $w=0.2$ and we take an initial condition $(x^0, w^0) := (10, 5, 0.2)$, of system (3.6), and similarly, we consider another nearby initial condition, $z^0 := (15, 10, 0.3)$ for the observer system (3.7). Figure 2 shows that the corresponding solution $z$ approaches the solution $x$ of the original system, and also correctly estimates the "unknown" parameter *w*.

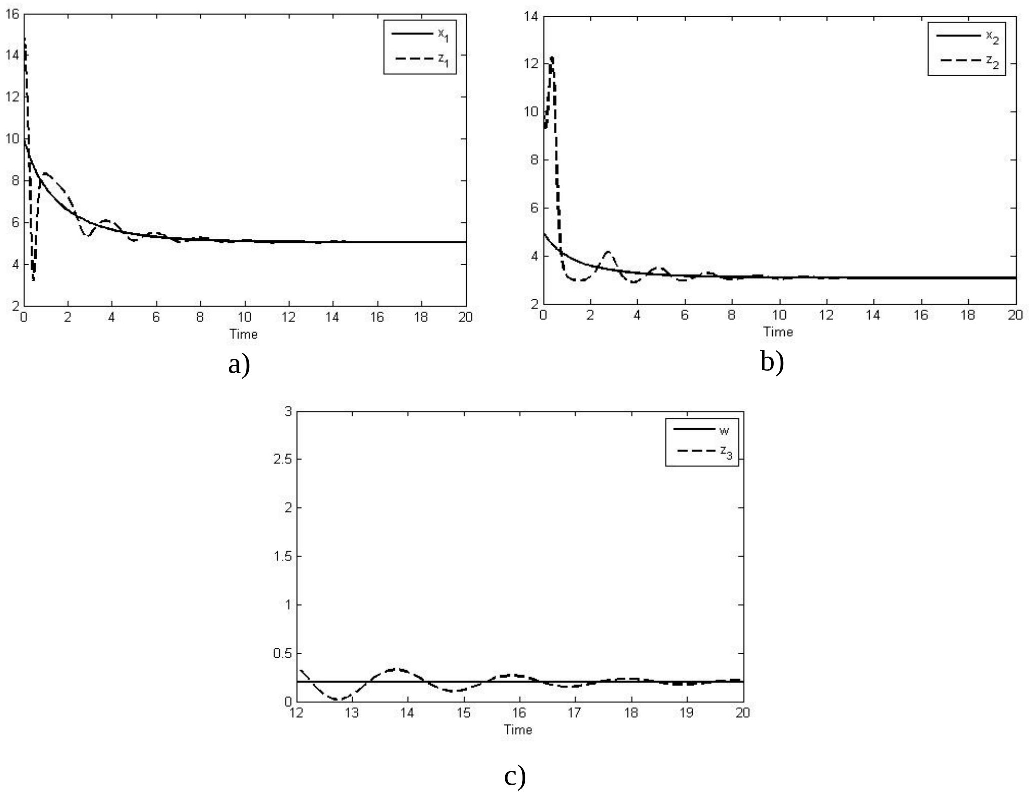



**Figure 2.** Simultaneous state and parameter estimation in system (3.6) with its observer (3.7). For a better scaling, in the graphical representation the graph of $z_3$ is plotted only after the relatively large transient values.

## 4. Open-loop equilibrium control by harvesting

An important issue in conservation ecology is controlling a population system into equilibrium in given time, and maintain it there. In this section we will deal with this problem in the framework of the dynamic fishing effort model. We also consider the case when, during this operation, the total harvested biomass with certain discount factor is maximized.

*4.1 Open-loop control into equilibrium by fishing effort in given time*

Let us suppose that the system is deviated from its equilibrium, and we want to steer it back into equilibrium by replacing the constant fishing effort by a time-dependent effort considered as control. *Open-loop control* means that we want to determine in advance a control as function of time, such that the corresponding time-dependent state of the system reaches the original equilibrium in given time. (The *closed-loop controls* to be considered in the next section will depend on the current state of the system.)

Let us suppose that the total effort applied for harvesting the fish population is controlled in function of time in the form $E+u(t)$. Here, with the notation of the Appendix, we can consider control functions $u \in U_{\varepsilon_0}[0,T]$ defined on a fixed interval $[0,T]$, with $s:=1$. Throughout this section, it will be is supposed that $\varepsilon_0 \le E$, which means that there is only harvesting and no release of fish is allowed. Then our model (2.1)-(2.2) takes the form

$$\begin{aligned} \dot{x}_1 &= r_1 x_1\left(1-\frac{x_1}{K_1}\right) - m_{12}x_1 + m_{21}x_2 - q(E+u(t))x_1 \\ \dot{x}_2 &= r_2 x_2\left(1-\frac{x_2}{K_2}\right) + m_{12}x_1 - m_{21}x_2 \end{aligned} \tag{4.1}$$

Then (4.1) can be considered as a control system, and in terms of the notation of Appendix, with

$$F:\ \mathbf{R}^3 \to \mathbf{R}^2, \quad F(x_1,x_2,u) := \begin{bmatrix} r_1 x_1\left(1-\frac{x_1}{K_1}\right) - m_{12}x_1 + m_{21}x_2 - qux_1 \\ r_2 x_2\left(1-\frac{x_2}{K_2}\right) + m_{12}x_1 - m_{21}x_2 \end{bmatrix},$$

control system (4.1) takes the form

$$\dot{x} = F(x, u^* + u(t)). \tag{4.2}$$

Here $u(t)$ is interpreted as an *additional fishing effort*. Obviously, to $u^* := E$ and $u(t) := 0$ $(t \in [0,T])$, there corresponds the non-trivial ecological equilibrium $x^*$ of dynamic system (2.1)-(2.2).

