The Brauer group of some quasitriangular Hopf algebras

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Abstract

We show that the Brauer group $BM(k, H_\nu, R_{s,\beta})$ of the quasitriangular Hopf algebra $(H_\nu, R_{s,\beta})$ is a direct product of the additive group of the field k and the classical Brauer group $B_{\theta_s}(k, \mathbf{Z}_{2\nu})$ associated to the bicharacter θ_s on $\mathbf{Z}_{2\nu}$ defined by $\theta_s(x, y) = \omega^{sxy}$, with ω a 2ν-th root of unity.

1 Introduction

Let k be a field and H be a Hopf algebra over k with bijective antipode. The Brauer group of H, denoted by $BQ(k, H)$, was introduced in [4] and later on

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studied in [5], [23], and [25]. This Brauer group is a special case of the Brauer group of a braided monoidal category introduced in [24]. In fact, $BQ(k,H)$ is the Brauer group of the category $\mathcal{YD}_H$ of Yetter-Drinfel'd modules over H. If (H,R) is a quasitriangular Hopf algebra the category of left H-modules ${}_H\mathcal{M}$ is a braided monoidal subcategory of $\mathcal{YD}_H$ and $Br({}_H\mathcal{M})$ is a subgroup of $BQ(k,H)$, denoted by $BM(k,H,R)$. Dually, if (H,r) is a coquasitriangular Hopf algebra, the category $\mathcal{M}^H$ of right H-comodules is a braided monoidal subcategory of $\mathcal{YD}_H$. The Brauer group $Br(\mathcal{M}^H)$ is a subgroup of $BQ(k,H)$, denoted by $BC(k,H,r)$. In this paper we compute BM and BC for all the quasitriangular structures (and coquasitriangular structures) of the family of Hopf algebras $H_\nu = \langle g,x : g^{2\nu}=1, x^2=0, gx+xg=0\rangle$ with ν an odd natural number, g a group-like element and x a $(g^\nu,1)$-primitive element. The antipode is defined by $S(g)=g^{-1},\ S(x)=g^\nu x$. This family of Hopf algebras was introduced by Radford in [20] and they are a generalization of Sweedler Hopf algebra H_4. The Hopf algebras H_ν have a family of quasitriangular structures

$$R_{s,\beta} = \frac{1}{2\nu}\left(\sum_{i,l=0}^{2\nu-1}\omega^{-il}g^i\otimes g^{sl}\right) + \frac{\beta}{2\nu}\left(\sum_{i,l=0}^{2\nu-1}\omega^{-il}g^ix\otimes g^{sl+\nu}x\right), \qquad (1.1)$$

where $\beta\in k$ and $1\le s\le 2\nu$ is odd. The Brauer group of Sweedler Hopf algebra H_4 and the quasitriangular structure R_0, in our notation $\nu=1,\ R_{1,0}$, was computed in [24]. It turns out to be a direct sum of the additive group of the field $(k,+)$ and the classical Brauer-Wall group of k. The Brauer group corresponding to $\nu=1$ and $t=\beta$ is isomorphic to the aforementioned, as it was shown in [7].

Since H_ν is self-dual each quasitriangular structure $R_{s,\beta}$ can be seen as a coquasitriangular structure $r_{s,\beta}$ and $BC(k,H_\nu,r_{s,\beta})\cong BM(k,H_\nu,R_{s,\beta})$. In order to compute $BM(k,H_\nu,R_{s,\beta})$ we first prove that $BM(k,H_\nu,R_{s,\beta})$ and $BM(k,H_\nu,R_{s,0})$ are isomorphic. This is achieved showing that $(H_\nu,R_{s,0})$ and $(H_\nu,R_{s,\beta})$ are twist-equivalent. By general theory, the categories of comodules for both quasitriangular pairs are then equivalent as braided monoidal categories. Then the corresponding Brauer groups are isomorphic. Hence we are reduced to computing the Brauer group $BM(k, H_\nu, R_{s,0})$. The quasitriangular structure $R_{s,0}$ is also a quasitriangular structure on $k\mathbf{Z}_{2\nu}$ and the inclusion map $i : (k\mathbf{Z}_{2\nu}, R_{s,0}) \to (H_\nu, R_{s,0})$ is a quasitriangular map. On the other hand, the projection map $p : (H_\nu, R_{s,0}) \to (k\mathbf{Z}_{2\nu}, R_{s,0})$ is also a quasitriangular map. Since H_ν is a Radford's biproduct by $k\mathbf{Z}_{2\nu}$, we have that $p \circ i = \mathrm{id}_{k\mathbf{Z}_{2\nu}}$. Thus the maps induced at the Brauer group level

$$BM(k, H_\nu, R_{s,0}) \underset{p^*}{\overset{i^*}{\rightleftarrows}} BM(k, k\mathbf{Z}_{2\nu}, R_{s,0})$$

satisfy $i^* \circ p^* = 1\mathrm{id}$. We prove that $Ker(i^*)$ is isomorphic to $(k, +)$ and that $BM(k, k\mathbf{Z}_{2\nu}, R_{s,0})$ commutes with it. Then $BM(k, H_\nu, R_{s,0}) \cong (k, +) \times BM(k, k\mathbf{Z}_{2\nu}, R_{s,0})$. So the computation of $BM(k, H_\nu, R_{s,0})$ reduces to the computation of $BM(k, k\mathbf{Z}_{2\nu}, R_{s,0})$. The quasitriangular structure $R_{s,0}$ on $k\mathbf{Z}_{2\nu}$ can be viewed as a bicharacter θ_s on $\mathbf{Z}_{2\nu}$ and the Brauer group BM of $k\mathbf{Z}_{2\nu}$ with respect to $R_{s,0}$ is just the classical Brauer group $B_{\theta_s}(k, \mathbf{Z}_{2\nu})$ defined in [9],[13], which is a generalization of the Brauer-Wall group, see [26]. The Brauer group $B_{\theta_s}(k, G)$ for an abelian group G can be described by an exact sequence due to Childs, see [8] and the conceptual proof in [2].

2 Preliminaries

From now on k stands for a field of characteristic different from 2 and H is a finite dimensional Hopf algebra with antipode S.Unless otherwise stated all tensor products will be over the field k. For general facts on Hopf algebras we refer the reader to [14] and [18].

The Brauer group, see [4], [5]: In this paragraph we recall the construction of the Brauer group of a quasitriangular Hopf algebra. Suppose that $R = \sum R^{(1)} \otimes R^{(2)} \in H \otimes H$ is a quasitriangular structure on H. The category ${}_H\mathcal{M}$ of left H-modules is a braided monoidal category with braiding given by

$$\psi_{MN} : M \otimes N \to N \otimes M,\ m \otimes n \mapsto \sum (R^{(2)} \cdot n) \otimes (R^{(1)} \cdot m),$$

for all $m \in M, n \in N$. Given two H-module algebras A, B, the *braided product* of A and B, denoted by $A\#B$, is an H-module algebra and it is defined as follows: as an H-module, $A\#B = A \otimes B$, while the multiplication is given by

$$(a\#x)(b\#y) = a\psi_{BA}(x\#b)y = \sum a(R^{(2)} \cdot b)\#(R^{(1)} \cdot x)y,$$

for all $a, b \in A, x, y \in B$. The *H-opposite algebra* of A, denoted by $\overline{A}$, is equal to A as an H-module but with multiplication given by $ab = \sum (R^{(2)} \cdot b)(R^{(1)} \cdot a)$ for all $a, b \in A$. For a finite dimensional left H-module M, $End(M)$ is an H-module algebra with the H-structure defined by

$$(h \cdot f)(m) = \sum h_{(1)} \cdot f(S(h_{(2)}) \cdot m).$$

Similarly, $End(M)^{op}$ is a left H-module algebra with

$$(h \cdot f)(m) = \sum h_{(2)} \cdot f(S^{-1}(h_{(1)}) \cdot m).$$

An H-module algebra A is called *H-Azumaya* if it is finite dimensional and the following H-module algebra maps are isomorphisms:

$$\begin{aligned} F &: A\#\overline{A} \to End(A),\ F(a\#\bar{b})(c) = \sum a(R^{(2)} \cdot c)(R^{(1)} \cdot b), \\ G &: \overline{A}\#A \to End(A)^{op},\ G(\bar{a}\#b)(c) = \sum (R^{(2)} \cdot a)(R^{(1)} \cdot c)b. \end{aligned}$$

Let $Az(H)$ denote the set of isomorphism classes of H-Azumaya module algebras. We say that $A, B \in Az(H)$ are *Brauer equivalent*, denoted by $A \sim B$, if there exist finite dimensional H-modules M, N such that $A\#End(M) \cong B\#End(N)$ as H-module algebras. The relation $\sim$ is an equivalence relation and the quotient set $BM(k, H, R) = Az(H)/\sim$ is a group. Given $[A], [B] \in BM(k, H, R)$, the multiplication is $[A][B] = [A\#B]$, the inverse is $[A]^{-1} = [\overline{A}]$ and the neutral element is represented by $[End(M)]$ where M is a finite dimensional H-module.

