On The Average Concept

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1 Introduction

It can be observed in many papers on periodic differential equations that a lot of hypotheses are stated in terms of the fundamental concept of average. Unfortunately , this concept has not been satisfactorily extended to the non-periodic case. However, several authors have come up with ingenious ways of establishing adequate hypotheses.

As an example we cite [7], where Vance and Coddington studied the dissipativity of the logistic equation $x' = xF(t,x)$; $x \geq 0$; assuming the hypothesis:

H)There exist $R, T > 0$ such that

$$\int_t^{t+T} F(s.R)\, ds \leq 0 \quad \forall t \in R\,. \tag{1.1}$$

Inspired in the above hypothesis, we state the following definition of upper average

$$\overline{A}_{VC}(f) = \inf_{T>0} \sup_{t\in R} \frac{1}{T} \int_t^{t+T} f(s)\, ds \tag{1.2}$$

for each bounded continuous function $f : R \to R$. See [2]

A more flexible definition of upper average could be given by taking into account the possibility that the intervals considered are not necessarily of the some length T. This leads us to define

$$\overline{B}_{VC}\,(f) = \inf_{s\in\mathcal{S}} \sup_{n\in Z} \frac{1}{S(n+1)-S(n)} \int_{S(n)}^{S(n+1)} f\,(s)\,ds \tag{1.3}$$

where $\mathcal{S}$ is the set of all sequences $s : Z \to Z$ suchthat $s(n) \to \pm\infty \; as \; n \to \pm\infty$ and

$$\sup\{S(n+1)-S(n)\,:\,n\in Z\} < +\infty\,.$$

It is clear that

$$\overline{B}_{VC}\,(f) \leq \; \overline{A}_{VC}\,(f)\;. \tag{1.4}$$

Another interesting example can be found in Burton and Hustson [1], where they use the following condition in order to study the permanence of class in predator-prey systems :

C) There exist numbers $\delta > 0\,,\, \alpha_0 \in R$ and a sequence $t_k \to +\infty$ such that

$$\frac{1}{t_k}\int_{\alpha}^{\alpha+t_k} \mu(s)ds \geq \delta \;\; \forall\, k\in N,\;\; \forall\alpha\geq\alpha_0$$

for a bounded continuous function $\mu : R \to R$.

As we will see in Proposition 3.1 below, it is equivalent to:

D) There exist number $\tau, \delta > 0$ and $\alpha_0 \in R$ such that

$$\frac{1}{\tau}\int_{\alpha}^{\alpha+\tau} \mu(s)ds \geq \delta \quad \forall\alpha\geq\alpha_0\,.$$

This suggests the following definition for lower average:

$$\underline{A}_{BH}(\mu) = \sup\left\{\inf\left\{\frac{1}{\tau}\int_{\alpha}^{\alpha+\tau} \mu(s)ds\,:\,\alpha\geq\alpha_0\right\}\,:\,\alpha_0\in R,\,\tau>0\right\} \tag{1.5}$$

and, by analogy, also suggests the following definition for upper average

$$\overline{A}_{BH}(\mu) = \inf\left\{\sup\left\{\frac{1}{\tau}\int_{\alpha}^{\alpha+\tau}\mu(s)ds \,:\, \alpha \geq \alpha_0\right\} \,:\, \alpha_0 \in R,\, \tau > 0\right\} \qquad (1.6)$$

Note that $\underline{A}_{BH}(\mu) = -\overline{A}_{BH}(-\mu)$.

In Tineo [5] it is defined the upper average of a bounded continuous function $f : R \to R$ as

$$\overline{A}_T = \lim_{r\to+\infty}\sup_{t-s\geq r}\frac{1}{t-s}\int_s^t f(\sigma)d\sigma$$

and the lower average as $\underline{A}_T(f) = -\overline{A}_T(-f)$. These definitions were used to establish the existence of " *coexistence states* " for non-autonomous competition systems of Lotka-Volterra type. They also were used in Tineo [6] in order togeneralize to the non-periodic case a well known work of Mottoni andSchiaffino [3] concerning periodic systems for species in competition.