Now we show that control system (4.2), i.e. (4.1), is locally controllable to $x^*$ on $[0,T]$. For the application of Theorem A.7 of Appendix, let us calculate the Jacobians

$$A := D_1 F(x^*, 0) = \begin{bmatrix} r_1 - 2r_1 \dfrac{x_1^*}{K_1} - m_{12} - qE & m_{21} \\ m_{12} & r_2 - 2r_2 \dfrac{x_2^*}{K_2} - m_{21} \end{bmatrix}, \; B := D_2 F(x^*, 0) = \begin{bmatrix} q x_1^* \\ 0 \end{bmatrix}.$$

Since

$$\det[B \mid AB] = q^2 {x_1^*}^2 m_{12} \neq 0,$$

we get $\operatorname{rank}[B \mid AB] = 2$, and applying Theorem A.7 we obtain the local controllability of system (4.2) into $x^*$ on interval $[0,T]$.

The obtained local controllability means that from nearby states, the system can be steered into the equilibrium applying an appropriate small control. Now we proceed to the determination of such control.

Fix an initial state $x^0$ from a neighbourhood of local controllability of system (4.1), and for each control function $u$ small enough (i.e. $u \in U_\varepsilon[0,T]$ for appropriate $\varepsilon \in ]0, \varepsilon_0]$, see conditions of system (A.6)-(A.7) of Appendix), let $x$ be the solution of (4.2) defined on $[0,T]$ and corresponding to the initial value $x^0$. Then a control $\bar{u} \in U_\varepsilon[0,T]$ will steer initial state $x^0$ into equilibrium $x^*$, if and only if it minimizes functional

$$\Phi(u) := \left| x(T) - x^* \right|^2 \; (u \in U_\varepsilon[0,T]).$$

The above reasoning can be summarized in the following theorem:

**Theorem 4.1.** Suppose that the parameters of system (4.1) satisfy conditions (2.3a)-(2.3c). Then system (4.1) is locally controllable to equilibrium $x^*$ on interval $[0,T]$. An initial state $x^0$ from a neighbourhood of local controllability will be steered into $x^*$ by a control $\bar{u} \in U_\varepsilon[0,T]$ if and only if the latter is a solution of the following optimal control problem:

$$\Phi(u) := \left|x(T) - x^*\right|^2 \to \min\,, \tag{4.3}$$

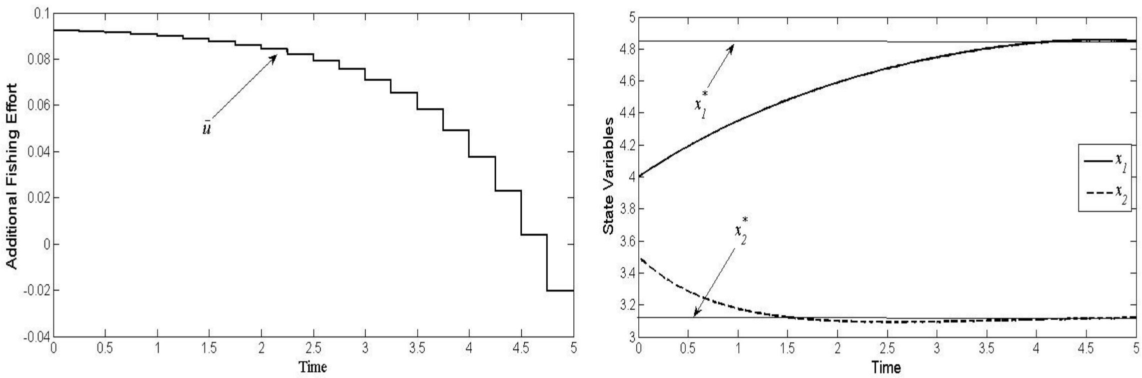
$$u \in U_\varepsilon[0,T],\ x(0) = x^0\,, \tag{4.4}$$

$$\dot{x} = F(x, u^* + u(t))\,. \tag{4.5}$$

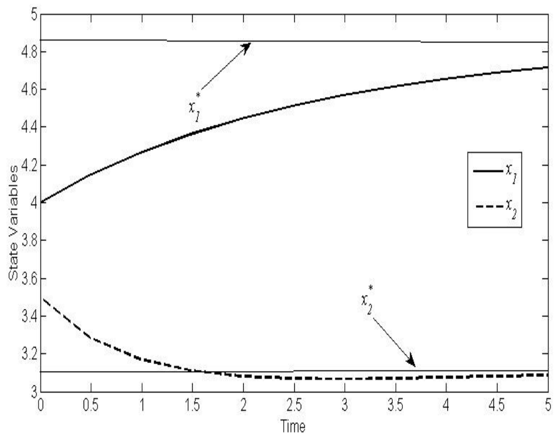
**Remark 4.2.** From the local controllability of control system (4.2), we know that the optimal control problem (4.3)-(4.5) has at least one solution.

