Persistence in the Mean of Some Competitive Systems

M. Gámez and R. Carreño

Departamento de Estadística y Matemática Aplicada.

Universidad de Almería. España.

1 de Noviembre 1997

Abstract

In this paper we study persistence in the mean for n-competing populations which are affected by some perturbation due to toxic effects.

Key words. Persistence in the mean, extinction, toxic effects, population models.

0 Introduction

The problem of estimating qualitatively the effects of a toxicant on a population by mathematical models has been studied by several authors [Hallam (1986), De Luna (1987), Huaping and Zhien (1991)], obtaining a survival threshold distinguishing between persistence in the mean and extinction of a single population. This problem includes the action of pesticide that check the growth of a population plague in a forming.

In this paper we study persistence, extinction and persistence in the mean for a system of n-competing species affected by the action of a toxic.

In the first section, we study the case of a single specie, getting accurate results about the persistence in the mean. We also give a specific example where the persistence in the mean holds, but the system is not persistent.

In the second section, we use an idea in A. Tineo [5] to obtain a result about the persistence in the mean of the following n-competing species problem.

$$x_i' = x_i \left[r_{i0} - r_{i1} c_0(t) \ - \sum_{j=1}^{n} b_{ij}\, x_j \right] \ ,$$

where $c_0(t)$ represents an exogenous factor (toxicant, artificial predator , pesticide,...) that can affect the demographic parameters ; r_{i0} is the intrinsic growth rate of the i th population in the absence of toxicant; r_{i1} the dose-response parameter of species i to the organisms..

In the case n=2, we also use an argument in Ahmad- Lazer [1], to study the extinction of one specie.

1 The Logistic Equation

In this section we study extinction and persistence in the mean [3] for the following logistic equation,

$$x' = x\,[a(t) - bx] \tag{1.1}$$

where $a : [0, \infty) \to \ R$ is bounded and continuous, and $b > \ 0$ is constant.

We recall that (1.1) is *persistent in the mean* if:

$$\liminf_{t\to+\infty} \frac{1}{t} \int_0^t x(s)ds > 0 \ ,$$

for any positive solution $x(t)$ of (1.1).

We say that a positive solution $x(t)$ of (1.1) *goes to extinction* if $x(t) \to 0 \ as \ \ t \to +\infty$.

We shall use the following notations:

Given a bounded function $g : [0, +\infty) \to R$, we define,

$$g^* = \limsup_{t\to+\infty} g(t) \quad ; \quad g_* = \liminf_{t\to+\infty} g(t)\,,$$

and given bounded continuous function

$f:(0,+\infty)\to R$,we define,

$$\langle f\rangle (t)=\frac{1}{t}\int_0^t f(s)ds\,.$$

In this paper we shall use several times the following result in [4].

Lemma 1.1 .- *Let* $f\in C\left[R_+,R_+^0\right]$; $R_+ := [0,+\infty)$, $R_+^0 := (0,+\infty)$.
a) If there exist positive constants λ,λ_0,t_0 *such that:*

$$\ln f(t)\le \lambda t-\lambda_0\int_0^t f(s)ds \;\; for\ all\, t\ge t_0 \Longrightarrow \langle f\rangle^* \le \lambda/\lambda_0$$

b) If there exist positive constants γ,γ_0,t_1 *such that:*

$$\ln f(t)\ge \gamma t-\gamma_0\int_0^t f(s)ds \;\; for\, all\ t\ge t_1 \Longrightarrow \langle f\rangle_* \ge \gamma/\gamma_0$$

Actually only part a) of the lemma is proved in [4] . But the proof of b) can be obtained by the same arguments in that paper, by reversing the corresponding inequalities.

Let x be a solution of (1.1) such that $x(0)>0$. It is easy to show that x is defined and bounded in $[0,\infty)$, moreover by theorem 2 of [3] we obtain.