More recently, Ortega and Tineo [4] and Mawhin [8] have used the averages $\overline{A}_T(f)$, $\underline{A}_T(f)$ to generalize some results related with the Landesman-Lazer condition. As a matter of fact, lemma 3 of [4] leads to the following definition of upper average $\overline{A}_{OT}(f)$ for a bounded continuous function $f : R \to R$. $\overline{A}_{OT}(f)$ is the infimun of those $\lambda \in R$ such that $f(t) \leq \lambda + B'(t)$ for some bounded and continuously differentiable function $B : R \to R$.

We believe that all these averages, apparently different, are equal, at least for a class sufficiently large in $\mathcal{C} = \{f : R \to R \,:\, f \text{ is continuous and bounded}\}$.

In fact, the second section of this article is devoted to prove the following result.

Theorem 1.1 .- $\overline{B}_{VC} = \overline{A}_{VC} = \overline{A}_T = \overline{A}_{OT}$.

In the third and last section, we will study the notion of upper average from an axiomatic point of view. Basically, an upper average will be a sublinear and monotone function $\overline{\mathcal{A}} : \mathcal{C} \to R$ such that $\overline{\mathcal{A}}(f) = f$ if f is constant and $\overline{\mathcal{A}}(f) = 0$ if f has a bounded primitive.

2 The Proof of Theorem 1.1

The proof of this theorem requires several intermediate results.. We begin the following proposition.

Proposition 2.1 .- $\overline{B}_{CV}(f) \leq \overline{A}_{CV}(f) \leq \overline{A}_T(f) \leq \overline{A}_{OT}(f)$ *for any bounded and continuous function* $f : R \to R$.

Proof. By (1.4) we have $\overline{B}_{CV}(f) \leq \overline{A}_{CV}(f)$ and by definition of $\overline{A}_T$ we have
$\overline{A}_T(f) = \inf\left\{\sup\left\{\frac{1}{T}\int_s^{s+T} f(t)dt \,:\, s \in R,\, T \geq r\right\} \,:\, r > 0\right\}$
Thus, $\overline{A}_{CV}(f) \leq \overline{A}_T(f)$.

Finally, given $\varepsilon > 0$ there exist a bounded and continuously differentiable function $B : R \to R$ such that $f(t) \leq \varepsilon + \overline{A}_{OT}(f) + B'(t)$, and hence,

$$\frac{1}{t-s}\int_s^t f(\sigma)d\sigma \leq \varepsilon + \overline{A}_{OT}(f) + \frac{B(t)-B(s)}{t-s} \qquad \forall t > s.$$

Since is bounded, there exists $r > 0$ such that

$$\frac{B(t)-B(s)}{t-s} \leq \varepsilon \quad if \quad t-s \geq r$$

and the proof follows easily.

Now, we shall show that $\overline{A}_{OT}(f) \leq \overline{B}_{CV}(f)$ and the proof of Theorem 1.1 will be complete. To this aim, we need five intermediate lemmas.

Lemma 2.2 .- *Given a continuous functions* $\alpha : [a,d] \to (-\infty, 0]$ *and a non negative real number* $\rho > \alpha(d)$, *there exists a continuous function* $\beta : [a,d] \to R$ *such that* $\alpha \leq \beta \leq \rho$; $\beta(a) = \alpha(a)$ *and* $\beta(d) = \rho$.

Proof.- Let $L : [a,d] \to R$ be the linear map determined by conditions $L(a) = \alpha(a)$ and $L(d) = \rho$. Since $L(d) > \alpha(d)$, there exists $c \in [a,d)$ such that $L(c) = \alpha(c)$ and $L > \alpha$ on $(c,d]$. Now, it suffices to define β by $\beta \equiv \alpha$ on $[a,c]$ and $\beta \equiv L$ on $[c,d]$.

Lemma 2.3 .- *Let* $\alpha : [a,c] \to R$ *be a continuous function such that* $\alpha < 0$ *on* $[a,c)$ *and* $\alpha(c) = 0$. *Then, given a positive real number* ε *there exists a continuous function* $\beta : [a,c] \to (-\infty, 0]$ *such that* $\beta(a) = \alpha(a)$, $\beta(c) = 0$, $\alpha \leq \beta$ *and* $\int_a^c \beta(t)dt \geq -\varepsilon$.

