Serial Coalgebras*

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Abstract

In this paper we extend the theory of serial and uniserial finite dimensional algebras to coalgebras of arbitrary dimension. Nakayama-Skorniakov Theorems are proved in this new setting and the structure of such coalgebras is determined up to Morita-Takeuchi equivalences. Our main structure theorem asserts that over an algebraically closed field k the basic coalgebra of a serial indecomposable coalgebra is a subcoalgebra of a path coalgebra $k\Gamma$ where the quiver Γ is either a cycle or a chain (finite or infinite). In the uniserial case, Γ is either a single point or a loop. For cocommutative coalgebras, an explicit description is given, serial coalgebras are uniserial and these are isomorphic to a direct sum of subcoalgebras of the divided power coalgebra.

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Introduction

In the last years different types of coalgebras have been investigated in connection with some properties of their categories of comodules. For example, semiperfect coalgebras, quasi-coFrobenius coalgebras, hereditary coalgebras, or pure-semisimple coalgebras; see [16], [12], [19], and [21] respectively. This line of research is continued in a very natural way with the study of serial coalgebras, which is done in this paper.

A coalgebra C is said to be *right serial* if every indecomposable injective right C-comodule has a unique composition series (finite or infinite). We prove that Nakayama-Skorniakov Theorem holds in this setting, that is, C is serial (left and right serial) if and only if the category $\mathcal{M}^C$ of all right C-comodules is a uniserial category in the sense of [2] (i.e., every indecomposable object of finite length has a unique composition series); see Theorem 1.8. Consequently, every finite-dimensional comodule is a direct sum of uniserial comodules. It is shown in Proposition 1.13 that such a decomposition holds for any comodule over a serial coalgebra with finite coradical filtration. Examples of serial coalgebras are provided in Corollary 1.11 as

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finite dual coalgebras of either serial or hereditary prime noetherian algebras. In particular, A^0 is serial for every Dedekind domain A.

In Proposition 2.3 we prove that every right serial coalgebra with coseparable coradical is isomorphic to a subcoalgebra of a cotensor coalgebra over a right serial bicomodule. In the right semiperfect case, the coalgebra may be represented in the cotensor coalgebra by an admissible sequence associated to its diagram; see Theorem 2.5. Therefore, the corresponding theory for finite dimensional right serial algebras (see, e.g., [9]) is extended.

Indecomposable serial coalgebras C are then characterized in terms of its diagram $\mathcal{D}(C)$, which, a fortiori, turns out to be its Ext-quiver. Concretely, we prove in Theorem 2.9 that a coalgebra C with coradical decomposition $C_0 = \oplus_{i\in I} M^c(D_i, n_i)$ (the D_i^*'s are division algebras) is serial if and only if I is contable, $\mathcal{D}(C)$ is a chain or a cycle and $D_i \cong D_j$ for every $i, j \in I$. In the algebraically closed case, indecomposable serial coalgebras are recognized, up to Morita-Takeuchi equivalence, as subcoalgebras of the path coalgebra $k\Gamma$, where Γ is a cycle or a chain. If, in addition, C is hereditary, then it is isomorphic to $k\Gamma$, Theorem 2.10.

A serial coalgebra C is said to be *uniserial* if each uniserial comodule has a homogeneous composition series. Uniserial coalgebras are described as direct sums of matrix coalgebras with coefficients in colocal serial coalgebras. It is also shown that a coalgebra C is uniserial if and only if every right coideal is coprincipal. Equivalently, its diagram $\mathcal{D}(C)$ consists of isolated points and loops, see Theorem 3.1. In the cocommutative case, the notion of serial and uniserial coincide, as shown in Theorem 3.2. In such a case, the dual of each irreducible component is a local noetherian algebra whose lattice of ideals consists of the powers of the principal maximal ideal. Under the hypothesis of the field k being algebraically closed, we prove that cocommutative coalgebras are isomorphic to a direct sum of subcoalgebras of the divided power coalgebra.

Let us to fix some notation and present some preliminaries. Throughout this paper k is a fixed ground field. All algebras, coalgebras, vector spaces and $\otimes$, Hom, etc., are over k. Every map is a k-linear map. The reader is expected to be familiar with coalgebra theory. Basic references are [1], [8], [18], and [22]. In the sequel C always stands for a coalgebra and Δ, ε will denote its comultiplication and counit respectively. The category of right (resp. left) C-comodules is denoted by $\mathcal{M}^C$ (resp. ${}^C\mathcal{M}$); for an object M of $\mathcal{M}^C$, its comodule structure map is denoted by ρ_M. The fundamental properties of the categories of comodules can be found in several places, see e.g. [13], [14], and [23]. We emphasize that $\mathcal{M}^C$ is a locally finite category and every injective indecomposable comodule is given as the injective envelope $E(S)$ of a simple comodule S. We will use that $C_C = \oplus_{i\in I} E(S_i)^{(n_i)}$ where $\{E(S_i)\}_{i\in I}$ is a full set of injective indecomposable right comodules, and the $n_i's$ are finite cardinals. For aspects of equivalences between comodule categories, in particular for the definition and properties of the co-endomorphism coalgebra, we refer to [23].

We recall from [6] the definition of a path coalgebra. Let Γ denote a quiver, the *path coalgebra* $k\Gamma$ is the k-vector space generated by the paths in Γ with comultiplication and counit given by

$$\Delta(\alpha) = \sum_{\beta\gamma=\alpha} \beta \otimes \gamma, \qquad \varepsilon(\alpha) = \begin{cases} 0 & if\ |\alpha| > 0 \\ 1 & if\ |\alpha| = 0 \end{cases},$$

where $\beta\gamma$ is the concatenation of paths and $|\alpha|$ the length of α.

1 Serial coalgebras: definition and first properties

Every right C-comodule M has a filtration $\{0\} \subset soc(M) \subset soc^2(M) \subset ....$, called the *Loewy series* of M, defined as follows: $soc(M)$ is the socle of M, and for $n > 1$, $soc^n(M)$ is the unique subcomodule of M satisfying $soc^{n-1}(M) \subset soc^n(M)$ and $soc(M/soc^{n-1}(M)) = soc^n(M)/soc^{n-1}(M)$, see [14, 1.4]. There is an alternative description of this series. Let $\{C_n\}_{n\in\mathbb{N}}$ be the coradical filtration of C, then $soc^{n+1}(M) = \rho_M^{-1}(M\square_C C_n)$.

Definition 1.1 *A right C-comodule M is called uniserial if its lattice of subcomodules is a chain (finite or infinite).*

Lemma 1.2 *The following assertions are equivalent for $M \in \mathcal{M}^C$:*

i) M is uniserial.

ii) The Loewy series is a composition series for M (and each term is finite dimensional).

iii) Every finite dimensional subcomodule of M is uniserial.

Proof: *i) $\Rightarrow$ ii)* Suppose that there is a non simple factor $soc^n(M)/soc^{n-1}(M) = soc(M/soc^{n-1}(M))$. It contains two simple comodules S_1, S_2. The subcomodules T_1, T_2 of M such that $S_i = T_i/soc^n(M),\ i = 1, 2$ would be incomparable, contrary to the hypothesis.