The Brauer group $BQ(k,H)$ of the category of Yetter-Drinfel'd modules is just the Brauer group $BM(k,D(H),R)$ where $D(H)$ is the Drinfel'd double of H and R its canonical quasitriangular structure.

For a coquasitriangular Hopf algebra (H,r) the category $\mathcal{M}^H$ of right H-comodules is a braided monoidal category with braiding defined by

$$M\otimes N \to N\otimes M,\ m\otimes n \mapsto \sum n_{(0)}\otimes m_{(0)}r(n_{(1)}\otimes m_{(1)}),$$

for all $m\in M, n\in N$. Since H is finite dimensional, the group $BC(k,H,r)$ is isomorphic to $BM(k,H^*,R)$ where H^* is the dual Hopf algebra of H and R is the quasitriangular structure of H^* induced by r. When H is the group algebra of an abelian group G then, identifying kG with $(kG)^*$ a dual quasitriangular structure r on $(kG)^*$ is nothing but a bicharacter r on G. It turns out that $BM(k,\, kG,\, r^*)\cong B_\phi(k,G)$, the Brauer group of graded Azumaya algebras introduced in [9] and [13]. The group $B_\phi(k,G)$ is described by an exact sequence having the classical Brauer group of the field $Br(k)$ as a kernel and a group of $G\times G$-graded Galois extensions $Gal_\phi(k,\, G\times G)$ as a cokernel (see [8]).

An equivalence of categories: Recall that a convolution invertible map $\sigma: H\otimes H\to k$ is called a 2-*cocycle* if it satisfies the following equalities:

i) $\sigma(h\otimes 1)=\sigma(1\otimes h)=\varepsilon(h)1$,

ii) $\sum\sigma(g_{(1)}\otimes h_{(1)})\sigma(g_{(2)}h_{(2)}\otimes m)=\sum\sigma(h_{(1)}\otimes m_{(1)})\sigma(gh_{(2)}m_{(2)})$,

for all $g,h,m\in H$. It is well-known that a new Hopf algebra H_σ, called the σ-*twist* of H, can be associated to H. As a coalgebra $H_\sigma=H$ while the multiplication is defined by

$$a\cdot b=\sum\sigma(a_{(1)}\otimes b_{(1)})a_{(2)}b_{(2)}\sigma^{-1}(a_{(3)}\otimes b_{(3)}) \tag{2.1}$$

for all $a,b\in H$. If (H,r) is coquasitriangular, then (H_σ,r_σ) is coquasitriangular with $r_\sigma=\sigma\tau * r * \sigma^{-1}$ where τ is the usual flip map and $*$ is the convolution product. It is also well-known that $\mathcal{M}^H$ is equivalent to $\mathcal{M}^{H_\sigma}$ as a braided monoidal category. As a consequence, their Brauer groups are isomorphic, i.e., $BC(k,H,r)\cong BC(k,H_\sigma,r_\sigma)$, see [7].

3 The Hopf algebra H_ν

Let ν be an odd number and let k be a field containing a primitive 2ν-th root of unity ω. Let H_ν denote the Hopf algebra over k generated by g and x such that

$$g^{2\nu} = 1, \qquad gx + xg = 0, \qquad x^2 = 0$$

with coproduct

$$\Delta(g) = g \otimes g, \qquad \Delta(x) = x \otimes g^\nu + 1 \otimes x$$

and antipode

$$S(g) = g^{-1}, \qquad S(x) = g^\nu x.$$

The Hopf algebras of type H_ν are a particular case of the family of pointed Hopf algebras constructed in [20, Section 5.1]. We use a simpler notation than Radford's because we consider only the quasitriangular ones. The Hopf algebra H_1 is just the Sweedler Hopf algebra H_4. On the other hand, for every ν, H_4 is a subHopf algebra of H_ν. Also H_4 may be viewed as a factor of H_ν by the Hopf ideal generated by $g - g^\nu$. This means that H_ν can be expressed as a Radford's biproduct where the Hopf algebra factor is isomorphic to H_4, see [22]. We can also consider H_ν as a Radford's biproduct where the Hopf algebra factor is the group algebra of $\mathbf{Z}_{2\nu}$, the cyclic group of order 2ν. Note that $k\mathbf{Z}_{2\nu}$ is a subHopf algebra of H_ν and a factor Hopf algebra by mapping x to 0.

In [14, Proposition 8] it is shown that H_ν is self-dual with isomorphism

$$\Theta \colon H_\nu \to H_\nu^*,\ g \mapsto G,\ x \mapsto X, \tag{3.1}$$

where G is the algebra homomorphism defined by $G(g) = \omega$ and $G(x) = 0$ and X is the linear map defined by $X(g^l x^m) = \delta_{1,m}$ for all $0 \leq l < 2\nu$ and $m \in \{0, 1\}$.

The quasitriangular structures on H_ν are computed in [14, Corollary 3].The quasitriangular structures are parametrized by pairs (s, β) where s is an odd positive integer $1 \leq s < 2\nu$ and $\beta \in k$. They are given by formula (1.1). Observe that $R_{s,0}$ can be viewed as a quasitriangular structure on $k\mathbf{Z}_{2\nu}$ and that the projection of H_ν onto $k\mathbf{Z}_{2\nu}$ maps $(H_\nu, R_{s,\beta})$ onto $(k\mathbf{Z}_{2\nu}, R_{s,0})$. The projection of H_ν onto H_4 mapping g^ν to the nontrivial grouplike element c of H_4 maps $(H_\nu, R_{s,\beta})$ onto (H_4, R_β) where

$$R_\beta = \frac{1}{2}(1 \otimes 1 + c \otimes 1 + 1 \otimes c - c \otimes c) + \frac{\beta}{2}(x \otimes x + x \otimes cx + cx \otimes cx - cx \otimes x).$$

It is not difficult to verify that $(H_\nu, R_{s,\beta})$ is *minimal if and only if* $\beta \neq 0$ and $(s, \nu) = 1$. It is proved in [20, Corollary 3,c] that $(H_\nu, R_{s,\beta})$ is *triangular if and only if* $s = \nu$. Hence H_ν does not admit minimal triangular structures unless $\nu \neq 1$.

Since H_ν is self-dual and quasitriangular, it is coquasitriangular, with a family of coquasitriangular structures parametrized again by the pairs (s, β) and given by $r_{s,\beta} := (\Theta \otimes \Theta)(R_{s,\beta})$. By direct computation one gets:

$$r_{s,\beta} = \sum_{n,m=0}^{2\nu-1} \omega^{snm} (g^n)^* \otimes (g^m)^* + \beta \sum_{n,m=0}^{2\nu-1} (-1)^m \omega^{snm} (g^n x)^* \otimes (g^m x)^*.$$

Since H_ν is self-dual and pointed, H_ν has the Chevalley property, see [1]. In particular this implies that each pair $(H_\nu, R_{s,\beta})$ is a Drinfel'd twist of a modified supergroup algebra, that is, we can twist the coproduct of H_ν in such a way that the quasitriangular structure $R_{\nu,\beta}$ gets twisted into the trivial quasitriangular structure or into a quasitriangular structure of the form $\frac{1}{2}(1 \otimes 1 + 1 \otimes a + a \otimes a - a \otimes a)$ for some grouplike a of order 2. As the Hopf algebra structure on H_ν is essentially unique once the algebra structure is fixed, it is natural to expect that this twist will not effect the coproduct but only the quasitriangular structure. We compute explicitly this twist in the dual perspective, using twists coming from cleft extensions. At the same time, we show that similar results hold for a general s, i.e., $(H_\nu, R_{s,\beta})$ *is always a twist of* $(H_\nu, R_{s,0})$. Besides we will show that $(H_\nu, R_{s,\beta})$ *can not be a twist of* $(H_\nu, R_{s',\beta})$ for $s \neq s'$.