As a consequence of this theorem, for an effective calculation of an equilibrium control $\bar{u}$, it is enough to solve the optimal control problem (4.3)-(4.5). To this end we can apply the toolbox developed for MatLab in Banga, et al. (2005) and Hirmajer et al. (2009). Actually, this program uses piecewise constant controls, providing in this way an approximate solution of the optimal control problem. Next, using this toolbox, we will illustrate the results of Theorem 4.1.

**Example 4.3.** Let us consider the parameters of Example 2.1. Taking as initial condition $x^0 := (4\ , 3.5)$ and time duration *T*:=5, we apply the MatLab toolbox mentioned above. Figure 3.a) shows the obtained optimal control $\bar{u}$; the corresponding solution $x$ ending up at equilibrium $x^* = (4.85, 3.12)$ can be seen in Figure 3.b).




a) b)




c)

**Figure 3.** a) Control function of system (4.1) for $T$=5, b) Solution of control system (4.1) for $T$=5, with initial value $x$(0)=(4, 3.5), c) Solution of the uncontrolled system (2.5) for $T$=5, with initial value $x$(0)=(4,3.5).

We note that, since by Remark 4.2, for the uncontrolled system, $x^*$ is asymptotically stable, the state would tend to $x^*$, reaching it in "*infinite time*", as seen in Figure 3.c. By our method the system state is steered into $x^*$ in *given finite time.*

*4.2. Open-loop equilibrium control by optimal fishing effort*

Since the equilibrium control of the previous section is usually not unique, it is reasonable to look for an equilibrium control that also maximizes the harvested biomass.
For a more flexible model, we will consider the corresponding integral with and without discount. Although for infinite time-horizon problems an exponential discount factor is a technical necessity (see. Clark 2010 and references therein), similar discount is also used in finite time-horizon models (see e.g. Chakraborty et al. 2011).
For $\varepsilon > 0$ of the previous subsection and arbitrarily fixed $\delta \geq 0$, with the same controlled population dynamics as (4.2), the corresponding optimal control problem is the following:

$$\Psi(u) := \int_0^T e^{-\delta t} q(E+u(t))x_1(t)dt \to \max, \tag{4.6}$$

$$u \in U_\varepsilon[0,T], \tag{4.7}$$

$$\dot{x} = F(x, u^* + u(t)), \tag{4.8}$$

$$x(0)=x^0,\quad x(T)=x^* \tag{4.9}$$

Now, for a numerical solution of this problem using the mentioned MatLab toolbox of Banga et al. (2005), piecewise constant controls are considered. More precisely, for fixed positive integer *N*, let $t_i := i\dfrac{T}{N}$ $(i\in\overline{0,N})$ be the uniform division of $[0,T]$, and let us define the set of controls as follows:

$S_{\varepsilon,N}[0,T] := \{u\in U_\varepsilon[0,T]:\ u \text{ is constant on each interval } ]t_{i-1},t_i[\ \ (i\in\overline{0,N})\}$.

Then, considering the set of admissible controls

$S^*_{\varepsilon,N}[0,T] := \{u\in S_{\varepsilon,N}[0,T]:\ u \text{ satisfies (4.8) and (4.9)}\}$,

$\varepsilon>0$ and $N$ are chosen as to guarantee that $S^*_{\varepsilon,N}[0,T]$ is not empty. Hence functional $\Psi$ in (4.6) can be defined on the compact set $S^*_{\varepsilon,N}[0,T]\subset \mathrm{R}^N$ and is the composition of two continuous mappings $S^*_{\varepsilon,N}[0,T]\to S^*_{\varepsilon,N}[0,T]\times C[0,T]$, assigning to each $u\in S^*_{\varepsilon,N}[0,T]$ the pair (*u*,*x*), where *x* is the solution of (4.8), corresponding to *u*, and mapping $S^*_{\varepsilon,N}[0,T]\times C[0,T]\to \mathrm{R}$, assigning to each pair $(u,x)\in S^*_{\varepsilon,N}[0,T]\times C[0,T]$ the integral $\int_0^T e^{-\delta t}q(E+u(t))x_1(t)dt$. As a result of the above reasoning, we obtain the following