Recall that a simple comodule is finite dimensional. Since the Loewy series is a composition series, and $soc(M), soc^2(M)/soc(M)$ are finite dimensional, $soc^2(M)$ is finite dimensional. By induction it follows that $soc^n(M)$ is finite dimensional for all $n \in \mathbb{N}$.

ii) $\Rightarrow$ iii) Let N be a finite dimensional subcomodule of M. There is $n \in \mathbb{N}$ such that $N \subset soc^n(M)$. Let $r = max\{s \in \mathbb{N} : soc^s(M) \subseteq N\}$ and assume that $N \neq soc^{r+1}(M)$. Then, $N \cap soc^{r+1}(M) = soc^r(M)$ and thus $(N/soc^r(M)) \cap soc(M/soc^r(M)) = \{0\}$. Hence $N = soc^r(M)$.

iii) $\Rightarrow$ i) For $m \in M$, let (m) be the subcomodule generated by m which is finite dimensional. Notice that M is uniserial if and only if either $(m) \subset (n)$ or $(n) \subset (m)$ for all $m, n \in M$. For any $n, m \in M$ we may compare $(n), (m)$ in (n, m) which is finite dimensional, and consequently uniserial by hypothesis. ∎

Definition 1.3 *Let C be a coalgebra.*

i) C is said to be a right serial coalgebra if its right injective indecomposable comodules are uniserial. A left serial coalgebra is defined in a symmetric way.

ii) C is called serial if it is right and left serial.

iii) C is uniserial if it is serial and the composition factors of each indecomposable injective comodule are isomorphic (homogeneous uniserial).

Example 1.4 1) Any cosemisimple coalgebra is uniserial.

2) Let C be the path coalgebra associated to the quiver $\mathcal{A}_{\infty}^{\infty}$

$$\cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots.$$

Denote by $\{g_i : i \in \mathbb{Z}\}$ the set of vertices. The family $\{kg_i\}_{i\in\mathbb{Z}}$ is a representative family of simple right (left) comodules. Let e_i be the idempotent in C^* defined as $\langle e_i, g_i\rangle = 1$ and zero elsewhere. The injective hull of kg_i (as a right comodule) is $E_i = Ce_i$ and it consists of all paths starting at g_i. The family $\{E_i\}_{i\in\mathbb{Z}}$ is a representative set of injective indecomposable right comodules. Every proper subcomodule of E_i is spanned by the paths connecting the vertex g_i and some fixed vertex g_j with $j \geq i$. Therefore, the subcomodules of E_i are clearly totally order and thus C is right serial. Similarly, C is left serial (the injective hull of kg_i as a left C-comodule is $F_i = e_iC$ consisting of all paths ending at g_i).

3) Let $C = k\{c_0, c_1, c_2, ...\}$ be the divided power coalgebra. Its comultiplication and counit are given by:

$$\Delta(c_n) = \sum_{i=0}^{n} c_i \otimes c_{n-i}, \qquad \varepsilon(c_n) = \delta_{n,0},$$

for all $n \in \mathbb{N}$, where $\delta_{n,0}$ denotes the Kronecker's delta. This coalgebra is just the path coalgebra associated to the quiver consisting of a unique vertex with a loop. We see that it is uniserial. Since C is cocommutative, every right (resp. left) subcomodule of C is a subcoalgebra. It is known that $C_i = k\{c_0, c_1, ..., c_i\}$ are the only subcoalgebras of C. The only simple comodule is C_0 and $E(C_0) = C$. Hence $C_C = {}_CC$ is a uniserial comodule with composition series $\{0\} \subset C_0 \subset C_1 \subset ....$

Proposition 1.5 *Let C be a coalgebra and D a subcoalgebra of C. If C is right serial (resp. uniserial), then D is right serial (resp. uniserial).*

Proof: Let D be a subcoalgebra of C and S a simple right D-comodule. Let $i : S \to E_D(S), j : S \to E_C(S)$ be the canonical embeddings. There is a C-comodule map $g : E_D(S) \to E_C(S)$ such that $gi = j$. Since i is essential and j injective, g is injective. Hence $E_D(S)$ is uniserial. For the uniserial case, notice that $soc^n(E_D(S))/soc^{n-1}(E_D(S))$ is a subcomodule of $soc^n(E_C(S))/soc^{n-1}(E_C(S))$. ∎

Proposition 1.6 *A finite dimensional coalgebra C is right serial (resp. uniserial) if and only if its dual algebra C^* is right serial (resp. uniserial).*

Proof: This fact follows from the duality between the categories of finite dimensional right C-comodules and finite dimensional right C^*-modules. It is defined as follows: For a finite dimensional right C-comodule N, its dual space N^* is a right C^*-comodule via the map $\cdot : N^* \to C^* \otimes N^*$ given by $\langle n^* \cdot c^*, n\rangle = \sum_{(n)} \langle n^*, n_{(0)}\rangle \langle c^*, n_{(1)}\rangle$ for all $n^* \in N^*, c^* \in C^*$ and $n \in N$. Moreover, the lattice of C-subcomodules of N is isomorphic to the lattice of C^*-submodules of N^* via the orthogonal space $(-)^{\perp(N^*)}$. It is proved in [14, 1.3a] and [16, Lemma 15] that S is a simple C-comodule if and only if S^* is a simple C^*-module, and I is the injective hull of S if and only if I^* is the projective cover of S^*, see [14, 1.3a] and [16, Lemma 15] for further details. ∎

Given a right C-comodule M, let $cf(M)$ denote the coefficients space of M, see [14, page 142]. It is known that $cf(M)$ is the smallest subcoalgebra of C such that the structure map $\rho_M : M \to M \otimes C$ factorises throughout $M \otimes cf(M)$, making M a right $cf(M)$-comodule. Note that a vector subspace N of M is a C-subcomodule of M if and only if it is a $cf(M)$-subcomodule of M. Then, the lattice of C-subcomodules of M coincides with the lattice of $cf(M)$-subcomodules of M. Hence M is uniserial as C-comodule if and only if it is uniserial as $cf(M)$-comodule. Finally, recall that if M is finite-dimensional, then so is $cf(M)$.

Proposition 1.7 *A coalgebra C is right serial if and only if C_1 is right serial.*

Proof: Assume that C_1 is right serial and let $E = E_C(S)$ be the injective hull of a simple right C-comodule S. Let N be a finite dimensional subcomodule of E. By Lemma 1.2, it suffices to prove that N is uniserial. Consider $D = cf(N)$ and J the Jacobson radical of D^*. We know that $J = D_0^{\perp(D^*)}$ and, since D is finite dimensional, $J^2 = D_1^{\perp(D^*)}$. By hypothesis and Proposition 1.5, D_1 is right serial. Proposition 1.6 yields that $D_1^* \cong D^*/J^2$ is a right serial algebra. By [9, Corollary 10.2.1], D^* is right serial and thus D is right serial. Since $soc(N) = S$, the subcomodule N embeds in $E_D(S)$. Hence N is uniserial as a D-comodule and so uniserial as a C-comodule. ∎

Theorem 1.8 *The following properties about a coalgebra C are equivalent:*

i) C is serial.

ii) Every finite dimensional right C-comodule is a direct sum of uniserial comodules.

iii) Every finite dimensional left C-comodule is a direct sum of uniserial comodules.

iv) Every finite dimensional indecomposable right C-comodule is uniserial.

v) Every finite dimensional indecomposable left C-comodule is uniserial.

vi) Every finite dimensional subcoalgebra of C is serial.

vii) C_1 is serial.