We choose the dual point of view and we want to twist the product and the coquasitriangular structure of H_ν by means of a 2-cocycle. Such cocycles correspond to H_ν-cleft extensions of k, i.e., convolution invertible maps $\phi\colon H_\nu \to B$ where B is an H_ν-comodule algebra such that k is the set of coinvariants of B (see [3] and [10]). With the same techniques as in [12] we can always make sure that ϕ satisfies:

$$\phi(g^j) = \phi(g)^j, \qquad \phi(g^j x) = \phi(g)^j \phi(x),$$

even though ϕ need not be an algebra map. Let us denote $\phi(g) = u$ and $\phi(x) = v$ and let ρ denote the H_ν-comodule structure map on B. We have:

$$\begin{aligned} \rho(v^2) &= \rho(\phi(x)^2) = \rho(\phi(x))^2 = ((\phi \otimes \mathrm{id})\Delta(x))^2 \\ &= (v \otimes g^\nu + 1 \otimes x)(v \otimes g^\nu + 1 \otimes x) \\ &= v^2 \otimes 1. \end{aligned}$$

Since the space of coinvariants is k, it follows that $v^2 = \mu \in k$. Similarly, one shows that there must hold $uv + vu = tu^\nu$ for some $t \in k$ and that $u^{2\nu} = \lambda \in k$ with λ invertible.

Therefore we get a family of comodule algebras $B(\mu, t, \lambda)$ parametrized by $\mu, t \in k$ and $\lambda \in k$, $\lambda \neq 0$. We can always choose ϕ such that $\lambda = \phi(1) = 1$. Therefore the extensions are given by the algebras $B(\mu, t, 1)$, i.e., the algebras generated by u and v with relations

$$u^{2\nu} = 1, \qquad uv + vu = tu^\nu, \qquad v^2 = \mu,$$

and with comodule structure

$$\rho(u) = u \otimes g, \qquad \rho(v) = v \otimes g^\nu + 1 \otimes x.$$

Since H_ν is pointed and $\phi(g)$ is invertible, ϕ is convolution invertible. The convolution inverse is given by:

$$\begin{aligned} &\phi^{-1}(g^j) = u^{-j}, \\ &\phi^{-1}(g^j x) = \begin{cases} u^{\nu-j} v - tu^{-j} & \text{for } j \text{ even}, \\ -u^{\nu-j} v & \text{for } j \text{ odd}. \end{cases} \end{aligned}$$

It can be directly checked that $B(\mu, t, 1)$ is indeed a H_ν-cleft extension of k, hence we can construct the corresponding 2-cocycles:

$$\sigma(a \otimes b) = \sum \phi(a_{(1)})\phi(b_{(1)})\phi^{-1}(a_{(2)}b_{(2)})$$

for all $a, b \in H_\nu$. We obtain:

$$\begin{aligned}
&\sigma(g^j \otimes g^m) = 1, \\
&\sigma(g^j \otimes g^m x) = 0, \\
&\sigma(g^j x \otimes g^m) = \begin{cases} 0 & \text{for } m \text{ even}, \\ t & \text{for } m \text{ odd}, \end{cases} \\
&\sigma(g^j x \otimes g^m x) = (-1)^m \mu.
\end{aligned}$$

The convolution inverse of σ is easily computed:

$$\begin{aligned}
&\sigma^{-1}(g^j \otimes g^m) = 1, \\
&\sigma^{-1}(g^j \otimes g^m x) = 0, \\
&\sigma^{-1}(g^j x \otimes g^m) = \begin{cases} 0 & \text{for } m \text{ even}, \\ -t & \text{for } m \text{ odd}, \end{cases} \\
&\sigma^{-1}(g^j x \otimes g^m x) = (-1)^{m+1} \mu.
\end{aligned}$$

The new product in the twisted Hopf algebra is given by formula (2.1) and it is:

$$g^r \cdot g^m = g^{r+m}$$

$$\begin{aligned}
x \cdot g &= \sigma(x \otimes g)g^{\nu+1}\sigma^{-1}(g^\nu \otimes g) + \sigma(1 \otimes g)xg\sigma^{-1}(g^\nu \otimes g) \\
&\quad + \sigma(1 \otimes g)g\sigma^{-1}(x \otimes g) \\
&= tg^{\nu+1} + xg + tg
\end{aligned}$$

$$\begin{aligned}
g \cdot x &= \sigma(g \otimes x)g^{\nu+1}\sigma^{-1}(g \otimes g^\nu) + \sigma(g \otimes 1)gx\sigma^{-1}(g \otimes g^\nu) \\
&\quad + \sigma(g \otimes 1)g\sigma^{-1}(g \otimes x) \\
&= gx
\end{aligned}$$

$$
\begin{aligned}
x\cdot x \;&= \sigma(x\otimes x)g^{\nu+\nu}\sigma^{-1}(g^\nu\otimes g^\nu)+\sigma(x\otimes 1)g^\nu x\sigma^{-1}(g^\nu\otimes g^\nu)\\
&\quad +\sigma(x\otimes 1)g^\nu\sigma^{-1}(g^\nu\otimes x)+\sigma(1\otimes x)xg^\nu\sigma^{-1}(g^\nu\otimes g^\nu)\\
&\quad +\sigma(1\otimes 1)x^2\sigma^{-1}(g^\nu\otimes g^\nu)+\sigma(1\otimes 1)xg\sigma^{-1}(g^\nu\otimes x)\\
&\quad +\sigma(1\otimes x)g^\nu\sigma^{-1}(x\otimes g^\nu)+\sigma(1\otimes 1)x\sigma^{-1}(x\otimes g^\nu)\\
&\quad +\sigma(1\otimes 1)1\sigma^{-1}(x\otimes x)\\
&= \mu+0-tx-\mu\\
&= -tx.
\end{aligned}
$$

When $t=0$ the product in H_ν remains unchanged by the twist. For the twists associated to $B(\mu,0,1)$, the coquasitriangular structure $r_{s,\beta}$ is twisted into $(\sigma\tau)*r_{s,\beta}*\sigma^{-1}$, which must be of the form $r_{s',\gamma}$ for some odd s' between 1 and $2\nu-1$ and some $\gamma\in k$. Since

$$
\omega^{s'jl}=r_{s',\gamma}(g^j\otimes g^l)=((\sigma\tau)*r_{s,\beta}*\sigma^{-1})(g^j\otimes g^l)=r_{s,\beta}(g^j\otimes g^l)=\omega^{sjl}
$$

for every j and l, it follows that $s'=s$. To find γ we compute

$$
((\sigma\tau)*r_{s,\beta}*\sigma^{-1})(g^jx\otimes g^lx)=r_{s,\gamma}(g^jx\otimes g^lx)=(-1)^l\omega^{skl}\gamma.
$$

We obtain

$$
\begin{aligned}
&((\sigma\tau)*r_{s,\beta}*\sigma^{-1})(g^jx\otimes g^lx)=\\
&\qquad = \sigma(g^lx\otimes g^jx)r_{s,\beta}(g^{j+\nu}\otimes g^{l+\nu})\sigma^{-1}(g^{j+\nu}\otimes g^{l+\nu})\\
&\qquad\quad +\sigma(g^l\otimes g^j)r_{s,\beta}(g^{j+\nu}\otimes g^lx)\sigma^{-1}(g^{j+\nu}\otimes g^\nu)\\
&\qquad\quad +\sigma(g^l\otimes g^jx)r_{s,\beta}(g^{j+\nu}\otimes g^l)\sigma^{-1}(g^{j+\nu}\otimes g^lx)\\
&\qquad\quad +\sigma(g^lx\otimes g^j)r_{s,\beta}(g^jx\otimes g^{l+\nu})\sigma^{-1}(g^\nu\otimes g^{l+\nu})\\
&\qquad\quad +\sigma(g^l\otimes g^j)r_{s,\beta}(g^jx\otimes g^lx)\sigma^{-1}(g^\nu\otimes g^\nu)\\
&\qquad\quad +\sigma(g^l\otimes g^j)r_{s,\beta}(g^jx\otimes g^l)\sigma^{-1}(g^\nu\otimes g^lx)\\
&\qquad\quad +\sigma(g^lx\otimes g^j)r_{s,\beta}(g^j\otimes g^{l+\nu})\sigma^{-1}(g^jx\otimes g^{l+\nu})\\
&\qquad\quad +\sigma(g^l\otimes g^j)r_{s,\beta}(g^j\otimes g^lx)\sigma^{-1}(g^jx\otimes g^\nu)\\
&\qquad\quad +\sigma(g^l\otimes g^j)r_{s,\beta}(g^j\otimes g^l)\sigma^{-1}(g^jx\otimes g^lx)\\
&\qquad = (-1)^k\mu r_{s,\beta}(g^{j+\nu}\otimes g^{l+\nu})\\
&\qquad\quad +r_{s,\beta}(g^jx\otimes g^lx)+(-1)^{l+1}r_{s,\beta}(g^j\otimes g^l)\mu\\
&\qquad = (-1)^l\omega^{sjl}(\beta-2\mu).
\end{aligned}
$$

Proposition 3.1 *The dual quasitriangular Hopf algebras $(H_\nu, r_{s,\beta})$ with $\beta\in k$ are all twist-equivalent to $(H_\nu, r_{s,0})$ for every odd s between 1 and $2\nu-1$. There is no 2-cocycle twisting $r_{s,\beta}$ into $r_{s',\gamma}$.*