**Theorem 4.4.** For any parameter choice satisfying conditions (2.3a)-(2.3c), the optimal control problem

$$\Psi(u) := \int_0^T e^{-\delta t}q(E+u(t))x_1(t)dt \to \max, \tag{4.9}$$

$$u\in S^*_{\varepsilon,N}[0,T] \tag{4.10}$$

$$\dot{x}=F(x,u^*+u(t)), \tag{4.11}$$

$$x(0)=x^0,\quad x(T)=x^* \tag{4.12}$$

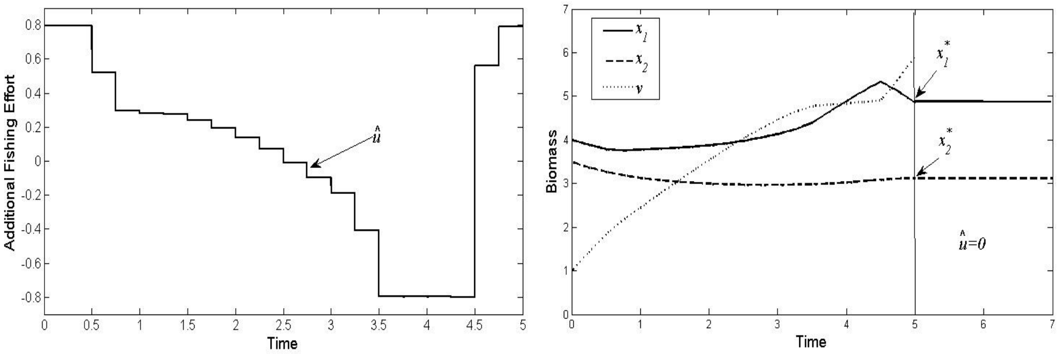
has a solution.

The obtained result shows a possible bargain between the bioeconomic and conservation aspects of fishery management. Now we proceed to the illustration of the above optimal control model for different discount parameter values.

**Example 4.5.** Now, with the same model parameter values of the previous examples, $T$:=5, initial state $x^0 := (4\ ,\ 3.5)$ and target equilibrium $x^* = (4.85\,,\,3.12)$, setting $\varepsilon := 0.8\ (< E = 0.9)$ and $N$:=20, we present the numerical realization of model (4.9)-(4.12), for discount parameters $\delta := 0;\ 0.5; 5$ and $50$, in Figures 4; 5; 6 and 7, respectively. For each case, the optimal control $\hat{u}$, the corresponding subpopulation biomasses $x_1$ , $x_2$ and the actual harvested biomass $v$ are plotted against time, where function $v$ is defined as

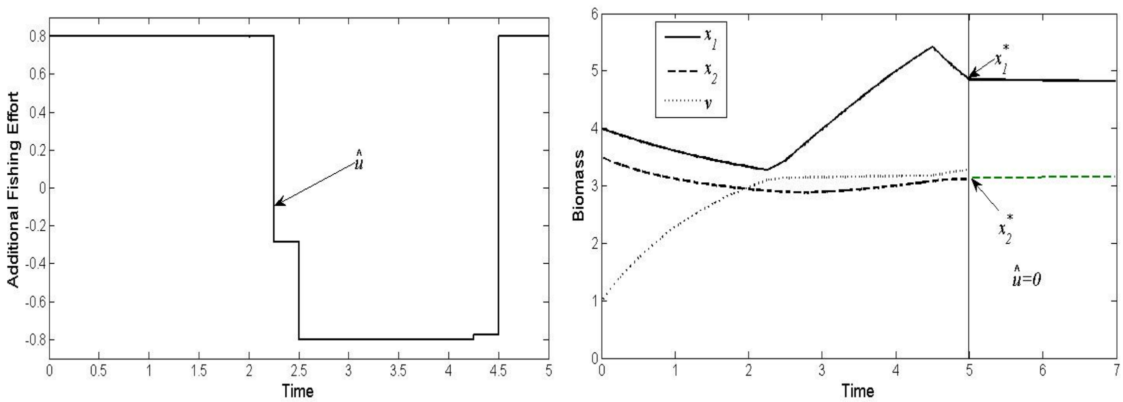
$$v(t) := \int_0^t e^{-\delta\tau} q(E + u(\tau)) x_1(\tau) d\tau \quad (t \in [0,t]).$$

It is also shown that once the system attains its required equilibrium $x^*$, this equilibrium is maintained with zero control (i.e. applying only the reference fishing effort).



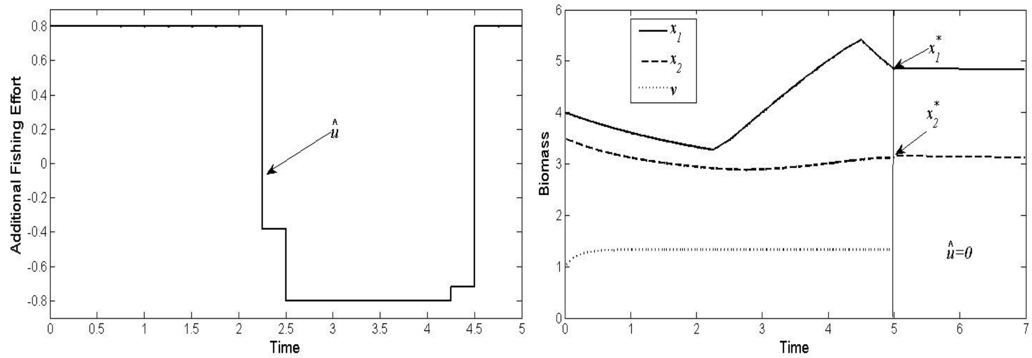

a) b)

**Figure 4**. a) Optimal control, b) corresponding subpopulation biomasses and harvested biomass as function of time; without discount, $\delta := 0$.