Proof: *i)* $\Rightarrow$ *ii)* Let M be a finite dimensional right C-comodule and $D = cf(M)$. Since D is a finite dimensional subcoalgebra of C, Propositions 1.5, 1.6 show that D^* is serial. In view of the isomorphism of categories $\mathcal{M}^D \cong {}_{D^*}\mathcal{M}$ and the Nakayama-Skorniakov Theorem ([9, Theorem 10.1.1]), N is a direct sum of uniserial D-comodules. But a uniserial D-comodule is uniserial as a C-comodule. Note that the same argument is valid to prove $vi) \Rightarrow ii)$.

ii) $\Rightarrow$ *iv)* Obvious.

iv) $\Rightarrow$ *i)* Let S be a simple right comodule and $E = E_C(S)$ its injective hull. It suffices to check that for any $m, n \in E$, either $(n) \subset (m)$ or $(m) \subset (n)$. This follows by comparing $(n), (m)$ in (n, m), which is finite dimensional and indecomposable.

i) $\Rightarrow$ *vi)* Straightforward from Proposition 1.5.

i) $\Leftrightarrow$ $vii)$ It follows from Proposition 1.7.

i) $\Leftrightarrow$ *iii)* $\Leftrightarrow$ *v)* is symmetric to i) $\Leftrightarrow$ ii) $\Leftrightarrow$ iv) by the Nakayama-Skorniakov Theorem for finite dimensional algebras, [9, Theorem 10.1.11]. ∎

We recall from [12] that a coalgebra C is called *left quasi-coFrobenius* if C as a left C^*-module embeds in a free left C^*-module. The coalgebra C is called *quasi-coFrobenius* if it is left and right quasi-coFrobenius. It has been shown in [12, Remarks 1.5 (a)] that a finite dimensional coalgebra C is quasi-coFrobenius if and only if the dual algebra C^* is quasi-Frobenius.

Theorem 1.9 *The following assertions about a coalgebra C are equivalent:*

i) *C is uniserial.*

ii) *Every finite dimensional right C-comodule is a direct sum of homogeneous uniserial comodules.*

iii) *Every finite dimensional left C-comodule is a direct sum of homogeneous uniserial comodules.*

iv) *Every finite dimensional indecomposable right C-comodule is homogeneous uniserial.*

v) *Every finite dimensional indecomposable left C-comodule is homogeneous uniserial.*

vi) *Every finite dimensional subcoalgebra of C is quasi-coFrobenius.*

vii) *C_1 is uniserial.*

Proof: Keeping in mind the characterization of uniserial algebras given in [9, Theorem 9.4.1, Corollary 9.4.5] and [11, Theorem 2.1], one may adapt the proof of Theorem 1.8 to uniserial coalgebras. ∎

Corollary 1.10 *Let C be a coalgebra and $\{C_i\}_{i\in I}$ a family of subcoalgebras such that $C = \oplus_{i\in I} C_i$. Then, C is serial (resp. uniserial) if and only if C_i is serial (resp. uniserial) for all $i \in I$.*

Proof: Assume that C_i is serial for all $i \in I$, and let M be a finite dimensional C-comodule. Then $M = \oplus_{j=1}^{n} M_{i_j}$ with $M_{i_j} = \rho_M^{-1}(M \square_C C_{i_j})$ for some $i_1, ..., i_n \in I$. Since M_{i_j} is a C_{i_j}-comodule and C_{i_j} is serial, Theorem 1.8 implies that M_{i_j} is a direct sum of uniserial C_{i_j}-comodules. Hence M is a direct sum of uniserial C-comodules. Theorem 1.8 implies that C is serial. The converse is deduced from Proposition 1.5.

For uniserial coalgebras the argument is analogous using Theorem 1.9. ∎

We recall that the *finite dual coalgebra* of an algebra A is $A^0 = \{f \in A^* : f(I) = 0$ for some ideal I of A with $dim(A/I) < \infty\}$.

Corollary 1.11 *Let A be an algebra and assume that every finite dimensional quotient algebra of A is serial. Then A^0 is serial. As a consequence:*

i) If A is serial, then A^0 is serial.

ii) If A is an hereditary noetherian prime algebra, then A^0 is serial. In particular, the finite dual of a Dedekind domain is serial.

Proof: Every finite dimensional subcoalgebra of A^0 is of the form $(A/I)^*$ where I is a cofinite two-sided ideal of A. Combining the hypothesis and Proposition 1.6 we have that every finite dimensional subcoalgebra of A^0 is serial. Theorem 1.8 now applies.

By [4, Theorem 32.2] every quotient algebra of a serial algebra is serial. This proves i). The assertion ii) follows from [10, Corollary 3.2] which claims that every proper quotient algebra of an hereditary noetherian prime algebra is serial. Dedekind domains are examples of such algebras. ∎

Remark 1.12 One of the most important results in the theory of serial rings is that for a left artinian serial ring every left and every right module decomposes as a direct sum of uniserial modules, see [4, Theorem 32.3]. We do not know whether this result holds for serial coalgebras. Although it is true for coalgebras having a finite coradical filtration, see Proposition 1.13. Note that this hypothesis assures that every comodule has finite Loewy length. In the artinian ring case this is a consequence of the nilpotency of the Jacobson radical.

Proposition 1.13 *Let C be a serial coalgebra and M a right C-comodule. Then $soc^n(M)$ is a direct sum of uniserial comodules for all $n \in \mathbb{N}$. In particular, if $C = C_n$ for some $n \in \mathbb{N}$, every right C-comodule is a direct sum of uniserial comodules.*

Proof: Let M be a right C-comodule such that $M = soc^{n+1}(M)$. Then $M = \rho_M^{-1}(M \square_C C_n)$ $= soc^{n+1}(M)$ and thus M is a C_n-comodule. We prove the result by induction on the length of the Loewy series $\{soc^i(M)\}_{i=1}^{n+1}$. It is clear that $soc(M)$ is a direct sum of uniserial comodules. Assume that $soc^n(M)$ is a direct sum of uniserial comodules. Let $M \neq soc^n(M)$, then there is $m \in soc^{n+1}(M) - soc^n(M)$. Let (m) be the subcomodule generated by m. By Theorem

1.8, $(m) = \oplus_{j=1}^{s} U_j$ where each U_j is uniserial. We claim that for some $j \in \{1, 2, ..., s\}$ it holds that $U_j \neq soc^n(U_j)$. If $U_j = soc^n(U_j)$ for all $j = 1, ..., n$, then

$$(m) = \oplus_{j=1}^{s} U_j = \oplus_{j=1}^{s} soc^n(U_j) = soc^n(\oplus_{j=1}^{s} U_j) = soc^n((m)) \subset soc^n(M).$$

From this, $m \in soc^n(M)$, a contradiction. Since U_j is uniserial and C is serial, $E_C(U_j)$ is uniserial. There is $h \in I\!N$ such that $U_j \cong E_{C_h}(U_j)$. It must be $h = n$ because U_j has length $n+1$. By Zorn's Lemma there exists a maximal independent family $\mathcal{U}$ of uniserial C_n-subcomodules of M of length $n+1$. Let $U = \oplus_{X \in \mathcal{U}} X$. Since U is injective as C_n-comodule, we can express $M = U \oplus L$ where U is a direct sum of uniserial comodules and L does not contain uniserial subcomodules of length $n+1$. We claim that $L = soc^n(L)$. If there is $y \in soc^{n+1}(L) - soc^n(L)$, then reasoning as in the above paragraph we would find a uniserial subcomodule of L of length $n+1$. Applying the hypothesis to L we are done. ∎

We recall from [16] that a coalgebra is called *right semiperfect* if the injective hull of a right simple comodule is finite dimensional.