Proof: The first statement is obtained taking the cocycle associated to the H_ν-cleft extension $B(\frac{\beta}{2}, 0)$. For the second statement, suppose that there is a 2-cocycle twisting $r_{s,\beta}$ into $r_{s',\gamma}$ for $s \neq s'$. Then, by composition of twists, there would be a 2-cocycle σ twisting $r_{s,0}$ into $r_{s',0}$. This would imply that

$$r_{s',0}(g^j \otimes g^l) = \omega^{s'jl} = (\sigma\tau * r_{s,0} * \sigma^{-1})(g^j \otimes g^l) = \sigma(g^l \otimes g^j)\sigma(g^j \otimes g^l)^{-1}\omega^{sjl}.$$

Since the restriction of a 2-cocycle on H_ν to the group algebra of the cyclic group generated by g is necessarily symmetric, $\sigma(g^l \otimes g^j)\sigma(g^j \otimes g^l)^{-1} = 1$ and therefore $s = s'$. $\square$

From Proposition 3.1 the category of right H_ν-comodules with braiding induced by $r_{s,\beta}$ is tensor equivalent to the category of right H_ν-comodules with braided induced by $r_{s,0}$. The invariance of the Brauer group under equivalences implies:

Corollary 3.2 *For any $\beta \in k$ and any odd $1 \leq s \leq 2\nu$, $BC(k, H_\nu, r_{s,\beta}) \cong BC(k, H_\nu, r_{s,0})$. Dually, $BM(k, H_\nu, R_{s,\beta}) \simeq BM(k, H_\nu, R_{s,0})$.*

4 The Brauer group of $(H_\nu, R_{s,\beta})$

In this section we compute the Brauer group $BM(k, H_\nu, R_{s,\beta})$ for each s and β. By Corollary 3.2 we are reduced to computing the Brauer group $BM(k, H_\nu, R_{s,0})$. Our calculation of this group is based on the ideas used in [23] where the Brauer group of Sweedler Hopf algebra is computed.

Let $i : \mathbf{Z}_{2\nu} \to H_\nu$ and $p : H_\nu \to \mathbf{Z}_{2\nu}$ be the canonical inclusion and projection respectively. Considering $R_{s,0}$ as a quasitriangular structure on $k\mathbf{Z}_{2\nu}$, these maps are quasitriangular maps. They induces group homomorphism at the Brauer group level

$$BM(k, H_\nu, R_{s,0}) \underset{p^*}{\overset{i^*}{\rightleftarrows}} BM(k, k\mathbf{Z}_{2\nu}, R_{s,0}).$$

Note that for any H_ν-Azumaya module algebra A, i^* maps $[A]$ into $[A]$ with A considered as a $\mathbf{Z}_{2\nu}$-Azumaya module algebra. Since $p \circ i = \mathrm{id}$, $i^* \circ p^* = \mathrm{id}$, and thus i^* is surjective. So we need to compute $Ker(i^*)$.

Let α, β and $\gamma \in k$. We denote by $A(\alpha, \beta, \gamma)$ the *generalized quaternion algebra* generated by u and v with relations $u^2 = \alpha$, $v^2 = \beta$ and $uv + vu = \gamma$. This algebra can be endowed with a natural H_4-action, the *standard* H_4-action, given by

$$g \rightharpoonup u = -u, \qquad g \rightharpoonup v = -v, \qquad x \rightharpoonup u = 0, \qquad x \rightharpoonup v = 1. \tag{4.1}$$

If the discriminant $d = \gamma^2 - 4\alpha\beta \neq 0$, the generalized quaternion algebra is called *nonsingular*. By [23, Proposition 5], $A(\alpha, \beta, \gamma)$ is a H_4-Azumaya algebra if and only if it is nonsingular.

Lemma 4.1 *Let $A = A(\alpha, \beta, \gamma)$ be an H_ν-module algebra for which the action of the subHopf algebra generated by g^ν and x is the standard H_4-action. Then:*

i) If $\alpha \neq 0$ or $\gamma \neq 0$ the action of g necessarily coincides with the action of g^ν;

ii) If $\alpha = \gamma = 0$ also the possibility $g \rightharpoonup u = \omega^t u$ and $g \rightharpoonup v = -v + \lambda u$ for $\lambda \in k$ and t odd and different from ν can occur.

Proof: *i)* Let us write $g \rightharpoonup u = x_1 + x_2 v + x_3 u + x_4 uv$ with $x_1, \ldots, x_4 \in k$. The condition $(gx + xg) \rightharpoonup u = 0$ yields $x_2 = x_4 = 0$. It is easily computed that for every $m \geq 1$

$$g^m \rightharpoonup u = x_1 (\sum_{l=0}^{m-1} x_3^l) + x_3^m u. \tag{4.2}$$

By assumption $g^\nu \rightharpoonup u = -u$, hence formula (4.2) for $m = \nu$ implies that

$$x_1 (\sum_{l=0}^{\nu-1} x_3^l) = 0, \qquad x_3^\nu = -1.$$

Since x_3 is not a ν^{th}-root of unity, $(\sum_{l=0}^{\nu-1} x_3^l) \neq 0$ and thus $x_1 = 0$. As $x_3^{2\nu} = 1$, there is an odd positive integer $t < 2\nu$ such that $x_3 = \omega^t$, hence $g \rightharpoonup u = \omega^t u$.

Let us set $g \rightharpoonup v = y_1 + y_2 v + y_3 u + y_4 uv$, with $y_1, \ldots, y_4 \in k$. The condition $(gx+xg) \rightharpoonup v = 0$ yields $y_2 = -1$ and $y_4 = 0$. An easy computation shows that

$$g^m \rightharpoonup v = \begin{cases} v - y_3 u(\sum_{l=0}^{m-1}(-1)^l \omega^{lt}) & \text{if } m \text{ is even,} \\ y_1 - v + y_3 u(\sum_{l=0}^{m-1}(-1)^l \omega^{lt}) & \text{if } m \text{ is odd.} \end{cases} \tag{4.3}$$

By hypothesis $g^\nu \rightharpoonup v = -v$, so formula (4.3) for $m = \nu$ implies

$$y_1 = 0; \qquad y_3(\sum_{l=0}^{\nu-1}(-1)^l \omega^{lt}) = 0. \tag{4.4}$$

Assume that $\alpha \neq 0$. From the equality $\alpha = g \rightharpoonup u^2 = (g \rightharpoonup u)^2 = \omega^{2t}\alpha$, we conclude that $\omega^t = -1$, hence $t = \nu$. Replacing $\omega^t = -1$ in (4.4) one gets that $y_3 \nu = 0$, so $y_3 = 0$ and therefore $g \rightharpoonup v = -v$, i.e., if $\alpha \neq 0$ the action of g coincides with the action of g^ν.

Suppose now that $\alpha = 0$. Then from (4.3) and (4.4) we obtain only

$$g \rightharpoonup u = \omega^t u, \qquad g \rightharpoonup v = -v + y_3 u.$$

Assume that $\gamma \neq 0$. From the equality

$$\beta = g \rightharpoonup v^2 = (g \rightharpoonup v)^2 = (y_3 u - v)(y_3 u - v) = -y_3(uv + vu) + \beta$$

we get $y_3 = 0$. But then $\gamma = g \rightharpoonup (uv + vu) = -\omega^t(uv + vu) = -\omega^t \gamma$. It follows that $\omega^t = -1$, i.e., that $t = \nu$. So the first statement is proved.

ii) It is easily checked that in case $\alpha = \gamma = 0$ also the action defined by

$$g \rightharpoonup u = \omega^t u, \qquad g \rightharpoonup v = -v + \lambda u, \qquad x \rightharpoonup u = 0, \qquad x \rightharpoonup v = 1$$

for $\lambda \in k$ and $t < 2\nu$ an odd nonnegative integer different from ν yields an H_ν-module algebra structure on A for which $g^\nu \rightharpoonup u = -u$ and $g^\nu \rightharpoonup v = -v$. $\square$

Lemma 4.2 *Let A and B be two H_ν-module algebras. The braided product $A\#B$ with respect to the quasitriangular structure $R_{s,0}$ is the same as the θ_s-twisted $\mathbf{Z}_{2\nu}$-graded product of $\mathbf{Z}_{2\nu}$-graded algebras where θ_s is the $\mathbf{Z}_{2\nu}$-bicharacter given by $\theta_s(x,y) = \omega^{sxy}$. The H_ν-opposite algebra $\bar{A}$ of A is the same as the $\mathbf{Z}_{2\nu}$-graded θ_s-twisted opposite algebra.*

Proof: The braiding in $A \otimes B$ is determined by the action of $R_{s,0}$ and it is

$$\psi_{AB}(c \otimes b) := \frac{1}{2\nu} \sum_{i,l=0}^{2\nu-1} \omega^{-il}(g^{sl} \rightharpoonup b) \otimes (g^i \rightharpoonup c).$$