Proof .- Let us fix $\delta \in (0, c-a)$ such that $\int_a^{a+\delta} \alpha(t)dt \geq -\varepsilon$. By Lemma 2.2 there exists a continuous function $\gamma : [a, a+\delta] \to (-\infty, 0]$ such that $\alpha \leq \gamma$ on $[a, a+\delta]$, $\gamma(a) = \alpha(a)$, and $\gamma(a+\delta) = 0$. Now, we define $\beta : [a,c] \to R$ by $\beta \equiv \gamma$ on $[a, a+\delta]$ and $\beta \equiv 0$ on $[a+\delta, c]$ and the proof follows easily.

Lemma 2.4 .- *Let $\alpha : [a,b] \to R$ be a continuous function such that $\int_a^b \alpha(t)dt \leq 0$. Then, given a positive number $M \geq \max(\alpha)$, there exists a continuous function $\beta : [a,b] \to R$ such that $\alpha \leq \beta \leq M$, $\beta(a) = \alpha(a)$, $\beta(b) = \alpha(b)$ and $\int_a^b \beta(t)dt = 0$.*

Proof.- If $\int_a^b \alpha(t)dt = 0$, it suffices to take $\beta \equiv \alpha$. Thus, we can assume that $\int_a^b \alpha(t)dt < 0$.

Let $\mathcal{C}_0$ be the space of all continuous functions $\beta : [a,b] \to R$ provided with the usual sup norm and let $\mathcal{F}$ be the subset $\mathcal{C}_0$ consisting of all points β such that $\alpha \leq \beta \leq M$ and

$$\beta \equiv \alpha \quad on \quad \{a,b\} \cup \alpha^{-1}\left([0,\infty)\right).$$

It is clear that $\mathcal{F}$ is convex and it contains α. On the other hand, the function $I : \mathcal{F} \to R$; $I(\beta) = \int_a^b \beta(t)dt$; is continuous and $I(\alpha) \leq 0$.Thus, it suffices to find $\beta_* \in \mathcal{F}$ such that $I(\beta_*) \geq 0$.

Note that if $\alpha(a)$, $\alpha(b) \geq 0$, then $\alpha^+ \in \mathcal{F}$ and $I(\alpha_+) \geq 0$, where $\alpha^+(t) = \max\{\, 0\,,\, \alpha(t)\,\}$. So, we can assume that either $\alpha(a) < 0$ or $\alpha(b) < 0$. Now let us consider the following cases and subcases.

Case 1. $\alpha^+ \not\equiv 0$.

Subcase 1.1. $\alpha(a) < 0 \leq \alpha(b)$. In this case, there exists $c \in (a,b]$ such that $\alpha < 0$ on $[a,c]$ and $\alpha(c) = 0$, and by Lemma 2.3, there exists a continuous function $\gamma : [a,c] \to (-\infty, 0]$ such that $\gamma(a) = \alpha(a)$, $\gamma(c) = 0$ and $\int_a^c \gamma(t)dt \geq -\int_a^b \alpha^+(t)dt = -\int_0^b \alpha^+(t)dt$. Now, it suffices to define $\beta_* \equiv \gamma$ on $[a,c]$ and $\beta_* \equiv \alpha^+$ on $[c,b]$.

Subcase 1.2. $\alpha(a) \geq 0 > \alpha(b)$. The proof of this case is similar to Subcase 1.1.

Subcase 1.3. $\alpha(a) < 0$ and $\alpha(b) < 0$. Since $\alpha^+ \not\equiv 0$,there exists $c < d$ in (a,b) such $\alpha < 0$ on $[a,c) \cup (d,b]$ and $\alpha(c) = \alpha(d) = 0$. By Lemma 2.3 there exists continuous functions $\gamma_0 : [a,c] \to (-\infty, 0]$, $\gamma_1 : (d,b] \to (-\infty, 0]$ such that $\gamma_0(a) = \alpha(a)$, $\gamma_0(c) = \gamma_1(d) = 0$, $\gamma_1(d) = \alpha(d)$,

$$2\int_a^c \gamma_0(t)dt \geq -\int_a^b \alpha^+(t)dt \quad and \quad 2\int_d^b \gamma_1(t)dt \geq -\int_a^b \alpha^+(t)dt.$$

To end the proof of this case, it suffices to define $\beta_* \equiv \gamma_0$ on $[a,c]$, $\beta_* \equiv \alpha^+$ on $[c,d]$ and $\beta_* \equiv \gamma_1$ on $[d,b]$.