Proposition 1.14 *Let C be a serial coalgebra. Every subcoalgebra with finite coradical filtration is semiperfect (both sides). In particular, if C has finite coradical filtration, C is semiperfect.*

Proof: Let S be a simple C-comodule and $E = E_C(S)$. It is not difficult to see that $E_{C_n}(S) = \rho_M^{-1}(E \square_C C_n) = soc^{n+1}(E_C(S))$. By Lemma 1.2 we get that $E_{C_n}(S)$ is finite dimensional for all $n \in I\!N$. ∎

2 Structure theorems for serial coalgebras

Let C be a coalgebra and let $\{S_i\}_{i \in I}$ be a full set of simple right C-comodules. The set $\{E(S_i)\}_{i \in I}$ is a representative family of injective indecomposable right C-comodules. For each S_i there is a primitive idempotent $\overline{e_i}$ in C_0^* such that $S_i \cong C_0\overline{e_i}$. Each idempotent $\overline{e_i}$ may be lifted to a primitive idempotent $e_i \in C^*$ such that $E(S_i) \cong Ce_i$, see [7] for more details. The family $\{e_i\}_{i \in I}$ is called a *basic set of idempotents* for C. For a fixed e_i the right C_0-comodule C_1e_i/C_0e_i is cosemisimple, then

$$C_1e_i/C_0e_i \cong \oplus_{j \in I} C_0 e_j^{(m_{ij})},$$

where the $m_{ij}'s$ are cardinal numbers. The *right diagram* of C is defined as the directed graph $\mathcal{D}(C)$ with vertex set $\{e_i\}_{i \in I}$ and with m_{ij} arrows between e_i and e_j. From the definition it is clear that $\mathcal{D}(C) = \mathcal{D}(C_1)$.

Proposition 2.1 *A coalgebra C is right serial if and only if there is at most one arrow starting at each vertex in $\mathcal{D}(C)$.*

Proof: Assume that C is right serial and let e_i be a vertex of $\mathcal{D}(C)$. Since Ce_i is uniserial, C_1e_i/C_0e_i is simple or zero. If it is non zero, then $C_1e_i/C_0e_i \cong C_0e_j$ for some e_j. Thus there is an arrow from e_i to e_j. There is only one because $\{e_i\}_{i\in I}$ is basic. Conversely, given e_i such that C_1e_i is not simple, there is a unique e_j such that $C_1e_i/C_0e_i \cong C_0e_j$ by hypothesis. Hence C_1e_i is uniserial which gives that C_1 is right serial. That C is right serial follows from Proposition 1.7. ∎

We now compare the right diagram of C with the right Ext-quiver of C. We recall from [17] that the *right Ext-quiver* of C is the diagram $\Gamma(C)$ whose vertices are the simples $\{S_i\}_{i\in I}$ in $\mathcal{M}^C$ and for $i,j \in I$ there is an arrow $S_i \to S_j$ if there is an indecomposable C-comodule P and an exact sequence $0 \to S_i \to P \to S_j \to 0$.

Two simple right C-comodules S_i and S_j are said to be *connected* if there is a path in $\Gamma(C)$ (as an undirected quiver) from S_i to S_j. It was proved in [17] that there is a family $\{C_\alpha\}_{\alpha\in\Delta}$ of subcoalgebras of C such that $C = \oplus_{\alpha\in\Delta}C_\alpha$ where each C_α is associated to a connected component of $\Gamma(C)$.

Remark 2.2 When C is a right serial coalgebra then $\mathcal{D}(C)$ and $\Gamma(C)$ are isomorphic.

Let S_i and S_j be two simple right C-comodules and e_i, e_j two primitive idempotents of C^* such that $S_i \cong C_0e_i$ and $S_j \cong C_0e_j$. If there is an arrow from e_i to e_j in $\mathcal{D}(C)$, then it is clear that there is an arrow from S_i to S_j in $\Gamma(C)$.

Assume that there is an arrow from S_i to S_j in $\Gamma(C)$. Let P be the indecomposable C-comodule appearing in a short exact sequence with S_i and S_j as extremes. Then $S_i = soc(P)$. Taking $E_{C_1}(P) \cong C_1e_i$ we have that S_j embeds in $E_{C_1}(P)/soc(E_{C_1}(P))$. But this is simple by hypothesis. Hence $E_{C_1}(P)/soc(E_{C_1}(P)) \cong S_j$ and thus there is an arrow from e_i to e_j in $\mathcal{D}(C)$.

We recall from [20] the definition of the cotensor coalgebra and some related properties. Let C be a coalgebra and M a C-bicomodule. The *cotensor coalgebra* of M over C, denoted by $T^c_C(M)$ is defined as follows: as a vector space $T^c_C(M) = \oplus_{n\in\mathbb{N}}M^{\Box_C n}$ where $M^{\Box_C n}$ denotes the cotensor product of M with itself n times. When $n=0$, $M^{\Box_C n} = C$. The comultiplication $\Delta : M^{\Box_C n} \to \sum_{i+j=n} M^{\Box_C i} \otimes M^{\Box_C j}$ is given by the comodule structure map of M when $i=0$ or $j=0$, and when $i,j>0$ it is induced by the map $x_1 \otimes ... \otimes x_n \to \sum_i (x_1 \otimes ... \otimes x_i) \otimes (x_{i+1} \otimes ... \otimes x_n)$. The counit is $\varepsilon_C\pi$ where ε_C is the counit of C and π is the projection from $T^c_C(M)$ into C. The coradical filtration of $T = T^c_C(M)$ is given by $T_n = \oplus_{i=0}^n M^{\Box_C i}$. The path coalgebra associated to a quiver is a particular case of the cotensor coalgebra, see [6, Remark 4.2].

The space $P = C_1/C_0$ is a (C_0, C_0)-bicomodule via the maps

$$\rho^+_P : P \to P \otimes C_0,\ c + C_0 \mapsto \textstyle\sum_{(c)} (c_{(1)} + C_0) \otimes c_{(2)}$$
$$\rho^-_P : P \to C_0 \otimes P,\ c + C_0 \mapsto \textstyle\sum_{(c)} c_{(1)} \otimes (c_{(2)} + C_0)$$

When the coradical of C is *coseparable* (the dual of each simple subcoalgebra is separable over k), C is isomorphic to a subcoalgebra of $T^c_C(P)$, see [24, 4.6]. We say that a C_0-bicomodule M is *right serial* if Me is simple or zero for any primitive idempotent $e \in C_0^*$.

Proposition 2.3 *Let C be a coalgebra and $P = C_1/C_0$.*

i) C is right (resp. left) serial if and only if P is right (resp. left) serial.

ii) If C is right serial and C_0 is coseparable, then C is isomorphic to a subcoalgebra of $T^c_{C_0}(P)$ where P is a right serial C_0-bicomodule.

iii) If M is a right serial C_0-bicomodule, then any subcoalgebra of $T^c_{C_0}(M)$ is right serial.

Proof: *i)* Recall from [7, Proposition 1.17] that an idempotent $e \in C^*$ is primitive if and only if $\overline{e} = e\mid_{C_0}$ is primitive. If C is right serial, then Ce is uniserial for any primitive idempotent $e \in C^*$. Hence $soc^2(Ce)/soc(Ce) = C_1e/C_0e \cong (C_1/C_0)e$ is simple. Conversely, assume that Pe is simple or zero for any primitive idempotent $e \in C_0^*$. By lifting e to an idempotent $e' \in C_1^*$ we have that C_1e' is uniserial. Hence C_1 is right serial, and from Proposition 1.7, C is right serial.

ii) It follows from *i*).

iii) Let $T = T^c_{C_0}(M)$. Since $T_1/T_0 \cong M$ as T_0-comodules, T is right serial. Now it is just to apply Lemma 1.5. ∎

In view of the foregoing result, in order to characterize right serial coalgebras, it suffices to find the subcoalgebras of $T = T^c_C(P)$ where C is a cosemisimple and coseparable coalgebra and P is a C-bicomodule that is serial as a right C-comodule. We may assume that C is basic. Then $C = \oplus_{i\in I} C_i$ where C_i^* is a division algebra. For each $i \in I$ let e_i be the primitive and central idempotent such that $C_i = Ce_i$. We are able to characterize the right semiperfect subcoalgebras of $T = T^c_C(P)$. Suppose that D is a right semiperfect subcoalgebra of T. Then, $D_D = \oplus_{j\in J} E_D(C_j)$. Since $E_D(C_j) \subset Te_j$ and Te_j is uniserial, $E_D(C_j) = T_{l_j}e_j$ for some $l_j \in I\!N$. Here T_{l_j} denotes the l_j-th term of the coradical filtration of T. Hence any right semiperfect subcoalgebra of T is of the form $D = \oplus_{j\in J} D_j$ where $D_j = T_{l_j}e_j$ for some set of index $J \subset I$.