The cyclic group $\mathbf{Z}_{2\nu} =< g >$ acts on A and B and since $g^{2\nu} = 1$ and $\omega \in k$, the action of g on A and B is diagonalizable. The algebras A and B inherit a $\mathbf{Z}_{2\nu}$-grading from the eigenspace decomposition for the action of g, which are in fact algebra gradings because A and B are H_ν-module algebras. We denote by A_j the eigenspace corresponding to the eigenvalue ω^j and we say that $c \in A$ has *degree* j if $c \in A_j$. Similarly for B. Then, for $c \in A_m$ and $b \in B_n$ we have

$$\begin{aligned}
\psi_{AB}(c \otimes b) &= \tfrac{1}{2\nu} \textstyle\sum_{i,l=0}^{2\nu-1} \omega^{-il} \omega^{sln} \omega^{im} b \otimes c \\
&= \tfrac{1}{2\nu} \textstyle\sum_{l=0}^{2\nu-1} \omega^{sln} \sum_{i=0}^{2\nu-1} \omega^{i(m-l)} b \otimes c \\
&= \omega^{smn} b \otimes c.
\end{aligned}$$

Hence the braiding is the θ_s-twisted $\mathbf{Z}_{2\nu}$-graded braiding. Since the braided product and the braided opposite product are completely determined by the braiding and the product in the algebras, we have the statement. $\square$

Remark 4.3 Observe that the braiding is in fact a $\mathbf{Z}_{\frac{2\nu}{(s,\nu)}}$-braiding because the effect of the braiding on homogeneous elements depends only on the class modulo $\frac{2\nu}{(s,\nu)}$ of the degrees. Another way to say this is to define the degrees as $deg'(a) = sh$ if a is an eigenvector of g of eigenvalue ω^h. Then it is clear that the grading is a $\mathbf{Z}_{\frac{2\nu}{(s,\nu)}}$-grading because a degree appears if and only if it is a multiple of s in $\mathbf{Z}_{2\nu}$. With this new definition of grading we see that the braiding induced by $R_{s,0}$ can also be seen as the $\mathbf{Z}_{\frac{2\nu}{(s,\nu)}}$-graded θ_1-twisted flip operator with bicharacter $\theta_1(t \otimes y) = \omega^{ty}$.

As the braiding ψ_{BA} induced by the quasitriangular structure $R_{s,0}$ is nothing but a $\mathbf{Z}_{2\nu}$-graded and θ_s-twisted flip operator, we can view the Brauer group $BM(k, k\mathbf{Z}_{2\nu}, R_{s,0})$ as the Brauer group $B_{\theta_s}(k, \mathbf{Z}_{2\nu})$ which is a generalization of the Brauer-Wall group for any cyclic group $\mathbf{Z}_n$ with respect to a bicharacter on $\mathbf{Z}_n$ (see [9], [13], [19] and [6, pages 329, 341, 423, 434]). In fact, since $k\mathbf{Z}_{2\nu} \simeq (k\mathbf{Z}_{2\nu})^*$, the dual quasitriangular structures $r_{s,0}$ on $(k\mathbf{Z}_{2\nu})^*$ induce the bicharacter θ_s on $\mathbf{Z}_{2\nu}$. Then

$$BM(k, k\mathbf{Z}_{2\nu}, R_{s,0}) \simeq BC(k, (k\mathbf{Z}_{2\nu})^*, r_{s,0}) \simeq B_{\theta_s}(k, \mathbf{Z}_{2\nu})$$

where the last isomorphism is explained in [5, Lemma 1.2].

We denote by $A(\alpha, \beta, \gamma; H_4)$ the generalized quaternion algebra $A(\alpha, \beta, \gamma)$ together with the standard action of H_4. If $A(\alpha, \beta, \gamma)$ is nonsingular then this uniquely determines a H_ν-module algebra structure on $A(\alpha, \beta, \gamma)$, which we call again *standard* and denote by $A(\alpha, \beta, \gamma; H_\nu)$. We want to describe which H_ν-module algebras with underlying algebra of type $A(\alpha, \beta, \gamma)$ are H_ν-Azumaya algebras.The following lemma shows that $A(\alpha, \beta, \gamma; H_4)$ with the action extended to H_ν in a nonstandard way is *not* H_ν-Azumaya.

Lemma 4.4 *The algebra $A = A(0, \beta, 0)$ with the action given by $g \rightharpoonup u = \omega^t u$ with t odd and $t \neq \nu$, $g \rightharpoonup v = -v + \lambda u$ for $\lambda \in k$, $x \rightharpoonup u = 0$ and $x \rightharpoonup v = 1$, i.e., with the action of Lemma 4.1 ii), is* not *a H_ν-Azumaya algebra.*

Proof: First we observe that if $\lambda \neq 0$ we can replace v by $v' = v - \frac{\lambda}{(\omega^t+1)} u$ obtaining

$$(v')^2 = \beta, \qquad uv' + v'u = 0, \qquad u^2 = 0$$

and

$$g \rightharpoonup u = \omega^t u, \qquad g \rightharpoonup v' = -v', \qquad x \rightharpoonup u = 0, \qquad x \rightharpoonup v' = 1.$$

The decomposition of A into eigenspaces with respect to the action of g is given by $A_0 = k$, $A_\nu = kv'$, $A_t = ku$ and $A_{t+\nu} = kuv' = kuv$. If A were an H_ν-Azumaya algebra then its left H_ν-center, i.e., the set

$$\{b \in A \mid by = m_A\psi_{AA}(b \otimes y), \quad \forall y \in A\}$$

with m_A the product in A, would be trivial. But it is easy to check that $y = \mu + \mu' u$ for μ, $\mu' \in k$ belongs to the H_ν-center of A because

$$\begin{aligned} &uy = \mu u = \mu u + 0 = m_A\psi_{AA}(u \otimes (\mu + \mu' u)),\\ &v'y = \mu v' - \mu' uv = \mu v' + \mu' \omega^{s\nu t} uv = m_A\psi_{AA}(v' \otimes (\mu + \mu' u)),\\ &uvy = \mu uv + 0 = m_A\psi_{AA}(uv \otimes (\mu + \mu' u)). \end{aligned}$$

Hence A is not H_ν-Azumaya. □

The next lemma shows when $A(\alpha, \beta, \gamma)$ with the standard H_ν-action is H_ν-Azumaya.

Lemma 4.5 *The algebra $A(\alpha, \beta, \gamma; H_\nu)$ is H_ν-Azumaya if and only if $d \neq 0$.*

Proof: The H_ν-action on $A(\alpha, \beta, \gamma; H_\nu)$ is the standard action and it is in fact an action of the quotient $H_\nu / < g^\nu - g > \simeq H_4$. Since the quasitriangular structure $R_{s,0}$ is mapped to the quasitriangular structure R_0 of H_4 under the projection, the braiding with respect to *any* $R_{s,0}$ is nothing but the braiding induced by R_0, i.e., the $\mathbf{Z}_2$-graded flip operator. The algebra $A(\alpha, \beta, \gamma; H_\nu)$ is H_ν-Azumaya with respect to the quasitriangular structure $R_{s,0} = \sum R^1_{s,0} \otimes R^2_{s,0}$ if and only if the H_ν-module algebra maps

$$\begin{aligned} &F\colon A(\alpha, \beta, \gamma) \# \overline{A(\alpha, \beta, \gamma)} \to End(A(\alpha, \beta, \gamma)),\\ &F(a \# \bar{b})(c) = \sum a(R^2_{s,0} \rightharpoonup c)(R^1_{s,0} \rightharpoonup b), \end{aligned}$$

and

$$\begin{aligned} &G\colon \overline{A(\alpha, \beta, \gamma)} \# A(\alpha, \beta, \gamma) \to End(A(\alpha, \beta, \gamma))^{op},\\ &G(\bar{a} \# b)(c) = \sum (R^2_{s,0} \rightharpoonup a)(R^1_{s,0} \rightharpoonup c)b, \end{aligned}$$

are isomorphisms. Since the actions of g and of g^ν coincide, the maps F and G coincide with the similar maps with respect to H_4 and R_0. Hence they are isomorphisms if and only if $A(\alpha, \beta, \gamma; H_4)$ is H_4-Azumaya. By [23, Proposition 5], this happens if and only if $d \neq 0$. $\square$

If H_ν acts on an Azumaya algebra A which is a H_ν-module algebra, then the action is *inner* by [16], i.e., there is a convolution invertible element $\pi \in Hom_k(H_\nu, A)$ for which

$$h \rightharpoonup b = \sum \pi(h_{(1)}) b \pi^{-1}(h_{(2)})$$

for every $h \in H_\nu$ and every $b \in A$. In general this action is not *strongly inner*, i.e., π is not necessarily an algebra homomorphism.