Case 2. $\alpha \leq 0$. Let us fix $\delta > 0$ such that $2\delta < b - a$ and

$$2\delta\left[M-\min(\alpha)\right]\leq M(b-a).$$

By Lemma 2.2, there exists continuous functions $\gamma_0:[a,a+\delta]\to R$ and $\gamma_0:[b-\delta,b]\to R$ such that $\gamma_0(a)=\alpha(a)$, $\gamma_0(a+\delta)=\gamma_1(b-\delta)=M$, $\gamma_1(b)=\alpha(b)$, $\alpha\leq\gamma_0\leq M$ on $[a,a+\delta]$ and $\alpha\leq\gamma_1\leq M$ on $[b-\delta,b]$. To end the proof it suffices to define $\beta_*\equiv\gamma_0$ on $[a,a+\delta]$, $\beta_*\equiv M$ on $[a+\delta,b-\delta]$ and $\beta_*\equiv\gamma_1$ on $[b-\delta,b]$.

Lemma 2.5 .- *Let $\phi:R\to R$ be a bounded above continuous function and suppose that there exists* S$\in\mathcal{S}$ *such that*

$$\int_{S(n)}^{S(n+1)}\phi(t)\,dt\leq 0\qquad\forall n\in Z.$$

Then, there exists a bounded and continuously differentiable function B : $R\to R$ such that $\phi(t)\leq B'(t)$ $for\ \ all$ $t\in R$.

Proof . If $M:=\max(\phi)\leq 0$, it suffices to take $B\equiv 0$. So, we can assume $M>0$.

Let us fix $n\in Z$. By Lemma 2.4, there exists a continuous function $\beta_n:[S(n),S(n+1)]\to R$ such that $\beta_n(S(n))=\phi(S(n))$, $\beta_n(S(n+1))=\phi(S(n+1))$, $\phi\leq\beta_n\leq M$ on $[S(n),S(n+1)]$ and

$$\int_{S(n)}^{S(n+1)}\beta_n(t)dt=0. \tag{2.1}$$

Now, it is easy to show that the function $\beta:R\to R$ defined by $\beta\equiv\beta_n$ on $[S(n),S(n+1)]$, is continuous and $\phi\leq\beta\leq M$.

To end the proof it suffices to show that the function $B(t)=\int_{S(0)}^{t}\beta(s)ds$ is bounded. To this aim, let us first remark that for each $t\in R$ there exists a unique $n\in Z$ such that $S(n)\leq t<S(n+1)$. In this case, we write $[t]=S(n)$ and by (2.1),

$$B(t)=\int_{[t]}^{t}\beta(s)ds. \tag{2.2}$$

Assume now that there exists a sequence $\{t_n\}$ in R such that $|B(t_k)|\to\infty$ as $k\to\infty$, then $|t_k|\to\infty$, since B is continuous. On the other hand, by (2.2), there exists $\sigma_k\in[[t_k],t_k]$ such that

$$B(t_k) = (t_k - [t_k])\beta(\sigma_k), \tag{2.3}$$

and hence, $|\beta(\sigma_k)| \to \infty$ as $k \to \infty$, since $\{t_k - [t_k]\}$ is bounded and non negative sequence. But $\beta \leq M$ and so, $\beta(\sigma_k) \to -\infty$. From this,

$$B(t_k) \to -\infty. \tag{2.4}$$

Given $t \in R$, we write $(t) = S(n+1)$ if $[t] = S(n)$. Since $\int_{[t]}^{(t)} \beta(s)ds = 0$ and (2.2) holds, we have

$$B(t) = -\int_t^{[t]} \beta(s)ds$$

and hence, $B(t_k) = -([t_k]-t_k)\beta(s_k)$ for some $s_k \in [t_k, [t_k]]$. By (2.4), $\beta(s_k) \to +\infty$ since $\{[t_k] - t_k\}$ is bounded sequence. This contradicts the fact that β is bounded above and the proof is complete.