Given a family $\{l_j\}_{j\in J}$ of natural numbers we study when $D = \oplus_{j\in J} T_{l_j}e_j$ is a subcoalgebra of T. We need some previous facts.

Lemma 2.4 *Let C be a coalgebra, $e \in C^*$ a central idempotent, and M a left C-comodule.*

i) For any $c \in C$, $\sum_{(c)}\langle e, c_{(1)}\rangle c_{(2)} = \sum_{(c)}\langle e, c_{(2)}\rangle c_{(1)}$.

ii) $Ce\square_C M = Ce\square_C Me$.

iii) If N is a C-bicomodule, then $(N\square_C M)e = Ne\square_C M$.

Proof: *i)* Let $\lambda_C : C \to C^{**}$ be the canonical embedding defined as $\langle \lambda_C(c), c^*\rangle = \langle c^*, c\rangle$ for all $c \in C, c^* \in C^*$. Since e is central, $ec^* = c^*e$ for all $c^* \in C^*$. Then,

$$\begin{aligned}\langle \lambda_C(\textstyle\sum_{(c)}\langle e, c_{(1)}\rangle c_{(2)}), c^*\rangle &= \textstyle\sum_{(c)}\langle e, c_{(1)}\rangle\langle c^*, c_{(2)}\rangle \\ &= \textstyle\sum_{(c)}\langle e, c_{(2)}\rangle\langle c^*, c_{(1)}\rangle \\ &= \langle \lambda_C(\textstyle\sum_{(c)}\langle e, c_{(2)}\rangle c_{(1)}), c^*\rangle.\end{aligned}$$

The injectivity of λ_C yields the claim.

ii) Let $x = \sum_{l=1}^{n} c_l e \otimes m_l \in Ce\Box_C M$. By definition of the cotensor product,

$$\sum_{l=1}^{n}\sum_{(c_l)} c_{l(1)}e \otimes c_{l(2)} \otimes m_l = \sum_{l=1}^{n}\sum_{(m_l)} c_l e \otimes m_{l(-1)} \otimes m_{l(0)}. \tag{1}$$

Now,

$$\begin{aligned} x = \textstyle\sum_{l=1}^{n} c_l e \otimes m_l &= \textstyle\sum_{l=1}^{n}\sum_{(c_l)} \langle e, c_{l(2)}\rangle c_{l(1)} \otimes m_l \\ &= \textstyle\sum_{l=1}^{n}\sum_{(c_l)} \langle e, c_{l(1)}\rangle c_{l(2)} \otimes m_l \quad \textit{since } e \textit{ is central} \\ &= \textstyle\sum_{l=1}^{n}\sum_{(c_l)} \langle e, c_{l(1)}\rangle c_{l(2)} \langle e, c_{l(3)}\rangle \otimes m_l \\ &= \textstyle\sum_{l=1}^{n}\sum_{(c_l)} c_{l(1)}e \otimes \langle e, c_{l(2)}\rangle m_l \\ &= \textstyle\sum_{l=1}^{n}\sum_{(m_l)} c_l e \otimes \langle e, m_{l(-1)}\rangle m_{l(0)} \quad \textit{by applying } 1\otimes e\otimes 1 \textit{ to } (1) \\ &= \textstyle\sum_{l=1}^{n} c_l e \otimes m_l e. \end{aligned}$$

iii) Let $x = (\sum_{l=1}^{n} n_l \otimes m_l)e \in (N\Box_C M)e$. Then,

$$x = \sum_{l=1}^{n} \langle e, (n_l \otimes m_l)_{(-1)}\rangle (n_l \otimes m_l)_{(0)} = \sum_{l=1}^{n} \langle e, n_{l(-1)}\rangle n_{l(0)} \otimes m_l = \sum_{l=1}^{n} n_l e \otimes m_l. \quad \blacksquare$$

For C a cosemisimple coalgebra and P a right serial (C, C)-bicomodule one can construct a diagram $\mathcal{D}(C, P)$ associated to C and P in a similar manner to $\mathcal{D}(C)$. Indeed, if C is right serial, then $\mathcal{D}(C) \cong \mathcal{D}(C_0, P)$ where $P = C_1/C_0$. A vertex is called a *sink* when it is not the tail of any arrow.

Theorem 2.5 *Let C be a basic cosemisimple and coseparable coalgebra and $\{e_i\}_{i\in I}$ be a family of primitive central idempotents of C^* such that $\varepsilon_C = \sum_{i\in I} e_i$. Let P be a right serial C-bicomodule and $T = T_C^c(P)$. Finally, let $\{l_i\}_{i\in I}$ be a family of natural numbers, $D_i = T_{l_i} e_i$ and $D = \oplus_{i\in I} D_i$. Then, D is a subcoalgebra of T if and only if $l_i = 0$ for every sink e_i in $\mathcal{D}(C, P)$ and $l_i \leq l_j + 1$ for every arrow from e_i to e_j in $\mathcal{D}(C, P)$.*

Proof: If $e_{i_1} \to e_{i_2} \to \ldots \to e_{i_n}$ is a path in $\mathcal{D}(C, P)$, then $Pe_{i_s} \cong Ce_{i_s+1}$ for all $s = 1, \ldots, n-1$. Using the above lemma,

$$(P\Box_C P)e_{i_1} = Pe_{i_1}\Box_C P \cong Ce_{i_2}\Box_C P = Ce_{i_2}\Box_C Pe_{i_2} \cong Pe_{i_1}\Box_C Pe_{i_2}.$$

In general, $P^{\Box_C h} e_{i_1} \cong Pe_{i_1}\Box_C Pe_{i_2}\Box_C \ldots \Box_C Pe_{i_h}$ for $h \leq n$. We identify these two spaces. Then,

$$\begin{aligned} \Delta(P^{\Box_C h} e_i) &\subseteq \textstyle\sum_{a+b=h} P^{\Box_C a} e_i \Box_C P^{\Box_C b} = \sum_{a+b=h} (P^{\Box_C a}\Box_C P^{\Box_C b}) e_i \\ &= Ce_i \Box_C P^{\Box_C h} e_i + Pe_i \Box_C P^{\Box_C (h-1)} e_{i_1} + P^{\Box_C 2} e_i \Box_C P^{\Box_C (h-2)} e_{i_2} + \ldots + \\ &\quad P^{\Box_C h-1} e_i \Box_C Pe_{i_{h-1}} + P^{\Box_C h} e_i \Box_C C \qquad (*) \end{aligned}$$

$\Leftarrow$) Assume that i is a sink and $l_i = 0$. Then $D_i = Ce_i$ and it follows that $\Delta(D_i) \subset D_i \otimes D_i$. If there is an arrow $e_i \to e_j$ and $l_i \le l_j + 1$, then $T_{l_i - 1} e_j \subset T_{l_j} e_j$. From $(*)$ we get for $h \le l_i$,

$$\begin{aligned} \Delta(P^{\square_C h} e_i) & \subset Ce_i \otimes T_h e_i + T_1 e_i \otimes T_{h-1} e_{s_1} + ... + T_{h-1} e_i \otimes T_1 e_{s_h - 1} + T_h e_i \otimes C \\ & \subset Ce_i \otimes D_i + D_i \otimes D_{s_1} + D_i \otimes D_{s_2} + ... + D_i \otimes D_{s_{h-1}} + D_i \otimes C \\ & \subset D \otimes D \end{aligned}$$

where $e_i \to e_{s_1} \to ... \to e_{s_h}$. This gives $\Delta(D_i) \subseteq D \otimes D$ and so D is a subcoalgebra of T.