Let us define the *induced subalgebra* with respect to the action as the (uniquely determined) algebra generated by $u := \pi^{-1}(g^\nu)$ and $v := \pi^{-1}(x)$. It turns out that this algebra is of the form $A(\alpha, \beta, \gamma)$ with $\alpha \neq 0$. By [23, Lemma 1], the action of H_4 is *strongly inner* if and only if $d = 0$ and α is a square in k. The action of H_4 on A is given by:

$$g^\nu \rightharpoonup b = u^{-1} b u, \qquad x \rightharpoonup b = bv - vu^{-1} bu. \tag{4.5}$$

Lemma 4.6 *Let H_ν act on an Azumaya algebra A. If the action of H_ν is not strongly inner but the action of g is strongly inner, then the restriction of the action to H_4 is not strongly inner and the action of g^ν is strongly inner.*

Proof: If H_ν acts on an Azumaya algebra A then there is a convolution invertible element $\pi \in Hom_k(H_\nu, A)$ for which

$$h \rightharpoonup b = \sum \pi(h_{(1)}) b \pi^{-1}(h_{(2)})$$

for every $h \in H_\nu$ and every $b \in A$. Since the action of g is strongly inner there exists $\pi\colon H_\nu \to A$ for which the restriction to $k\mathbf{Z}_{2\nu}$ is an algebra homomorphism. This implies that the action of g^ν is strongly inner. It suffices to prove that if π is not an algebra homomorphism then the restriction of π to H_4 cannot be an algebra homomorphism. If π is not an algebra homomorphism it will not preserve at least one of the relations $x^2 = 0$ and

$gx+xg=0$. If $\pi(x)^2 \neq 0$ then $\pi|_{H_4}$ is not an algebra homomorphism and we are done. Suppose that π does not preserve $gx+xg=0$ and that π preserves $g^\nu x + xg^\nu = 0$. We will get to a contradiction. Since

$$(gx) \rightharpoonup b = g \rightharpoonup (x \rightharpoonup b), \qquad \forall b \in A$$

we get

$$\pi(g)b\pi^{-1}(gx) + \pi(gx)b\pi^{-1}(g^{\nu+1}) = \pi(g)(b\pi^{-1}(x) + \pi(x)b\pi^{-1}(g^\nu))\pi(g)^{-1}$$

for all $b \in A$. As π restricted to $k\mathbf{Z}_{2\nu}$ is an algebra homomorphism we have

$$\pi^{-1}(x) = -\pi(x)\pi(g)^{-\nu}, \qquad \pi^{-1}(gx) = -\pi(g)^{-1}\pi(gx)\pi(g)^{-\nu-1}.$$

Hence

$$\pi(g)[-b\pi(x)+\pi(x)b]\pi(g)^{-\nu-1} = \pi(g)[-b\pi(g)^{-1}\pi(gx)+\pi(g)^{-1}\pi(gx)b]\pi(g)^{-\nu-1}$$

for all $b \in A$. Since $\pi(g)$ is invertible we get

$$b[-\pi(x) + \pi(g)^{-1}\pi(gx)] = [-\pi(x) + \pi(g)^{-1}\pi(gx)]b \qquad \forall b \in A.$$

Since A is central, there exists $t_1 \in k$ such that $\pi(g)\pi(x) = \pi(gx) + t_1\pi(g)$. Similarly using

$$(xg) \rightharpoonup b = -(gx) \rightharpoonup b = x \rightharpoonup (g \rightharpoonup b), \qquad \forall b \in A$$

one shows that there exists $t_2 \in k$ for which $\pi(g)\pi(x) = -\pi(gx) + t_2\pi(g)$. Therefore,

$$\pi(x)\pi(g) + \pi(g)\pi(x) = (t_1 + t_2)\pi(g).$$

It can be proved by induction on m that

$$\pi(g)^m\pi(x) = \begin{cases} \pi(x)\pi(g)^m & \text{for } m \text{ even}, \\ -\pi(x)\pi(g)^m + (t_1+t_2)\pi(g)^m & \text{for } m \text{ odd}. \end{cases}$$

Hence if π restricted to H_4 were an algebra map, this would mean that $(t_1+t_2)\pi(g)^\nu = 0$. Since $\pi(g)$ is invertible this would imply that $t_1+t_2=0$, i.e., the relation $gx+xg=0$ would be preserved by π, a contradiction. $\square$

Lemma 4.7 *Let A be an H_ν-module Azumaya algebra such that A is an Azumaya algebra. Assume that the action of g is strongly inner but the action of H_ν is not strongly inner. Then there exist a nonsingular generalized quaternion algebra $A(\alpha,\beta,\gamma) \subset A$ and an Azumaya subalgebra B of A commuting with $A(\alpha,\beta,\gamma)$ such that*

$$A \simeq A(\alpha,\beta,\gamma) \otimes B$$

as H_ν-module algebras.

The action of g^ν on $A(\alpha,\beta,\gamma)$ coincides with the action of g, the action of g^ν and x on B is trivial and the action of g on B is a $\mathbf{Z}_\nu$-action. Hence the action on A is completely determined by an H_4-action on $A(\alpha,\beta,\gamma)$ and by a $\mathbf{Z}_\nu$-action on B.

Proof: By Lemma 4.6, H_4 does not act on A in a strongly inner way but g^ν does. By [23, Corollary 2], $A \simeq A(\alpha,\beta,\gamma) \otimes B$ as H_4-module algebras where $A(\alpha,\beta,\gamma)$ is the (nonsingular) induced subalgebra and B commutes with $A(\alpha,\beta,\gamma)$. It is Azumaya and the action of H_4 on $A(\alpha,\beta,\gamma)$ is given by (4.5) while the action of H_4 on B is trivial. We need to show that the induced subalgebra $A(\alpha,\beta,\gamma)$ and the subalgebra B are preserved by the action of g. Since the action of g is strongly inner and that g is grouplike, there exists an invertible $w = \pi^{-1}(g) \in A$ for which

$$w^\nu = \pi^{-1}(g)^\nu = \pi(g)^{-\nu} = \pi(g^{-\nu}) = \pi(g^\nu)^{-1} = \pi^{-1}(g^\nu) = u$$

and $g \rightharpoonup b = w^{-1}bw$ for every $b \in A$. Multiplying the equality

$$0 = (gx + xg) \rightharpoonup b = w^{-1}bvw - w^{-1}vu^{-1}buw + w^{-1}bwv - vu^{-1}w^{-1}bwu \quad (4.6)$$

by w on the left and using the fact that u and w commute, we obtain

$$b(vw + wv) = (vw + wv)w^{-1}u^{-1}bwu.$$

This formula for $b = w$ yields $w^2v = vw^2$, hence w^2 commutes with $A(\alpha,\beta,\gamma)$. Therefore w^2 belongs either to k or to B.

- If $w^2 \in k$ then
$$u = w^\nu = w^{1+2\frac{\nu-1}{2}} = tw$$
for some $t \in k$. Hence $w \in A(\alpha,\beta,\gamma)$ so $A(\alpha,\beta,\gamma)$ is H_ν-stable. Besides, for every $b \in B$ $g \rightharpoonup b = w^{-1}bw = u^{-1}bu = b$. Hence g acts trivially on B.

- If $w^2 \in B$ then $u = w^\nu = w\bar{b}$ for $\bar{b} = w^{2\frac{\nu-1}{2}} \in B$. Since w is invertible, $\bar{b}$ is invertible. The action of g on u is trivial because w commutes with u and the action of g on v is given by:
$$g \rightharpoonup v = w^{-1}vw = \bar{b}u^{-1}vu\bar{b}^{-1} = (u^{-1}vu) = g^\nu \rightharpoonup v,$$
so the action of g on the induced subalgebra coincides with the action of g^ν. Hence $A(\alpha,\beta,\gamma)$ is H_ν-stable. For $b \in B$ we have
$$g \rightharpoonup b = w^{-1}bw = \bar{b}u^{-1}bu\bar{b}^{-1} = \bar{b}(g^\nu \rightharpoonup b)\bar{b}^{-1} = \bar{b}b\bar{b}^{-1}.$$
Since $\bar{b}^\nu b\bar{b}^{-\nu} = g^\nu \rightharpoonup b = b$, it follows that $\bar{b}^\nu \in k$. Hence the action of g on B is determined by a $\mathbf{Z}_\nu$-action on B.

In particular the action of H_ν on an Azumaya algebra is completely determined by an H_4-action on a quaternion algebra and a $k\mathbf{Z}_\nu$-action on the Azumaya subalgebra B. $\square$

Remark 4.8 Observe that this proof recovers the result of Lemma 4.1 that if $\alpha \neq 0$ then the action of g on a generalized quaternion algebra must coincide with the action of g^ν.