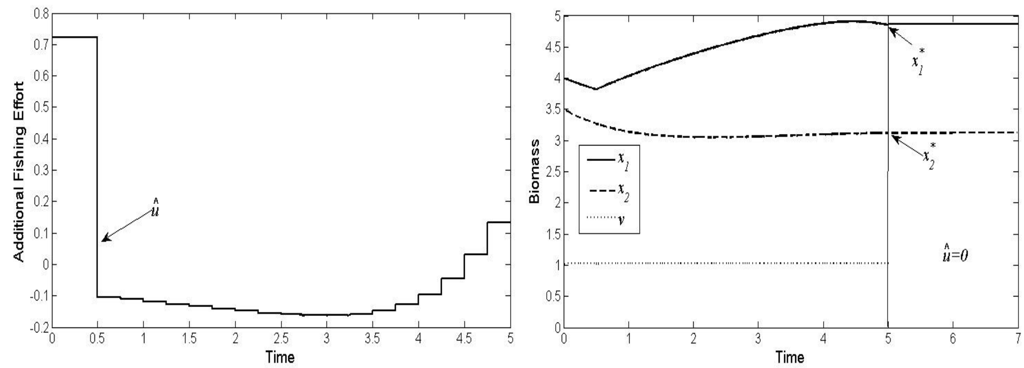
a) b)

**Figure 5.** a) Optimal control, b) corresponding subpopulation biomasses and harvested biomass as function of time; with discount parameter $\delta := 0.5$.




a) b)

**Figure 6.** a) Optimal control, b) corresponding subpopulation biomasses and harvested biomass as function of time; with discount parameter $\delta := 5$




a) b)

**Figure 7.** a) Optimal control, b) corresponding subpopulation biomasses and harvested biomass as function of time; with discount parameter $\delta := 50$

## 5. Closed-loop control steering the population system asymptotically into equilibrium

In this section we suppose that the environmental authority decides to intervene in the reserve area, controlling the biomass $x_2(t)$ of the corresponding subpopulation, on the basis of the actual system state vector $x(t)$. More concretely, the objective is to find a feedback control that steers the population of fish population inside the unreserved area to a desired level $x_1^* = x_{1d}$. In order to solve this problem, the method of Rafikov et al.

(2008) will be applied. To this end, based on the fishery resource model (2.1)-(2.2), we consider the following control system

$$\begin{aligned} \dot{x}_1 &= r_1 x_1\left(1-\frac{x_1}{K_1}\right) - m_{12}x_1 + m_{21}x_2 - qEx_1 \\ \dot{x}_2 &= r_2 x_2\left(1-\frac{x_2}{K_2}\right) + m_{12}x_1 - m_{21}x_2 + U. \end{aligned} \tag{5.1}$$

Our objective is to find a feedback control that steers the fish population inside the unreserved area to a desired level $x_1^* = x_{1d}$. The corresponding value $x_2^* = x_{2d}$ and $u^*$ can be calculated solving the following system of linear equations:

$$\begin{aligned} & r_1 x_1^*\left(1-\frac{x_1^*}{K_1}\right) - m_{12}x_1^* + m_{21}x_2^* - qEx_1^* = 0 \\ & r_2 x_2^*\left(1-\frac{x_2^*}{K_2}\right) + m_{12}x_1^* - m_{21}x_2^* + u^* = 0. \end{aligned} \tag{5.2}$$

We note that $u^*$ is interpreted as a constant intervention rate in the fish population inside the reserve area that would maintain the desired level $x_1^* = x_{1d}$ of fish population inside the free area.

Now, following section A.3 of Appendix, we rewrite the feedback control version of system (5.1) in the form

$$\dot{x} = Lx + g(x) + BU,$$

where $U$ is a continuous control function,

$$L := \begin{pmatrix} r_1 - m_{12} - qE & m_{21} \\ m_{12} & r_2 - m_{21} \end{pmatrix},\ g(x) := \begin{pmatrix} -\dfrac{r_1}{K_1}x_1^2 \\ -\dfrac{r_2}{K_2}x_2^2 \end{pmatrix},\ B := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and

$$q(x) := g(y + x^*) - g(x^*) = \begin{pmatrix} -\dfrac{r_1}{K_1}(y_1^2 + 2y_1x_1^*) \\ -\dfrac{r_2}{K_2}(y_2^2 + 2y_2x_2^*) \end{pmatrix}.$$

Assume that to constant control $u^* \in \mathbf{R}$, there corresponds an equilibrium state $x^*$, i.e.

$$Lx^* + g(x^*) + Bu^* = 0.$$

Then, for the new variables

$$y := x - x^*; \quad u := U - u^*$$

we have

$$y = Ly + q(y) + Bu. \tag{5.3}$$

Now, for the construction of the required linear feedback, we apply Theorem A.8 of Appendix with illustrative numerical data:

**Example 5.1.** Considering the the same model parameters of the previous examples, $r_1$= 0.7, $r_2$= 0.5, $q$= 0.25, $E$=0.9, $m_{12}$= 0.2, $m_{21}$= 0.1, $K_1$= 10 and $K_2$= 2.2; we obtain that system (2.1)-(2.2) has an asymptotically stable positive equilibrium, where $x_1$=4.85. Then, we suppose that the objective is to increase the fish population in the unreserved area, for example to a level $x_{1d}$= 6. To this end, from system (5.2) we calculate $x_{2d}$= 8.7 and $u^*$ =12.52.