Proposition 1.2 .- $\langle x\rangle_*>0$ *if* $\langle a\rangle_*>0$ *and* $x^*=0$ *if* $\langle a\rangle^*<0$.

In Proposition 1.3 and Remark 1.4 below, we obtain a more precise estimating for $\langle x\rangle_*$ and $\langle x\rangle^*$.

Proposition 1.3 .- *If* $\langle a\rangle_*>0$ *then* $\langle x\rangle^* = b^{-1}\ \langle a\rangle^*$.
Proof. By (1.1) weobtain,

$$\langle a\rangle (t)=b\,\langle x\rangle (t)+\frac{1}{t}\ln\frac{x(t)}{x(0)} \quad for\ all\ t>0\ . \tag{1.2}$$

Since x is bounded on $[0,+\infty)$ it follows

$$\limsup_{t\to\infty}\frac{1}{t}\ln\frac{x(t)}{x(0)}\le 0,$$

and using (1.2) we have,

$$\langle a\rangle^*\le b\,\langle x\rangle^*\ . \tag{1.3}$$

On the other hand, given $\mu > \langle a \rangle^*$,there exists $t_0 > 0$ such that $\langle a \rangle (t) \leq \mu$ for all $t \geq t_0$, and by (1.2) ,

$$\frac{1}{t} \ln \frac{x(t)}{x(0)} \leq \mu - b \langle x \rangle (t) \quad for\ all\ t \geq t_0.$$

From Lemma 1.1, $\langle x \rangle^* \leq \mu / b$, and consequently, $b \langle x \rangle^* \leq \langle a \rangle^*$.

The proof follows now from (1.3) .

□

Remark 1.4 .- Using Lemma 1.1 b) in the previous proposition, we also obtain

$$b \langle x \rangle_* \geq \langle a \rangle_* \ .$$

In particular (1.1) *is permanentin mean* if $\langle a \rangle_* > 0$.

We recall that (1.1) is permanent in the mean if there exists $m > 0$ such that $\langle x \rangle_* \geq m$ for any positive solution x of (1.1) .

Note also that if $\langle a \rangle^* = \langle a \rangle_* > 0$, then $\langle x \rangle^* = \langle x \rangle_* > 0$, for any positive solution of (1.1) .

In the following example we exhibit a logistic equation which is *permanent in mean* but is not persistent.

Example.- There exists a continuous function $a : [0, +\infty) \to [0, 1]$ and a sequence $t_n \to +\infty$ $(t_n \geq 0)$ such that.

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t a(s) ds = \frac{1}{2} \tag{1.4}$$

$$\lim_{t_n \to +\infty} x(t_n) = 0$$

for any positive solution $x(t)$ of the following logistic equation.

$$x' = x \left[a(t) - x \right] \tag{1.5}$$

Proof. Given an interval $I = [\alpha, \beta]$ such that $\beta - \alpha > 2$ wedefine $a_I : I \to [0, 1]$ by

$$a_I(t) = \begin{cases} t - \alpha & if & \alpha \leq t \leq \alpha + 1 \\ 1 & if & \alpha + 1 \leq t \leq \alpha + l + 1 \\ \alpha + l + 2 - t & if & \alpha + l + 1 \leq t \leq \alpha + l + 2 \\ 0 & if & \alpha + l + 2 \leq t \leq \beta \end{cases}$$

where $2l = \beta - \alpha - 2$.