$\Rightarrow$) Suppose that D is a subcoalgebra of T and let e_i be a sink in $\mathcal{D}(C, P)$. Since P is right serial, $Pe_i = \{0\}$. We thus get $T_{l_i} e_i = Ce_i$, and hence $l_i = 0$. Let $e_i \to e_j$ be an arrow in $\mathcal{D}(C, P)$. Since D is a subcoalgebra, from $(*)$ it follows that $l_i - 1 \le l_j$. ∎

Definition 2.6 *Let C be a coalgebra and $\mathcal{D}(C)$ its diagram. An admissible sequence for $\mathcal{D}(C)$ is a map $l : \mathcal{D}(C) \to \mathbb{N}$, $i \mapsto l_i$ such that $l_i = 0$ if e_i is a sink and $l_i \le l_j + 1$ if there is an arrow from e_i to e_j.*

Corollary 2.7 *Let C be a right serial and right semiperfect basic coalgebra with coseparable coradical. Then C is determined by the following data: C_0, C_1/C_0, and an admissible sequence for $\mathcal{D}(C)$.*

Proposition 2.8 *Let $D = (D_0, P, \{l_i\}_{i \in I})$ and $D' = (D'_0, P', \{l'_i\}_{i \in I'})$ be two right serial and right semiperfect basic coalgebras. The coalgebras D and D' are isomorphic if and only if there is a coalgebra isomorphism $\phi : D_0 \to D'_0$ and a bicomodule isomorphism $\overline{\phi} : P \to P'$ verifying that $l_i = l'_{\theta(i)}$ where θ is the diagram isomorphism induced by ϕ and $\overline{\phi}$.*

Proof: Assume that $\phi : D \to D'$ is a coalgebra isomorphism, then ϕ induces a coalgebra isomorphism $\phi_0 : D_0 \to D'_0$ and a bicomodule isomorphism $\overline{\phi} : D_1/D_0 \to D'_1/D'_0$. Consider the dual map $\phi^* : D'^* \to D^*$. If $\{e'_i\}_{i \in I'}$ is a basic set of idempotents for D', then $\{\phi^*(e'_i)\}_{i \in I'}$ is a basic set of idempotents for D. Moreover, for $e_i = \phi^*(e'_i)$, $\phi(De_i) = D'e'_i$. Thus ϕ also induces a diagram isomorphism $\theta : \mathcal{D}(D) \to \mathcal{D}(D')$. Since l_i is the length of De_i, it holds that $l_i = l'_{\theta(i)}$.

Conversely, a coalgebra isomorphism $\varphi : D_0 \to D'_0$ and a bicomodule isomorphism $f : P \to P'$ induces a coalgebra isomorphism $\phi : T^c_{D_0}(P) \to T^c_{D'_0}(P')$. From the hypothesis it follows that $\phi(D) = D'$. ∎

Theorem 2.9 *Let C be an indecomposable coalgebra and $C_0 = \oplus_{i \in I} M^c(D_i, n_i)$ be a decomposition of its coradical with $\{D_i^*\}_{i \in I}$ a family of division algebras.*

i) If I is finite, then C is serial if and only if $\mathcal{D}(C)$ is a cycle or a finite chain and $D_i \cong D_j$ for any vertex e_i, e_j in $\mathcal{D}(C)$.

ii) If I is infinite, then C is serial if and only if I is contable, $\mathcal{D}(C)$ is a chain, and $D_i \cong D_j$ for any pair of vertex e_i, e_j in $\mathcal{D}(C)$.

Proof: This proof is inspired in [9, Theorem 10.3.1] and the methods used there for finite dimensional algebras may be extended for infinite dimensional coalgebras as we next show. We can assume that C is basic since the notion of serial coalgebra is invariant under Morita-Takeuchi equivalences, and in this case $\mathcal{D}(C)$ is also so.

Since C is basic, $\varepsilon = \sum_{i\in I} e_i$ where $\{e_i\}_{i\in I}$ is a basic set of idempotents for C. The idempotents $\overline{e_i} = e_i \mid_{C_0}$ are central and $C_0\overline{e_i} = D_i$, see [7, Theorem 3.10, Corollary 3.12]. By hypothesis and Proposition 2.3 i), Pe_i and e_iP are simple or zero right and left C_0-comodules respectively, where $P = C_1/C_0$. We claim that two different arrows in $\mathcal{D}(C)$ may not have the same head. Let e_h, e_i, e_j be vertex of $\mathcal{D}(C)$ and $e_i \to e_j, e_h \to e_j$ be arrows. Then $e_jPe_i \neq \{0\}$ and $e_jPe_h \neq \{0\}$. These would be two direct summands of the simple left C-comodule e_jP yielding a contradiction. Combining this fact and Lemma 2.1 we know that there is at most one arrow starting at each vertex and a vertex can not be the head of two arrows.

If I is finite, then $\mathcal{D}(C)$ is a cycle or a chain. Assume that I is infinite. Since C is serial, the Ext-quiver coincides with $\mathcal{D}(C)$. As C is indecomposable, $\mathcal{D}(C)$ is connected. Fix a vertex i_0. We know that for any vertex i there is a path from i to i_0 or from i_0 to i. Let n be the length a such path. In the first case we label i with $-n$ and in the second one i with n. Thus we have a map from I to $\mathbb{Z}$. This map is injective in view of the above observation. Hence I is contable and $\mathcal{D}(C)$ has to be a chain.

Let $e_j \to e_i$ be an arrow in $\mathcal{D}(C)$. Then $P_{ij} = e_iPe_j \neq \{0\}$ and $e_iP = P_{ij} = Pe_j$. As a right (resp. left) C_0-comodule, P_{ij} is isomorphic to D_i (resp. D_j). Hence P_{ij} is a (D_j, D_i)-bicomodule. By the universal property of the co-endomorphism coalgebra, there is a coalgebra map $\lambda : e_{-C_0}(Pe_j) \to D_j$. Since $Pe_j \cong C_0e_i$ as right C_0-comodules, $e_{-C_0}(Pe_j) \cong e_{-C_0}(C_0e_i) \cong D_i$. Hence we have a coalgebra map $\lambda : D_i \to D_j$. It is an isomorphism because D_i^*, D_j^* are division algebras verifying $dim(D_i) = dim(P_{ij}) = dim(D_j)$.