Using Lemma 2.5, it is easy to show that $\overline{A}_{OT}(f) \leq \overline{B}_{VC}(f)$ and the proof of Theorem 1.1 is complete.

3 Axiomatic

In this section, $\mathcal{C}$ denotes the space of all bounded continuous functions $f : R \to R$ provided with the usual sup norm $\|f\|_\infty = \sup\{|f(t)| : t \in R\}$.

We identify R with the subset of $\mathcal{C}$ consisting of all constant functions.

We say that a function $\overline{\mathcal{A}} : \mathcal{C} \to R$ is an upper average if:

A$_1$) $\overline{\mathcal{A}}(f+g) \leq \overline{\mathcal{A}}(f) + \overline{\mathcal{A}}(g)$.

A$_2$) $\overline{\mathcal{A}}(\lambda f) = \lambda \overline{\mathcal{A}}(f)$ *for all* $\lambda \in (0,\infty)$.

A$_3$) $\overline{\mathcal{A}}(f) \leq \overline{\mathcal{A}}(g)$ *if* $f \leq g$.

A$_4$) $\overline{\mathcal{A}}(f) = 0$ *if f has a bounded primitive.*

A$_5$) $\overline{\mathcal{A}}(\lambda) = \lambda$ *if* $\lambda \in R$.

It is easy to show that $\overline{A}_T$ is upper average and by the results in [2], the same holds for $\overline{A}_{VC}$. We shall see that $\overline{A}_{BH}$ is also upper average.

Proposition 3.1 .- *Suppose that there exists* $\tau > 0,\ \mu \in R,\ s_0 \geq -\infty$ *such that*

$$\frac{1}{\tau}\int_s^{s+\tau} f(\sigma)d\sigma \leq \mu \quad \forall\, s \geq s_0.$$

Then

$$\frac{1}{k\tau}\int_{s}^{s+k\tau} f(\sigma)d\sigma \le \mu \quad \forall\, s \ge s_0,\ \forall\, k \in N.$$

Proof. For all integers $k \ge 1$ we have,

$$\int_{s}^{s+k\tau} f(\sigma)d\sigma = \int_{s}^{s+\tau} f(\sigma)d\sigma + \int_{s+\tau}^{s+2\tau} f(\sigma)d\sigma + ... + \int_{s+k\tau-\tau}^{s+k\tau} f(\sigma)d\sigma$$

and the proof follows easily.

Proposition 3.2 .- $\overline{A}_{BH}(f+g) \le \overline{A}_{BH}(f) + \overline{A}_{BH}(g)$.

Proof. Given $\varepsilon > 0$ there exists $\tau > 0$ and $s_0 \ge -\infty$ such that

$$\frac{1}{\tau}\int_{s}^{s+\tau} f(\sigma)d\sigma \le \frac{\varepsilon}{2} + \overline{A}_{BH}(f) \quad \forall s > s_{0.}$$

On the other hand

$$\frac{1}{\tau+s}\int_{s}^{s+\tau+\delta} f(\sigma)d\sigma \longrightarrow \frac{1}{\tau}\int_{s}^{s+\tau} f(\sigma)d\sigma \quad as \quad \delta \to 0$$

uniformly on $s \in R$, and hence, there exists a positive rational number T such that

$$\frac{1}{T}\int_{s}^{s+T} f(\sigma)d\sigma \le \varepsilon + \overline{A}_{BH}(f) \ \ \forall\, s > s_{0.}$$

From this and Proposition 3.1, then exists on integer $N \ge 1$such that

$$\frac{1}{N}\int_{s}^{s+N} f(\sigma)d\sigma \le \varepsilon + \overline{A}_{BH}(f) \ \ \forall\, s > s_{0.}$$