For each $n \in N$, let us define $I_n = [t_{n-1}, t_n] := [\,(n-1)\,(n+2)\,, n\,(n+3)\,]$ and $a(t) = a_{I_n}(t)$ if $t \in I_n$. Evidently $a : [0, +\infty) \to [0,1]$ is a well defined continuous function, and by induction it is easy to show that

$$\langle a \rangle (t_n) = \frac{1}{2} \quad for \;\; all \;\; n \in N.$$

On the other hand we have:

$$0 \leq \frac{1}{2} - \langle a \rangle (t) \leq \frac{1}{2t_n} \quad if \quad t_n \leq t \leq t_n + 1,$$

$$0 \leq \langle a \rangle (t) - \frac{1}{2} \leq \frac{1+n}{2\,(t_n + n)} \quad if \quad t_n + 1 \leq t \leq t_{n+1},$$

and the proof of (1.4) is complete

Let x be a solution of (1.5) such that $x(0) > 0$. In $[t_{n+1} - n, t_{n+1}]$ we have $a \equiv 0$, and hence $x' = -x^2$ in this interval. That is , $\left(\frac{1}{x}\right)' = -1$ on $[t_{n+1} - n, t_{n+1}]$, and writing $\xi_n = x\,(t_{n+1} - n)$ we have,

$$x(t_{n+1}) = \frac{1}{n + \frac{1}{\xi_n}} < \frac{1}{n} \to 0 \qquad as \qquad n \to +\infty$$

$\square$

2 Persistence in the Mean for some Competitive Systems

Let us consider the n- competing systems ; $n \geq 2$;

$$x_i' = x_i \left[a_i(t) - \sum_{j=1}^{n} b_{ij}\, x_j \right] \;, \tag{2.1}$$

where $a_i : [0, +\infty) \to R$ is a bounded continuous function and b_{ij} is a positive constant. It is easy to show that if $x = (x_1, ..., x_n)$ is a solution of (2.1) and $x(0) > 0$ then x is defined and bounded on $[0, \infty)$.

Theorem 2.1 .- *Assume that for some* $1 \leq i \leq n$,

$$\langle a_i \rangle_* > \sum_{j \in J_i} b_{ij} \frac{\langle a_j \rangle^*}{b_{jj}} \tag{2.2}$$

where $J_i = \{j = 1, 2, ..., n \ : \ i \neq j\}$. Then,

$$b_{ii} \langle x_i \rangle_* \geq \langle a_i \rangle_* - \sum_{j \in J_i} \frac{b_{ij}}{b_{jj}} \langle a_j \rangle^* ,$$

for any positive solution $(x_1, ..., x_n)$ *of* (2.1). *In particular, if* (2.2) *holds for alli* $= 1, ..., n$, *then* (2.1) *is permanent in the mean.*

Proof. Let X_j be the solutionof

$$X' = X \left[a_j(t) - b_{jj} X\right] \qquad X(0) = x_j(0) .$$

We know that $X_j \geq x_j$ on $[0 , +\infty)$ and $\langle X_j \rangle^* = \frac{\langle a_j \rangle^*}{b_{jj}}$. See Proposition 1.3. In particular

$$x_i'(t) \geq x_i(t) \left[a_i(t) - b_{ii} x_i(t) - \sum_{j \in J_i} b_{ij} X_j(t) \right] .$$

That is, x_i is a supersolution of the system

$$y' = y \left[\eta_i - b_{ii} y\right] \tag{2.3}$$

where $\eta_i := a_i(t) - \sum_{j \in J_i} b_{ij} X_j(t)$. Note also that

$$\langle \eta_i \rangle_* \geq \langle a_i \rangle_* - \sum_{j \in J_i} b_{ij} \langle X_j \rangle^* = \langle a_i \rangle_* - \sum_{j \in J_i} b_{ij} \frac{\langle a_j \rangle^*}{b_{jj}}$$

and by (2.2) $\langle \eta_i \rangle_* > 0$. From this , if y denotes the solution of (2.3) determined by the initial condition $y(0) = x_i(0)$,we have $x_i(t) \geq y(t)$ and the proof follows from Remark 1.4 .

□

Theorem 2.2 .- *Assume that* $\langle a_i \rangle_* = \langle a_i \rangle^*$ *and that* (2.2) *hold for all* $i = 1, ...n$.

Then there exists $p \in R^n$; $p > 0$; such that $\langle x \rangle (t) \to p$ as $t \to +\infty$ for all positive solutions x of (2.1).