Conversely, assume that $\mathcal{D}(C)$ is a cycle or a chain and $D_i \cong D_j$ for any pair of vertex e_i, e_j in $\mathcal{D}(C)$. From Lemma 2.1, C is right serial. If e_i is a sink, then $e_iP = \{0\}$. Let $e_j \to e_i$ be an arrow, then $Pe_j \cong C_0e_i \cong D_i$ as right C_0-comodules. On the other hand, since P is semisimple as a left C_0-comodule $P \cong \oplus_{l\in I} D_l^{(T_l)}$. Then $e_iP \cong D_i^{(T_i)}$ as left C_0-comodules. Thus $e_iP = e_iPe_j \cong D_i^{(T_i)}$ as left C_0-comodules. Now, $e_iP \cong \oplus_{l\in I} D_l^{(S_l)}$ as a right C_0-comodules. Hence $e_iPe_j = D_j^{(S_j)}$. The isomorphism $Pe_j \cong D_i \cong D_j$ forces $S_j = 1$. Summarizing, we have that $D_i^{(T_i)} \cong D_i$ as vector spaces. So $T_i = 1$ and e_iP is simple as a left C_0-comodule. From Proposition 2.3, C is left serial. ∎

Theorem 2.10 *Let C be an indecomposable serial coalgebra over an algebraically closed field.*

i) C is Morita-Takeuchi equivalent to a subcoalgebra B of $k\Gamma$ where Γ is one of the following quivers:

a) A cycle or a finite chain.

b) $\mathcal{A}_\infty$: $\quad 0 \to 1 \to 2 \to 3 \to ...$

c) $\mathcal{A}^\infty$: $\quad ... \to -3 \to -2 \to -1 \to 0.$

d) $\mathcal{A}_\infty^\infty$: $\quad ... \to -3 \to -2 \to -1 \to 0 \to 1 \to 2 \to 3 \to ...$

ii) If C is right semiperfect, then B is determined by an admissible sequence associated to Γ.

iii) If C is hereditary, then $B \cong k\Gamma$.

Proof: *i)* Let B be the basic coalgebra associated to C and Γ the Ext-quiver of C. By [6, Theorem 4.3], B may be taken as a subcoalgebra of $k\Gamma$ containing to $(k\Gamma)_1$. Since C is serial, $\Gamma = \mathcal{D}(C)$, and from Theorem 2.9, Γ is one of the listed quivers.

ii) Since k is algebraically closed, B_0 is pointed, [6, Corollary 2.5]. Thus each simple subcoalgebra is given by a group-like element, and $V = D_1/D_0$ consists of the non trivial skew-primitive elements. Then $T^c_{B_0}(P) \cong k\Gamma$ ([6, Remark 4.2]), and from Theorem 2.5, B is determined by an admissible sequence.

iii) It follows from [5, Theorem 1] which claims that an hereditary basic coalgebra is isomorphic to the path coalgebra of its Ext-quiver. ∎

Remark 2.11 The form of the Ext-quiver in Theorem 2.10 may be deduced from [2]. It is claimed without proof in page 86 that the Ext-quiver of an uniserial connected category is a cycle or a chain. For a serial indecomposable coalgebra C its category of right comodules $\mathcal{M}^C$ is a uniserial connected category, Theorem 1.8.

Remark 2.12 If the coalgebra C is assumed to be pointed, then C is isomorphic to a subcoalgebra of $k\Gamma$ and the hypothesis of k being algebraically closed in Theorem 2.10 is not necessary.

Remark 2.13 In case $\mathcal{D}(C) = \mathcal{A}_\infty$, the coalgebra C is *left pure semisimple*, see [21, Corollary 2.5]. This means that any left C-comodule decomposes as a direct sum of finite dimensional comodules. Applying Theorem 1.8 to each summand, any left C-comodule is a direct sum of uniserial comodules. This partially answers the problem proposed in Remark 1.12. Similarly, if $\mathcal{D}(C) = \mathcal{A}^\infty$, then C is right pure semisimple and thus every right C-comodule is a direct sum of uniserial comodules.

We finish this section by giving a characterization of serial coalgebras in terms of its coradical being a coprincipal coideal.

Definition 2.14 *A right coideal I of a coalgebra C is said to be coprincipal if there is a right C-comodule map $f : C \to C$ such that $ker(f) = I$.*

Example 2.15 Let $C = k\{c_0, c_1, c_2, ...\}$ be the power divided coalgebra. Consider the map $f : C \to C, c_0 \mapsto 0, c_i \mapsto c_{i-1}$. It is easy to check that f is a C-comodule map. For $i \in I\!\!N$, let f^i denote the composition of f with itself i times. Then $ker(f^i) = C_i$ and so C_i is a coprincipal coideal.

Lemma 2.16 *Let I be a right coideal of C.*

i) I is coprincipal in C if and only if $I^{\perp(C^*)}$ is a principal ideal in C^*.

ii) If D is a subcoalgebra of C and I is coprincipal, then $I \cap D$ is a coprincipal coideal of D.

Proof: *i)* Assume that I is coprincipal and let $f : C \to C$ be a right C-comodule map such that $ker(f) = I$. The dual map $f^* : C^* \to C^*$ is a right C^*-module map and $Im(f^*) = Ker(f)^{\perp(C^*)} = I^{\perp(C^*)}$. Then $I^{\perp(C^*)} = \psi C^*$ where $\psi = f^*(\varepsilon)$. Conversely, suppose that $I^{\perp(C^*)} = \psi C^*$ for some $\psi \in C^*$. We define $f : C \to C, c \mapsto \sum_{(c)} \langle \psi, c_{(1)} \rangle c_{(2)}$. It is routine to check that f is a right C-comodule map and $ker(f) = I$.

ii) Let $i : D \to C$ be the inclusion map and consider $i^* : C^* \to D^*$ the dual map. We have that $i^*(I^{\perp(C^*)}) = i^{-1}(I)^{\perp(D^*)} = (I \cap D)^{\perp(D^*)}$. By hypothesis and the above item, $I^{\perp(C^*)}$ is a principal right ideal of C^*. Hence $(I \cap D)^{\perp(D^*)}$ is a principal right ideal of D^*, and from i) we conclude that $I \cap D$ is coprincipal in D. ∎

Theorem 2.17 *Let C be a basic coalgebra. Then, C is serial if and only if C_0 is a coprincipal right and a coprincipal left coideal.*

Proof: The socle of the right C-comodule $V = C/C_0$ is equal to C_1/C_0. Since C is serial, each simple comodule appears only once in V. This also happens in C because C is basic. Taking injective envelopes, we get that $E(V)$, and hence V, embeds in C as a right comodule. This implies that C_0 is a coprincipal right coideal. A symmetric argument proves that C_0 is a coprincipal left ideal.

Conversely, suppose that C_0 is a coprincipal right coideal of C. In light of the above lemma C_0 is a coprincipal coideal of C_1. Then C_1/C_0 embeds in C_1 as a right C_1-comodule. Since C is basic, each simple of C_1/C_0 appears with multiplicity one. Hence for each primitive idempotent $e \in C_0^*$, $(C_1/C_0)e$ is simple or zero. From Proposition 3.4 i), it follows that C is right serial. Analogously, C is left serial. ∎

Example 2.18 Let C be the path coalgebra associated to the quiver $\mathcal{A}_\infty^\infty$. For each vertex i let S_i^n be the path of length n starting at i. We define $f : C \to C$ by

$$f(S_i^0) = 0, \qquad f(S_i^n) = S_{i+1}^{n-1}.$$

It is not difficult to check that f is a bicomodule map whose kernel is C_0. Hence C_0 is a coprincipal right and left coideal of C. Note that under the canonical isomorphism $Com_{-C}(C, C) \cong C^*$, $f \mapsto \varepsilon f = f^*(\varepsilon)$, the space of bicomodule maps $Com_{C-C}(C, C) \cong Z(C^*)$, the center of C^*. Thus $\phi = f^*(\varepsilon) \in Z(C^*)$ and $C_0^{\perp(C^*)} = C^*\phi = \phi C^*$.

3 Uniserial coalgebras characterized

A coalgebra C is said to be *colocal* if C_0 is the dual of a finite dimensional division algebra.

Theorem 3.1 *Let C be a coalgebra. The following assertions are equivalent:*

i) C is uniserial.

ii) $C \cong \oplus_{\alpha\in\Delta} M^c(D_\alpha, n_\alpha)$ where D_α is a colocal serial coalgebra.

iii) Every right (left) coideal of C is coprincipal.

iv) The diagram of C consists of isolated points and loops.