Furthermore, for matrices $L$ and $B$ we have

$$L := \begin{pmatrix} 0.275 & 0.1 \\ 0.2 & 0.4 \end{pmatrix}, \quad B := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and we also choose

$$Q := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \;\; R := (1).$$

Now, from the matrix Riccati equation

$$PL + L^T P - PBR^{-1}B^T P + Q = 0\,,$$

using the function LQR of MATLAB™ v7.0 we calculate

$$P := \begin{pmatrix} 103.024 & 7.796 \\ 7.796 & 2.049 \end{pmatrix}.$$

P and Q are obviously positive definite symmetric matrices. Furthermore, for the auxiliary function $l$ we have

$$l(y) = 14.42y_1 + (17.21 + 0.93y_2)y_2^2 + y_1^2(174.08 + 1.09y_2) + y_1y_2(74.76 + 3.54y_2).$$

Its first order partial derivatives are

$$D_1 l(y) = 43.27y_1^2 + y_1(348.16 + 2.18y_2) + y_2(74.76 + 3.54y_2)$$

$$D_2 l(y) = 1.09y_1^2 + y_2(34.41 + 2.79y_2) + y_1(74.76 + 7.09y_2).$$

Obviously

$$D_0 l(0) = D_1 l(0) = D_2 l(0) = 0\,,$$

and for the Hessian of $l$ at the origin we obtain

$$Hl(0) = \begin{bmatrix} 348.16 & 74.76 \\ 74.76 & 34.41 \end{bmatrix},$$

implying that the origin is a strict local minimum point of function $l$ . Applying Corollary A.11 of Appendix, we have the local asymptotic stability of the zero equilibrium of system (5.3). Therefore, applying (A.11) (see Appendix), we obtain the required feedback control for

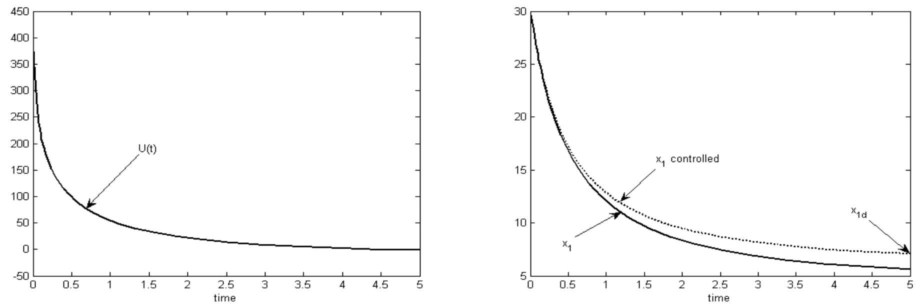
$$u = 7.796 y_1 + 2.049 y_2$$

Hence, from equalities $x = x^* + y$ and $U = u^* + u$, we can calculate the closed-loop control system for (5.1):

$$\begin{aligned} \dot{x}_1 &= 0.7x_1\left(1 - \frac{x_1}{10}\right) - 0.2x_1 + 0.1x_2 - 0.25 \cdot 0.9x_1 \\ \dot{x}_2 &= 0.5x_2\left(1 - \frac{x_2}{2.2}\right) + 0.2x_1 - 0.1x_2 + 7.796x_1 + 2.049x_2 - 52.08 \end{aligned} \tag{5.4}$$

Figure 8.a) shows the time-dependent control $U(t)$, which turned out to be always positive. This, in biological terms, is interpreted as release of hatchery-bred juvenile fish inside the reserved area.

In Figure 8.b) we show how the first coordinate of the solution of the controlled system asymptotically reaches the desired value $x_{1d}$=6.




a) b)

**Figure 8.** a) Time-dependent control $U(t)$ obtained for system (5.4), b) biomass in the unreserved area without and with control (with the same initial state $x^0$(30,120)), biomass approaches the desired value $x_{1d}$=6.

**6. Discussion**

Over the last decade, tools of mathematical systems theory has been successfully applied to both density-dependent multi-species and structured single-species dynamic population models, for a survey of our results on the subject see Varga (2008). The majority of stock assessment methods use a *statistical* approach, see Cadrin et al. (2005). Recently Guiro et al. (2009) used a global observer system for *deterministic* stock estimation. Our observer system method proposed in this paper is local, but can not only perform better near equilibrium, but it also turned out to be able to estimate stock and environmental parameters simultaneously. This method can be also extended to general spatially structured populations, as well as to stage-, age- or size structured populations. For example, in a size-structured model only fish of cachable size are harvested and therefore observed, and the total state vector containing all size classes are estimated. Similarly, the state of a population system in time-dependent environment can be also estimated with an appropriate observer system.