Analogously, there exists $s_1 \ge -\infty$ and an integer $P \ge 1$ such that

$$\frac{1}{P}\int_{s}^{s+P} g(\sigma)d\sigma \le \varepsilon + \overline{A}_{BH}(g) \ \ \forall\, s > s_1,$$

and by Proposition 7.1 once again,

$$\frac{1}{NP}\int_{s}^{s+PN} f(\sigma)d\sigma \le \varepsilon + \overline{A}_{BH}(f) \ \ and \ \ \frac{1}{NP}\int_{s}^{s+PN} g(\sigma)d\sigma \le \varepsilon + \overline{A}_{BH}(g)$$

if $s > \max\{s_0,\, s_1\}$. Therefore:

$$\frac{1}{NP}\int_s^{s+PN} [f(\sigma)+g(\sigma)]\, d\sigma \leq 2\varepsilon + \overline{A}_{BH}(f) + \overline{A}_{BH}(g)$$

for all $s > \max\{s_0,\, s_1\}$, and the proof is complete.

Using Proposition 3.2 it is easy to show that $\overline{A}_{BH}$ is an upper average. In the following, $\overline{\mathcal{A}} : \mathcal{C} \to R$ denotes an upper average. The function $\overline{\mathcal{A}} : \mathcal{C} \to R$ given by $\mathcal{A}_{-}(f) = -\overline{\mathcal{A}}(-f)$ will be referred as the *loweraverage associated* to $\overline{\mathcal{A}}$.

Proposition 3.3 .- $\underline{A}_T \leq \underline{\mathcal{A}} \leq \overline{\mathcal{A}} \leq \overline{A}_T$.

Proof. Let us fix $f \in \mathcal{C}$ and $\beta > \overline{A}_T(f)$. By lemma 3 of [4] , there exists a bounded and continuouslydifferentiable function $B : R \to R$ such that

$$f(t) \leq \beta + B'(t) \quad \forall\, t \in R,$$

and by $A_1)-A_5)$,

$$\overline{\mathcal{A}}(f) \leq \overline{\mathcal{A}}(\beta + B') \leq \overline{\mathcal{A}}(\beta) + \overline{\mathcal{A}}(B') = \overline{\mathcal{A}}(\beta) = \beta.$$

From this, $\overline{\mathcal{A}} \leq \overline{A}_T$.

On the other hand

$$\overline{\mathcal{A}}(f) + \overline{\mathcal{A}}(-f) \geq \overline{\mathcal{A}}(f-f) = \overline{\mathcal{A}}(0) = 0$$

and hence $\overline{\mathcal{A}}(f) \geq \underline{\mathcal{A}}(f)$. The rest of the proof follows easily since $\underline{A}_T(f) = -\overline{A}_T(-f)$.

Let $\mathcal{F} = \left\{ f \in \mathcal{C} \,:\, A_{-T}(f) = \overline{A}_T(f) \right\}$. It is clear that $\mathcal{F}$ is the subspace of $\mathcal{C}$ consisting of all points $f \in \mathcal{C}$ such that

$$\frac{1}{T}\int_s^{s+T} f(\sigma)d\sigma \to \lambda \quad as \;\; T \to +\infty \;\; uniformly \; on \; s \in R$$

for some $\lambda \in R$. In fact, $\lambda = \underline{A}_T(f) = \overline{A}_T(f)$. As a corollary of Proposition 3.3 we have:

Corollary 3.4 .- *If* $f \in \mathcal{F}$ *then*

$$\overline{\mathcal{A}}(f) = \lim_{T\to+\infty} \frac{1}{T}\int_s^{s+T} f(\sigma)d\sigma, \;\; uniformly \;\; on \; s \in R.$$

Remark 3.5.- $\mathcal{F}$ is a closed linear subspace of $\mathcal{C}$ which contains all periodic functions. In particular, it contains the space of all almost periodic functions. $\mathcal{F}$ also contains the space $\mathcal{B}_p$ of all points $f \in \mathcal{C}$ having a bounded primitive and the space $\mathcal{L}$ of all points $f \in \mathcal{C}$ such that $f(t)$ has a finite limit λ *as* $|t| \to +\infty$. In this case, $\overline{\mathcal{A}}(f) = \underline{\mathcal{A}}(f) = \lambda$.