Proof. Let us fix $i = 1, 2, ..., n$ and $0 < \varepsilon < \min \{ \langle a_i \rangle_* : 1 \le i \le n \}$. (Note that by (2.2) , $\langle a_i \rangle_* > 0$ *for all* i). We knowthat ,

$$\frac{1}{t} \ln \frac{x_i(t)}{x_i(0)} = \langle a_i \rangle (t) - b_{ii} \langle x_i \rangle (t) - \sum_{j \in J_i} b_{ij} \langle x_j \rangle (t)$$

and so, there exists $t_0 > 0$ such that

$$\frac{1}{t} \ln \frac{x_i(t)}{x_i(0)} \ge \langle a_i \rangle_* - \varepsilon - b_{ii} x_i(0) \frac{\langle x_i \rangle (t)}{x_i(0)} - \sum_{j \in J_i} b_{ij} \left(\langle x_j \rangle^* + \varepsilon \right) \quad for \ all \ t \ge t_0.$$

From this and Lemma 1.1 , it follows

$$\langle x_i \rangle_* \ge \frac{\langle a_i \rangle_* - \varepsilon - \sum\limits_{j \in J_i} b_{ij} \left(\langle x_j \rangle^* + \varepsilon \right)}{b_{ii}}$$

and letting $\varepsilon \to 0^+$, we obtain,

$$b_{ii} \langle x_i \rangle_* + \sum_{j \in J_i} b_{ij} \langle x_j \rangle^* \ge \langle a_i \rangle_* . \tag{2.4}$$

Similarly,

$$b_{ii} \langle x_i \rangle^* + \sum_{j \in J_i} b_{ij} \langle x_j \rangle_* \le \langle a_i \rangle^* . \tag{2.5}$$

Since $\langle a_i \rangle_* = \langle a_i \rangle^*$ we have

$$\sum_{j \in J_i} b_{ij} \left(\langle x_j \rangle^* - \langle x_j \rangle_* \right) \ge b_{ii} \left(\langle x_i \rangle^* - \langle x_i \rangle_* \right) .$$

Following [5] pag. 13 , we define $w_i := b_{ii} \left(\langle x_i \rangle^* - \langle x_i \rangle_* \right)$, and fix $k = 1, ..., n$ such that

$$\frac{w_k}{\alpha_k} \ge \max_j \left\{ \frac{w_j}{\alpha_j} : j = 1, ..., n \right\} . \tag{2.6}$$

Since $w_i \le \sum\limits_{i \in J_i} \frac{b_{ij}}{b_{jj}} w_j$ for all i then , by (2.6) ,

$$w_k \leq \sum_{j \in J_k} \frac{b_{kj}}{b_{jj}} w_j \leq \sum_{j \in J_k} \frac{b_{kj}}{b_{jj}} \alpha_j \frac{w_k}{\alpha_k} \, ,$$

and hence ,

$$\left[\alpha_k - \sum_{j \in J_k} \frac{b_{kj}}{b_{jj}} \alpha_j \right] w_k \leq 0 \, ,$$

so , $w_k = 0$, and by (2.6) , $\langle x_j \rangle_* = \langle x_j \rangle^*$ for all $j = 1, ..., n$.

Let us define $p_i = \langle x_i \rangle_* = \langle x_i \rangle^*$. By (2.4) $-$ (2.5) , we conclude that $p = (p_1, ..., p_n)$ is the solution of linear system,

$$\sum_{j=1}^{n} b_{ij} x_j = \langle a_i \rangle^* \, ,$$

and the proof is complete

$\square$

Using an idea in proposition 1.4 of [1]; we obtain .