It has been shown that the usual fishing activity, apart from purely profitable commercial activity, can be also applied for purposes of conservation ecology. Among all harvesting strategies steering the system into equilibrium, an economically optimal one can be also calculated, according to different discount parameters. In this way both bioeconomic and conservation tasks can be dealt with in our model, providing a complex management approach to fishing activity. On the basis of an appropriate dynamic population model, this method also applies to the management of other harvested populations.

Since, in general, the parameters of a model can be estimated with certain error, an important question of modelling methodology is to what extend the conclusions drown from the model would change due to this error. In our case, it is not hard to prove that both observability and controllability of the considered systems are robust against small changes in the model parameters.

As for our result on the optimal equilibrium control by fishing effort, we note that the considered optimization problems are typically non-convex, and then in their numerical solution the mathematical programming problem obtained by discretization is also not convex, and the usual algorithms may provide only a local extremum. In our case this is not a problematic issue, since serious we always consider local problems, near the equilibrium.

Finally, we found that the intervention of the competent authority in the reserve area can be also efficiently modelled by Rafikov’s approach to linear feedback control that already turned out to be efficient also in a cell population model of radiotherapy, see Gámez et al. (2009).

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# Appendix

*A.1. Observability and observer of nonlinear systems*

Given positive integers *m, n*, let

$$f : \mathbf{R}^n \to \mathbf{R}^n, \quad h : \mathbf{R}^n \to \mathbf{R}^m$$

be continuously differentiable functions and for some $x^* \in \mathbf{R}^n$ we have that $f(x^*) = 0$ and $h(x^*) = 0$.

We consider the following observation system

$$\dot{x} = f(x) \tag{A.1}$$

$$y = h(x) \ , \tag{A.2}$$

where $y$ is called the *observed function*.

**Definition A.1** Observation system (A.1)-(A.2) is called *locally observable* near equilibrium $x^*$, over a given time interval $[0,T]$, if there exists $\varepsilon > 0$, such that for any two different solutions $x$ and $\bar{x}$ of system (A.1) with $|x(t) - x^*| < \varepsilon$ and $|\bar{x}(t) - x^*| < \varepsilon$ $(t \in [0,T])$, the observed functions $h \circ x$ and $h \circ \bar{x}$ are different. ($\circ$ denotes the composition of functions. For brevity, the reference to $[0,T]$ is often suppressed).

For the formulation of a sufficient condition for local observability consider the linearization of the observation system (A.1)-(A.2), consisting in the calculation of the Jacobians

$$A := f'(x^*) \text{ and } C := h'(x^*).$$

**Theorem A.2** (Lee and Markus, 1971). Suppose that

$$rank[C \mid CA \mid CA^2 \mid ... \mid CA^{n-1}]^T = n. \tag{A.3}$$

Then system (A.1)-(A.2) is locally observable near $x^*$.

Now, we recall the construction of an observer system will be based on Sundarapandian (2002). Let us consider observation system (A.1)-(A.2).

**Definition A.3.** Given a continuously differentiable function $G: \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$, system

$$\dot{z} = G(z, y) \tag{A.4}$$

is called a *local asymptotic (respectively, exponential) observer for observation system* (A.1)-(A.2) if the composite system (A.1)-(A.2), (A.4) satisfies the following two requirements.

i) If $x(0) = z(0)$, then $x(t) = z(t)$, for all $t \geq 0$.

ii) There exists a neighbourhood $V$ of the equilibrium $x^*$ of $\mathbf{R}^n$ such that for all $x(0), z(0) \in V$, the estimation error $z(t) - x(t)$ decays asymptotically (respectively, exponentially) to zero.

**Theorem A.4.** (Sundarapandian, 2002). Suppose that the observation system (A.1)-(A.2) is Lyapunov stable at equilibrium, and that there exists a matrix $K$ such that matrix $A - KC$ is Hurwitz (i.e. its eigenvalues have negative real parts), where $A = f'(x^*)$ and $C = h'(x^*)$. Then dynamic system defined by

$$\dot{z} = f(z) + K[y - h(z)]$$

is a local exponential observer for observation system (A.1)-(A.2).

Now, for the estimation of a change in the dynamical parameters of an ecosystem, we recall that Sundarapandian (2002) also considered the possibility of an "input generator" determined by an external system called *exosystem* $w' = s(w)$, in terms of which we can form a composite (nonlinear) system with inputs of the form

$$\begin{aligned} \dot{x} &= F(x, u(w)) \\ \dot{w} &= s(w) \\ y &= h(x), \end{aligned} \tag{A.5}$$

where we suppose that $F:\mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}^n$, $s:\mathbf{R}^k \to \mathbf{R}^k$ are continuously differentiable and $F(x^*,0)=0$, $u(w^*)=0, s(w^*)=0$. Variable $u$ is interpreted as a time-dependent vector of system parameters of the original system (A.1), corresponding to right-hand side $f$. For the construction of an observer for the composite system we can apply the following

**Theorem A.5** (Sundarapandian, 2002). Suppose that observation system (A.5) is Lyapunov stable at equilibrium. If system (A.5) has a local exponential observer, and that there exists a matrix $K$ such that matrix $A-KC$ is stable (its eigenvalues have negative real parts), where $A=F'(x^*,w^*)$ and $C=h'(x^*)$. Then dynamic system defined by

$$\dot{z}=F(z,u(w))+K[y-h(z)]$$

is a local exponential observer for observation system (A.5).