Proof: *i) $\Rightarrow$ ii)* Since any coalgebra is a direct sum of indecomposable coalgebras, we may assume that C is indecomposable and uniserial. Suppose that $C = P^{(n)} \oplus P'$ where P is an indecomposable injective coideal and P' has no direct summands isomorphic to P. We prove that $P' = \{0\}$. If $P' \neq \{0\}$, then P' contains a simple coideal T such that T is not isomorphic to S, where $S = soc(P)$. As C is indecomposable, its Ext-quiver its connected. There are simple comodules $S_1, ..., S_n$ with $S_1 = S, S_n = T$ and indecomposable comodules $M_1, ..., M_{n-1}$ satisfying that $soc(M_i) \cong S_i$ and $M_i/S_i \cong S_{i+1}$. Hence $E(M_i)$ is indecomposable and, by hypothesis, homogeneous uniserial. Since $soc(E(M_i)/S_i) \cong S_i$, we get $S_{i+1} \cong S_i$. From this, $S \cong T$ which is a contradiction. Thus, $C = P^{(n)}$. Now, $C \cong e_{-C}(P^{(n)}) \cong M^c(e_{-C}(P), n)$ and $e_{-C}(P)$ is the basic coalgebra of C with S as unique simple coideal. We conclude that $e_{-C}(P)$ is colocal.

ii) $\Rightarrow$ iii) If J is a right coideal of C, then $J = \oplus_{\alpha\in\Delta} J_\alpha$ where J_α is a right coideal of $M^c(D_\alpha, n_\alpha)$. Since the direct sum of coprincipal coideals is coprincipal, we may assume that $C = M^c(D, n)$ where D is a colocal serial coalgebra. On the other hand, every right coideal of C is of the form $M^c(I, n)$ where I is a right coideal of D. Moreover, if I is coprincipal, then $M^c(I, n)$ is so. Hence, it suffices to prove that every right coideal of D is coprincipal. The only proper right coideals of D are of the form $D_n = soc^n(D)$ for $n \in \mathbb{N}$. Since D_0 is the only simple of D and D is serial, $soc(D/D_n) = D_{n+1}/D_n \cong D_0$. Then $E(D/D_n) \cong D$ and so there is an injective D-comodule map from D/D_n into D. This gives that D_n is coprincipal for all $n \in \mathbb{N}$.

iii) $\Rightarrow$ i) Let P be an injective indecomposable C-comodule. We can write $C = P^{(n)} \oplus P'$ where P' has no direct summands isomorphic to P. By hypothesis, $soc(P) \oplus \{0\}$ is a coprincipal coideal of C. Let $f : C \to C$ be the C-comodule map such that $ker(f) = soc(P) \oplus \{0\}$. Let $i : P \to C$ be the inclusion map and $\pi : C \to P$ the canonical projection. Consider $g = \pi f i$. We claim that $ker(g) = \{x \in P : f(x) \in P'\} = soc(P)$. Assume that there is $p \in P$ with $f((p, 0)) \neq 0$. Let I be a simple coideal of C such that $I \subset f(P)$ and $X = \{x \in P : f(p) \in I\}$.

On the other hand, f establishes an isomorphism between the set of simple coideals non isomorphic to $soc(P)$. There is a simple coideal J such that $f(J) = I$. Since $f(X) = f(J)$, we may pick $y \in J$ such that $f((p, 0)) = f((0, y))$. Then $(p, -y) \in soc(P) \oplus \{0\}$ yielding a contradiction. We conclude that $ker(g) = soc(P)$ and thus

$$soc^2(P)/soc(P) = soc(P/soc(P)) \cong soc(Im(g)) \cong soc(P).$$

Now $soc^2(P) \oplus \{0\}$ is a coprincipal coideal of C. Let $f : C \to C$ be a C-comodule map such that $ker(f) = soc^2(P) \oplus \{0\}$. Consider $g = \pi f i$. Arguing as before, it may be proved that $ker(g) = soc^2(P)$. Then

$$soc^3(P)/soc^2(P) = soc(P/soc^2(P)) \cong soc(Im(g)) \cong soc(P).$$

Continuing this process we see that the socle series is a composition series for P.

i) ⇒ iv) Note that for each primitive idempotent e_i, the comodule C_1e_i/C_0e_i is either zero or isomorphic to C_0e_i by hypothesis.

iv) ⇒ i) The assumption on $\mathcal{D}(C)$ implies that C_1 is uniserial. By Theorem 1.9, C is uniserial. ∎

Theorem 3.2 *Let C be a cocommutative coalgebra. The following assertions are equivalent:*

i) C is serial.

ii) The dual of each irreducible component of C is noetherian and its only ideals are the powers of the maximal ideal which is principal.

iii) C is uniserial.

When C is pointed and serial, each irreducible component is isomorphic to a subcoalgebra of the divided power coalgebra.

Proof: $i) \Rightarrow ii)$ Let $\{C_i\}_{i\in I}$ be a family of irreducible subcoalgebras such that $C = \oplus_{i\in I} C_i$. Since C_i is serial, $(C_i)_1/(C_i)_0$ is simple. Hence $(C_i)_1$ is finite dimensional. By [15, Theorem 5.2.1], C_i^* is noetherian. Any ideal I of C^* is finitely generated. From [15, Proposition 1.3.1] it must be of the form $I = D^{\perp(C^*)}$ for some subcoalgebra D of C. Since C_i is serial, $D = (C_i)_n$ for some $n \in I\!N$. Then $I = D^{\perp(C_i^*)} = (C_i)_n^{\perp(C_i^*)} = J^{n+1}$ where $J = (C_i)_0^{\perp(C_i^*)}$ is the only maximal ideal, [15, Corollary 4.1.2]. From Lemma 2.16 and Theorem 2.17, J is a principal ideal.

$ii) \Rightarrow iii)$ It follows from Theorem 3.1 and Corollary 1.10.

$iii) \Rightarrow i)$ Obvious.

If C_i is pointed and uniserial, its Ext-quiver Γ is an isolated point with a loop, Theorem 3.1. Hence $k\Gamma$ is isomorphic to the power divided coalgebra. Now Theorem 2.10 i) and Remark 2.12 apply. ∎

We give an application of this result to compute the finite dual of some Hopf algebras. It is known that a pointed cocommutative Hopf algebra H is isomorphic to the smash product $H_1 \# kG(H)$ where $G(H)$ is the set of group-like elements of H, H_1 is the irreducible component of H containing to 1, and $G(H)$ acts on H_1 by conjugation, see [18, Corollary 5.6.4]. If $char(k) = 0$, then $H_1 \cong U(P(H_1))$, the universal enveloping algebra of the primitive elements $P(H_1)$ of H_1, see [18, Theorem 5.6.5].

Let H be a commutative Hopf algebra such that H^0 is pointed and serial. By Theorem 3.2, $P(H_1^0)$ is one dimensional and hence $U(P(H_1^0)) \cong k[x]$. Then $H^0 \cong k[x]\#kG(H^0)$. If in addition H^0 is commutative, then $H^0 \cong k[x] \otimes kG(H^0)$.

Using this result we can give an alternative computation of $H = k[x]^0, k[x, x^{-1}]^0$ or $k[[x]]^0$ when k is algebraically closed and of characteristic zero. The group-like elements of H are the algebra maps from H to k. By Corollary 1.11 ii), H^0 is serial. Then $k[x]^0 \cong k[x] \otimes k(k, +)$ where $(k, +)$ is the additive group of k; $k[x, x^{-1}]^0 \cong k[x] \otimes k(k^*, \cdot)$, where $(k^*, \cdot)$ is the multiplicative group of k; and $k[[x]]^0 \cong k[x]$.

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