Theorem 2.3 .- *Suppose* $n = 2$ *and* $\langle a_i \rangle_* > 0$ *for* $i = 1, 2$. *If* ,

$$b_{22} \langle a_1 \rangle_* > b_{12} \langle a_2 \rangle^*$$

$$b_{11} \langle a_2 \rangle^* \leq b_{21} \langle a_1 \rangle_* \tag{2.7}$$

Then ,

$$\langle x_1 \rangle_* \geq \frac{\langle a_1 \rangle_*}{b_{11}} \quad \textit{and} \quad \langle x_2 \rangle^* = 0,$$

for any positive solution (x_1, x_2) *of* (2.1) .

Proof. By Theorem 2.1,

$$b_{11} \langle x_1 \rangle_* \geq \langle a_1 \rangle_* - \frac{b_{12}}{b_{22}} \langle a_2 \rangle^* > 0 \, ,$$

and by the arguments in Theorem 2.2 , we have

$$\langle a_1 \rangle_* \leq b_{11} \langle x_1 \rangle_* + b_{12} \langle x_2 \rangle^* \tag{2.8}$$

$$\langle a_2 \rangle^* \geq b_{21} \langle x_1 \rangle_* + b_{22} \langle x_2 \rangle^* \ .$$

From this ,

$$(b_{11}b_{22} - b_{12}b_{21}) \langle x_1 \rangle_* \geq b_{22} \langle a_1 \rangle_* - b_{12} \langle a_2 \rangle^* > 0 \tag{2.9}$$

$$(b_{11}b_{22} - b_{12}b_{21}) \langle x_2 \rangle^* \leq b_{11} \langle a_2 \rangle^* - b_{21} \langle a_1 \rangle_* \leq 0 \tag{2.10}$$

By (2.9) $\ 0 < b_{11}b_{22} - b_{12}b_{21}$ and by (2.10) $\langle x_2 \rangle^* = 0$. The proof follows now from (2.8).

□

Remark.- If in (2.7) ,we replace $\leq$ by $<$, then $x_2(t) \to 0$ *as* $t \to +\infty$.
Proof. From the equation

$$x_2' = x_2 \left[a_2 - b_{21}x_1 - b_{22}x_2 \right] \ ,$$

we obtain

$$\limsup_{t \to +\infty} \frac{1}{t} \ln \frac{x_2(t)}{x_2(0)} \leq \langle a_2 \rangle^* - b_{21} \langle x_1 \rangle_* - b_{22} \langle x_2 \rangle_* \leq \langle a_2 \rangle^* - b_{21} \langle x_1 \rangle_* \ ,$$

and by (2.8) , $\langle x_1 \rangle_* \geq \frac{\langle a_1 \rangle_*}{b_{11}}$. Thus ,

$$\limsup_{t \to +\infty} \frac{1}{t} \ln \frac{x_2(t)}{x_2(0)} \leq \langle a_2 \rangle^* - \frac{b_{21}}{b_{11}} \langle a_1 \rangle_* < 0 \ ,$$

and the proof follows easily.

□

ACKNOWLEDGEMENTS

We thank Professor A. Tineo for fruitful and valuable suggestions during his stay at University of Almeria. We also thank the Dirección General de Enseñanza Superior (ref. 1995-0675) that made this stay possible.

References

[1] S. Ahmad and A.C. Lazer, *Asymptotic behavior of solutions of periodic competition diffusion systems.* J. Nonlinear Analysis. Vol 13, N^o 3 (1989), 263-284.

[2] J. T. De Luna and T. G. Hallam, *Effects of toxicantsof population: a qualitative approach IV. Resource-Consumer-Toxicant models.* Ecol. Modelling. Vol 35 (1987), 249-273.

[3] T. G. Hallam and M. Zhien, *Persistence in populationmodels with demographic fluctuations.* J. Math. Biol. Vol 24 (1986), 327-339.

[4] L. Huaping and M. Zhien, *The threshold of survival for system of two species in a polluted environment.* J. Math. Biol. Vol 30 (1991), 49-61.

[5] A. Tineo, *An iterative scheme for n-competing speciesproblem.* J. of Differential Equations. Vol 116, N^o 1 (1995), 1-15.