Let us define $\mathcal{E} = \left\{ f \in \mathcal{C} \,:\, \overline{\mathcal{A}}(f) = \underline{\mathcal{A}}(f) \right\}$. By Proposition 3.3 we have

$$R \subset \mathcal{F} \subset \mathcal{E}. \tag{3.1}$$

Proposition 3.6 .- *a)* $\overline{\mathcal{A}}(f+g) = \overline{\mathcal{A}}(f) + \overline{\mathcal{A}}(g)$ *if* $g \in \mathcal{E}$.

b) $\left|\overline{\mathcal{A}}(f) - \overline{\mathcal{A}}(g)\right| \leq \|f-g\|_{\infty}$.

c) $\mathcal{E}$ *is a closed linear subspace of* $\mathcal{C}$.

Proof. Using A_1) we have

$$\overline{\mathcal{A}}(f) = \overline{\mathcal{A}}(f+g-g) \leq \overline{\mathcal{A}}(f+g) + \overline{\mathcal{A}}(-g) = \overline{\mathcal{A}}(f+g) - \mathcal{A}_-(g) = \overline{\mathcal{A}}(f+g) - \overline{\mathcal{A}}(g)$$

and the proof of a) follows from A_1).

To show b), let us write $\varepsilon = \|f-g\|_{\infty}$, then $g - \varepsilon \leq f \leq g + \varepsilon$ and the proof follow from part a), A_1) and A_5).

Let us fix $f, g \in \mathcal{E}$. Using part a) we have

$$-\overline{\mathcal{A}}(f) = \overline{\mathcal{A}}(-f) = \overline{\mathcal{A}}(-f+g-g) = \overline{\mathcal{A}}(-f-g) + \overline{\mathcal{A}}(g)$$

and hence, $\overline{\mathcal{A}}(-f-g) = -\overline{\mathcal{A}}(f) - \overline{\mathcal{A}}(g) = -\overline{\mathcal{A}}(f+g)$. Thus, $f+g \in \mathcal{E}$. The rest of proof is similar.

Proposition 3.7 .- *Suppose that*

A_6) $\overline{\mathcal{A}}(f) = \sup(f)$ *if* $f^{-1}(\sup(f))$ *contains a sequence of intervals* $\{[a_n, b_n]\}$
such that $b_n - a_n \to +\infty$.

If $f \in \mathcal{C}$ *and* $f(t)$ *has a finite limit* μ_+ (*resp.* μ_-) *as* $t \to +\infty$ (*resp.* $t \to -\infty$). *Then,*

$$\overline{\mathcal{A}}(f) = \max\{\mu_+ \,,\, \mu_-\}.$$

Proof. If $\mu_+ = \mu_-$The result follows from Remark 3.5 and (3.1) . Thus, we can suppose that $\mu_+ > \mu_-$.

Let $\phi \in \mathcal{C}$ be defined by $\phi(t) \equiv \mu_+$ in $[1, +\infty)$, $\phi(t) \equiv \mu_-$ in $(-\infty, -1]$ and $\phi(t)$ is linear in $[-1, 1]$.

Then $f(t) - \phi(t) \to 0 \quad as \quad |t| \to \infty$, and hence

$$\overline{\mathcal{A}}(f - \phi) = \overline{\mathcal{A}}(\phi - f) = 0.$$

On the other hand

$$\overline{\mathcal{A}}(f) = \overline{\mathcal{A}}(f - \phi - \phi) \leq \overline{\mathcal{A}}(f - \phi) + \overline{\mathcal{A}}(\phi) = \overline{\mathcal{A}}(\phi)$$

and by the same argument, $\overline{\mathcal{A}}(\phi) \leq \overline{\mathcal{A}}(f)$. From this and A_6), $\overline{\mathcal{A}}(f) = \overline{\mathcal{A}}(\phi) = \mu_+$, and the proof is complete.

Remark .- $\overline{A}_T$ is an upper average that satisfies A_6). However $\overline{A}_{BH}$ does not satisfies this condition as the following example shows.

Let $f : R \to [1, 2]$ be a continuous function such that $f \equiv 1$ on $[1, \infty)$ and $f \equiv 2$ on $(-\infty, -1]$. It is easy to show that $\overline{A}_{BH}\,(f) = 1$, and so, $\overline{A}_{BH}$ does not satisfies A_6).

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