Corollary 4.9 *Let $A(\alpha,\beta,\gamma)$ be a quaternion algebra with $d = \gamma^2 - 4\alpha\beta \neq 0$, which is an H_ν-module algebra. Then*
$$A(\alpha,\beta,\gamma) \simeq A(d, -\alpha d^{-1}, 0; H_\nu)$$
as H_ν-module algebras.

Proof: By [23, Lemma 3], $A(\alpha,\beta,\gamma) \simeq A(d, -\alpha d^{-1}, 0; H_4)$ as H_4-module algebras. Since $d \neq 0$, either α or γ is nonzero. Now Lemma 4.1 applies. $\square$

Corollary 4.10 *Under the hypothesis of Lemma 4.7 on A, the induced subalgebra $A(\alpha,\beta,\gamma)$ is always nonsingular and it is always a H_ν-Azumaya algebra.*

Proof: By the discussion at the end of [23, Lemma 1], $A(\alpha,\beta,\gamma)$ is always nonsingular. By Corollary 4.9, $A(\alpha,\beta,\gamma) \simeq A(d, -\alpha d^{-1}, 0; H_4)$ as H_ν-module algebras. The discriminant of $A(d, -\alpha d^{-1}, 0)$ is equal to $4\alpha \neq 0$ because $\alpha = \pi^{-1}(g^\nu)^2$ is invertible. By Lemma 4.5, $A(\alpha,\beta,\gamma)$ is H_ν-Azumaya. $\square$

Lemma 4.11 *Let A be an Azumaya algebra satisfying the hypothesis of Lemma 4.7. With notation as before, $A = A(\alpha,\beta,\gamma; H_\nu)\#B$ with respect to every quasitriangular structure of the form $R_{s,0}$ as H_ν-module algebras. Moreover, A is H_ν-Azumaya if and only if B is H_ν-Azumaya.*

Proof: By Lemma 4.7 we know that g^ν acts trivially on B and that g acts like g^ν on $A(\alpha,\beta,\gamma)$. Hence the gradings induced by the eigenspaces decomposition for the action of g are:

$$B = \bigoplus_{l=0}^{\nu} B_{2l}, \qquad A(\alpha,\beta,\gamma; H_\nu) = A(\alpha,\beta,\gamma; H_\nu)_0 \oplus A(\alpha,\beta,\gamma; H_\nu)_\nu,$$

i.e., the only eigenvalues for g on B are given by *even* powers of ω while the only eigenvalues of g on $A(\alpha,\beta,\gamma; H_\nu)$ are given by $\omega^0 = 1$ and $\omega^\nu = -1$. By Lemma 4.2 we know that the braided product $A(\alpha,\beta,\gamma; H_\nu)\#B$ with respect to the quasitriangular structure $R_{s,0}$ is the θ_s-twisted graded flip operator

$$(a\#b)(c\#d) = \omega^{s(\deg c)(\deg b)} ac\#bd$$

for homogeneous a and b. Since for these algebras $\omega^{s(\deg b)(\deg c)} = 1$ for every homogeneous c and b, the braided product coincides with the ordinary tensor product independently of s.

By definition A is H_ν-Azumaya with respect to $R_{s,0}$ if the H_ν-module algebra maps

$$F_A\colon A\#\bar{A} \to End(A), F_A(a\#\bar{b})(c) = \omega^{s\deg(b)\deg(c)}acb$$

for b and c homogeneous and

$$G_A\colon \bar{A}\#A \to End(A)^{op}, G_A(\bar{a}\#b)(c) = \omega^{s\deg(a)\deg(c)}acb$$

for a and c homogeneous are isomorphisms. By [4, Proposition 2.4.2(c)], as H_ν-module algebras

$$\bar{A} \simeq \overline{A(\alpha,\beta,\gamma)\otimes B} \simeq \overline{B}\#\overline{A(\alpha,\beta,\gamma)} \simeq \overline{B}\otimes\overline{A(\alpha,\beta,\gamma)},$$

where the second isomorphism χ is given on homogeneous elements by

$$\chi(\overline{a\#b}) = \omega^{-s(\deg a)(\deg b)}\bar{b}\#\bar{a} = \bar{b}\#\bar{a},$$

and the third isomorphism follows by the fact that the braiding between $\overline{A(\alpha,\beta,\gamma)}$ and $\overline{B}$ is again trivial. Moreover, if an algebra A is $\mathbf{Z}_{2\nu}$-graded, then also $End(A)$ will be $\mathbf{Z}_{2\nu}$-graded: here $f\in End_k(A)$ has degree d if for every homogeneous element $a\in A$, $f(a)$ is homogeneous of degree $d+\deg a$. By [4, Proposition 4.3], there is an isomorphism ξ between $End(A(\alpha,\beta,\gamma)\#B)$ and $End(A(\alpha,\beta,\gamma))\#End(B)$ given, on homogeneous elements, by

$$\xi(f\#f')(a\#b) = \omega^{-s(\deg a)(\deg f')}f(a)\#f'(b) = f(a)\#f'(b)$$

because the grading on $End(B)$ will only have even degrees. If A is H_ν-Azumaya, then

$$\xi\circ F_A\circ(\mathrm{id}_A\otimes\chi^{-1})\colon A\#\overline{B}\#\overline{A(\alpha,\beta,\gamma)} \to End(A(\alpha,\beta,\gamma))\#End(B)$$

and

$$G_A\circ(\chi^{-1}\otimes\mathrm{id}_A)\colon \overline{B}\#\overline{A(\alpha,\beta,\gamma)}\#A \to End(A)^{op}$$

are isomorphisms. On homogeneous elements a, c, $e \in A(\alpha,\beta,\gamma)$, (i.e., of degree 0 or ν) b, d, $f \in B$ (i.e., of even degree) one has

$$\begin{aligned}
&F_A \circ (\mathrm{id}_A \otimes \chi^{-1})((a\#b)\#(\bar{d}\#\bar{c}))(e\#f) = \\
&\quad = F_A((a\#b)\#(\overline{c\#d}))(e\#f) \\
&\quad = \omega^{s(\deg(c\#d))(\deg(e\#f))}(a\#b)(e\#f)(c\#d) \\
&\quad = \omega^{s(\deg(c)+\deg(d))(\deg(e)+\deg(f))}(aec\#bfd) \\
&\quad = \omega^{s\deg(c)\deg(e)}(aec\#bfd) \\
&\quad = \omega^{s\deg(c)\deg(e)}aec\#\omega^{s}\deg(d)\deg(f)bfd \\
&\quad = F_{A(\alpha,\beta,\gamma)}(a\#\bar{c})(e)\#F_B(b\#\bar{d})(f)
\end{aligned}$$

where the third equality follows from the first part of the lemma, the fifth follows from the fact that B has only even degrees and $A(\alpha,\beta,\gamma)$ has only degrees that are multiples of ν. Similarly one proves that

$$\xi \circ G_A \circ (\chi^{-1}\#\mathrm{id})((\bar{b}\#\bar{a})\#(c\#d))(e\#f) = G_{A(\alpha,\beta,\gamma)}(\bar{a}\#c)(e)\#G_B(\bar{b}\#d)(f).$$

So, since $A(\alpha,\beta,\gamma)$ is H_ν-Azumaya by Lemma 4.5 and since we are dealing with tensor products over the field k, F_A and G_A are isomorphisms if and only if F_B and G_B are so. $\square$

Theorem 4.12 *The Brauer group $BM(k,H_\nu,R_{s,0})$ is isomorphic to the direct sum of $(k,+)$ and $B_{\theta_s}(k,\mathbf{Z}_{2\nu})$ where $\theta_s\colon \mathbf{Z}_{2\nu}\times\mathbf{Z}_{2\nu}\to k$ is the bicharacter induced on $\mathbf{Z}_{2\nu}$ by $R_{s,0}$.*

Proof: We first show that there is a split exact sequence of groups

$$1 \longrightarrow (k,+) \longrightarrow BM(k,H_\nu,R_{s,0}) \longrightarrow B_{\theta_s}(k,\mathbf{Z}_{2\nu}) \longrightarrow 1. \qquad (4.7)$$

Then we will show that the subgroups on the right and on the left commute. We define a map $\Phi\colon (k,+)\to BM(k,H_\nu,R_{s,0})$ by $\Phi(0)=[M_2]$, the class of the algebra M_2 of 2×2 matrices with trivial action, and $\Phi(\alpha) = [A(\alpha^{-1},-\alpha^{-1},0;H_\nu)]$ for $\alpha\neq 0$. If $\alpha+\beta=\sigma\neq 0$, then by [23, Proposition