*A.2. Controllability of nonlinear systems*

Given $m,s\in\mathbf{N}$, let $F:\mathbf{R}^m \times \mathbf{R}^s \to \mathbf{R}^m$ be a continuously differentiable function. For a reference control value $u^*\in\mathbf{R}^s$ , let $x^*\in\mathbf{R}^m$ be such that $F(x^*,u^*)=0$. For technical reason we shall need a rather general class of controls. Let us fix a time interval $[0,T]$, and for each $\varepsilon\in\mathbf{R}^+$ define the class of essentially bounded $\varepsilon$- controls

$$U_\varepsilon[0,T]:=\left\{u\in L^s_\infty[0,T]\ |\ \|u(t)\|_\infty\le\varepsilon \quad \textit{for almost every} \quad t\in[0,T]\right\}.$$

Then it can be shown that there exists $\varepsilon_0\in\mathbf{R}^+$ such that for all $u\in U_{\varepsilon_0}[0,T]$ and $x^0\in\mathbf{R}^m$ with $\|x^0-x^*\|<\varepsilon_0$ the initial value problem

$$\dot{x}(t)=F(x(t),u^*+u(t)) \quad (\textit{for a.e. } t\in[0,T]) \tag{A.6}$$

$$x(0)=x^0 \tag{A.7}$$

has a unique solution. We notice that $x^*$ is an equilibrium state for the zero-control system.

**Definition A.6.** Control system (A.6)-(A.7) is said to be *locally controllable to* $x^*$ *on* $[0,T]$, if there exists $\varepsilon\in]0,\varepsilon_0]$ such that for all $x^0$ from the $\varepsilon$-neighbourhood of $x^*$, there is a control $u\in U_\varepsilon[0,T]$ that controls the initial state $x^0$ to equilibrium $x^*$, i.e. for the solution *x* of the initial value problem (A.6)–(A.7), equality *x*(*T*)= $x^*$ holds.

Let us linearize system (A.6)-(A.7) around $(x^*, u^*)$, introducing the corresponding Jacobians

$$A := D_1 F(x^*, u^*)\,, \quad B := D_2 F(x^*, u^*)\,.$$

Then we have the following sufficient condition for local controllability:

**Theorem A.7** (Lee and Markus, 1971)

If $rank[B|AB|...|A^{n-1}B] = n$ then system (A.6)-(A.7) is locally controllable to $x^*$ on $[0,T]$.

*A.3. Closed-loop asymptotic control into equilibrium in nonlinear systems*

For $n, r \in \mathbf{N}$, $L \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times r}$, and continuously differentiable function $g : \mathbf{R}^n \to \mathbf{R}^n$, consider the control system

$$\dot{x} = Lx + g(x) + BU \tag{A.8}$$

where $U$ is a continuous control function. Assume that to a constant control $u^* \in \mathbf{R}^r$, there corresponds an equilibrium state $x^*$, i.e.,

$$Lx^* + g(x^*) + Bu^* = 0 \tag{A.9}$$

Then, from (A.8) and (A.9), for the new variables

$$y := x - x^*\,; \quad u := U - u^*$$

we have

$$y = Ly + q(y) + Bu \quad with \quad q(y) := g(y + x^*) - g(x^*) \tag{A.10}$$

A feedback control will be given below which asymptotically steers system (A.10) into the zero equilibrium.

**Theorem A.8** (Rafikov et al., 2008) If there exist matrices $P$, $Q$, $R \in \mathbf{R}^{nxn}$; $P$ positive definite and $Q$ symmetric, such that the function

$$l(y) := y^T Q y - q^T(y) P y - y^T P h(y) \quad y \in R^n$$

is positive definite, and $P$ satisfies the equation

$$PL + L^T P - PBR^{-1}B^T P + Q = 0$$

Then the linear feedback

$$u(y) := R^{-1} B^T P y \tag{A.11}$$

asymptotically steers any initial state $y(0)$ to zero.

**Remark A.9** The statement $\lim_{\infty} y = 0$ is obviously equivalent to $\lim_{\infty} x = x^{*}$.

**Remark A.10** According to Rafikov et al. (2008), the feedback control (A.11) also minimizes the functional

$$\phi(y) := \int_{0}^{\infty} [l(y(t)) + u^{T}(y(t))Ru(y(t))dt$$

however, we do not use this statement.

**Corollary A.11** (M. Gámez et al. 2009). Using the notation of the previous theorem, let us suppose that function $l$ is *locally* positive definite. Then there exists a neighbourhood $V$ of zero in $\mathbf{R}^{n}$ such that for all $x(0) \in V$, for the solution $x$ of system (A.8) we have $\lim_{\infty} x = x^{*}$.