7] and by Lemma 4.7, $A(\alpha^{-1}, -\alpha^{-1}, 0; H_\nu)\#A(\beta^{-1}, -\beta^{-1}, 0; H_\nu)$ is isomorphic to $A(\sigma^{-1}, -\sigma^{-1}, 0; H_\nu) \otimes M_2$ with trivial H_4-action on M_2 and with g-action on M_2 given by conjugation by an invertible element $b \in M_2$ for which $b^\nu \in k$. By Cayley-Hamilton Theorem we know that $b^2 \in kb + k$, hence $b \in k$ because ν is odd, so the action of H_ν on M_2 is trivial. Hence $[M_2] = [End(P)] = 1$ for some H_ν-module P with trivial action. Therefore, for $\alpha + \beta \neq 0$

$$\Phi(\alpha)\#\Phi(\beta) = \big[A(\sigma^{-1}, -\sigma^{-1}, 0; H_\nu)\big] = \Phi(\alpha + \beta).$$

If $\alpha = -\beta$, again by [23, Proposition 7], the H_4-action on

$$A(\alpha^{-1}, -\alpha^{-1}, 0; H_\nu)\#A(-\alpha^{-1}, \alpha^{-1}, 0; H_\nu)$$

is strongly inner and the above algebra is a 4×4 matrix algebra isomorphic, as an H_4-module algebra, to $End(P)$ for some H_4-module P. The action on the vector space P is given by $g^\nu.p = up = u^{-1}p$ and $x.p = -vup$ for the induced elements u and v identified with the matrices. The action of g on $A(\alpha^{-1}, -\alpha^{-1}, 0; H_\nu)$ and on $A(-\alpha^{-1}, \alpha^{-1}, 0; H_\nu)$ coincides with the action of g^ν in view of Lemma 4.1. Then the action of g on their product coincides with the action of g^ν so that the action of H_ν on $A(\alpha^{-1}, -\alpha^{-1}, 0; H_\nu)\#A(-\alpha^{-1}, \alpha^{-1}, 0; H_\nu)$ is also strongly inner. The action of the matrices u and $-vu$ on P equips P with a H_ν-module structure so that

$$\big[A(\alpha^{-1}, -\alpha^{-1}, 0; H_\nu)\#A(-\alpha^{-1}, \alpha^{-1}, 0; H_\nu)\big] = [End(P)] = 1.$$

Hence Φ is a group homomorphism. It is injective because if we had

$$\Phi(\alpha) = [A(\alpha^{-1}, -\alpha^{-1}, 0; H_\nu)] = [1] = [End(X)]$$

for some H_ν-module X, then the action of H_4 would be strongly inner, which is impossible because $d \neq 0$.

Let $\Psi: BM(k, H_\nu, R_{s,0}) \to B_{\theta_s}(k, \mathbf{Z}_{2\nu})$ be the homomorphism given by forgetting the action of x using the identifications

$$BM(k, k\mathbf{Z}_{2\nu}, R_{s,0}) \simeq BC(k, k(\mathbf{Z}_{2\nu})^*, r_{s,0}) \simeq B_{\theta_s}(k, \mathbf{Z}_{2\nu})$$

where the second is [5, Lemma 1.2]. The homomorphism Ψ is surjective because a $\mathbf{Z}_{2\nu}$-Azumaya algebra becomes a H_ν-Azumaya algebra by taking the action of x to be zero and the braidings induced by $R_{s,0}$ and by θ_s are the same.
Hence we only need to prove that $\Phi(k,+) = Ker(\Psi)$. The kernel of Ψ consists of matrix algebras on which the action of g is strongly inner. We check that $\Phi(k,+) \subseteq Ker(\Psi)$. We know from Corollary 4.9 that

$$A(\alpha^{-1}, -\alpha^{-1}, 0; H_\nu) \simeq A(4\alpha^{-2}, -4^{-1}\alpha, 0; H_\nu).$$

Since $4\alpha^{-2}$ is a square the action of g^ν is strongly inner. By Lemma 4.1, the action of g and of g^ν coincide, hence the action of g is strongly inner. The quaternion algebra is a matrix algebra because $4\alpha^{-2}$ is a square.
Now suppose that A is an H_ν-Azumaya algebra such that $\Psi([A]) = 1$ and $[A] \neq 1$ in $BM(k, H_\nu, R_{s,0})$. We know that the action of g is strongly inner because $A \simeq End(X)$, a matrix algebra, for some X and the action of g on A is given by conjugation by the matrix representing the action of g on X. Hence A is Azumaya, and H_ν acts in a non-strongly inner way on A (otherwise $[A]$ would be 1 in $BM(k, H_\nu, R_{s,0})$). By Lemma 4.7, $A \simeq A(\alpha, \beta, \gamma)\#B$. Since g^ν acts in a strongly inner way, we can make sure that $\alpha = 1 \neq 0$ is a square, so the induced subalgebra is a matrix algebra. This implies that B is a matrix algebra too. The action of g on B is strongly inner and the action of x is trivial, hence $B = End(Y)$. By Lemma 4.11 and Corollary 4.10 both $A(1, \beta, \gamma)$ and B are both H_ν-Azumaya so that

$$[A] = [A(1, \beta, \gamma)\#B] = [A(1, \beta, \gamma)][End(Y] = [A(1, \beta, \gamma)]$$

and $A(1, \beta, \gamma)$ is nonsingular.
By Corollary 4.9, $[A(1, \beta, \gamma)] = [A(d, -d^{-1}, 0; H_\nu)]$, i.e., the class of A coin-

cides with the class of the nonsingular generalized quaternion algebra generated by u and v with relations $uv + vu = 0$, $u^2 = d$ and $v^2 = -d^{-1}$ and standard action. If we replace u by $u' = d^{-1}u$ then the action on the new basis is still standard and we have

$$A(d, -d^{-1}, 0; H_\nu) \simeq A(d^{-1}, -d^{-1}, 0; H_\nu).$$

Hence $[A] = [A(d^{-1}, -d^{-1}, 0; H_\nu)] = \Phi(d)$ so the sequence is exact. The sequence is split exact because the map

$$\Psi' \colon B_{\theta_s}(k, \mathbf{Z}_{2\nu}) \simeq BM(k, k\mathbf{Z}_{2\nu}, R_{s,0}) \to BM(k, H_\nu, R_{s,0})$$

obtained by extending the action of $k\mathbf{Z}_{2\nu}$ to H_ν by letting x act as 0 is a section for Ψ.

Let now A be a representative of a class in $(k, +)$ and B be a representative of a class in $B_{\theta_s}(k, \mathbf{Z}_{2\nu})$. We want to show that the corresponding classes commute in the Brauer group. By Lemma 4.2 we know that the braiding between the two algebras is the same as the θ_s-twisted $\mathbf{Z}_{2\nu}$-graded product where the grading on A and B is the eigenspace decomposition for the action of g. Besides we know that the only possible degrees in A are 0 and ν. Hence the braided product in $A\#B$ is given by

$$\begin{aligned}(a\#b)(c\#d) &= \omega^{s \deg(b) \deg(c)} ac\#bd = \omega^{s\nu \deg(b) \deg(c)} ac\#bd \\ &= (-1)^{\deg(b) \deg(c)} ac\#bd\end{aligned}$$

because ν and s are odd. Therefore

$$A\#B \simeq A \otimes_2 B \simeq B \otimes_2 A \simeq B\#A$$

where $\otimes_2$ denotes the $\mathbf{Z}_2$-graded tensor product and the second isomorphism holds because the $\mathbf{Z}_2$-graded flip is an algebra isomorphism (the category of $\mathbf{Z}_2$-graded modules with $\mathbf{Z}_2$-graded tensor product is symmetric). Hence the proof. □

Corollary 4.13 *Let ν be a product of r distinct primes $p_1, \ldots, p_r$ and let k be algebraically closed. Then*

$$BM(k, H_\nu, R_{s,0}) \simeq \underbrace{\mathbf{Z}_2 \times ... \times \mathbf{Z}_2}_{r+1 \ times} \times (k, +).$$

Proof: Following the idea of the proof of [15, Theorem 2.7] one checks that

$$B_{\theta_s}(k, \mathbf{Z}_{2\nu}) \simeq BW(k) \times B_{\theta_{s_1}}(k, \mathbf{Z}_{p_1}) \times ... \times B_{\theta_{s_r}}(k, \mathbf{Z}_{p_r})$$

where BW denotes the Brauer-Wall group of $\mathbf{Z}_2$-graded algebras and where $s_j = 2s\nu/p_j$ *mod.* p_j for $j = 1, \ldots, r$. By [13, Corollary 3.2], $BW(k) \simeq \mathbf{Z}_2$ and each $B_{\theta_{s_j}}(k, \mathbf{Z}_{p_j}) \simeq \mathbf{Z}_2$. □

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