ON HOPF ALGEBRAS WITH NON-ZERO INTEGRAL

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1 Introduction

One of the most important notions in Hopf algebra theory is the notion of integral, introduced by Sweedler in [27]. This notion has its origin in the Haar measure of the Hopf algebra $\mathcal{R}(G)$ of regular functions on a compact Lie group G, see [15]. If H denotes a Hopf algebra over a field k, a *left integral* for H is a linear map $\int_l \in H^*$ such that $h^* \int_l = h^*(1) \int_l$ for all $h^* \in H^*$. Hopf algebras having a non-zero left integral are called *co-Frobenius* and they have been extensively studied in the literature, see [27], [25], [26], [18], [6], [7], [11], [14], [5]. Co-Frobenius Hopf algebras are characterized by the following interesting finiteness condition: the injective hull of every simple left (or right) comodule is finite dimensional.

In [23, Corollary 2] Radford proved that if H is a co-Frobenius Hopf algebra whose coradical H_0 is a subalgebra, then H has finite coradical filtration. Andruskiewitsch and Dăscălescu investigated in [5] the relation between co-Frobenius Hopf algebras and the finiteness of the coradical filtration. They proved that a Hopf algebra with finite coradical filtration is necessarily co-Frobenius and they conjectured that any co-Frobenius Hopf algebra has fi-

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nite coradical filtration. In this paper we provide more evidences of the truthfulness of this conjecture. We give two new sufficient conditions for a co-Frobenius Hopf algebra to have finite coradical filtration, see Proposition 3.1. Using one of these conditions we prove in Section 3 that the conjecture holds for the Hopf algebra of rational functions of an algebraic group with integral over a perfect field. We also observe that it holds for a Hopf algebra over a perfect field whose coradical is cocommutative and for a Hopf algebra over a field of characteristic zero such that the restriction of the antipode to the coradical is an involution.

In Section 4 we characterize in several ways non-cosemisimple co-Frobenius Hopf algebras. If H is a co-Frobenius Hopf algebra whose coradical is a subalgebra, we prove that the following statements are equivalent: (i) H is not cosemisimple; (ii) $Rad(H)$, the radical of H (as a coalgebra), contains the unit; (iii) H_0 is contained in $Rad(H)$; (iv) no simple right (or left) H-comodule is injective. Finally in Section 5 we describe the head of an injective indecomposable comodule over a co-Frobenius Hopf algebra H. We show in Theorem 5.2 that the head of the injective hull $E(T)$ of a simple right H-comodule T is isomorphic to $kg \otimes T^{**}$ where g is the distinguished group-like element of H. As a consequence, the socle of $E(T)^*$ is isomorphic to $T^{***} \otimes kg^{-1}$, so [14, Corollary 2.4] is recovered. The proof of this latter result uses different methods to ours. It relies on the equivalence between the category of H-comodules and the category of unital modules over the non-unital algebra $Rat(H^*)$.

We next fix some notation and conventions and present some preliminaries needed in the sequel. The reader is referred to [1], [8], [22] and [28] for basic facts about coalgebras and Hopf algebras. Unless otherwise stated, we will always work over a fixed ground field k. All vector spaces, linear maps, and unadorned tensor product are over k. Throughout C will be a coalgebra and H a Hopf algebra, both over k. The antipode of H will be denoted by S. By C^* we denote the dual algebra of C and $\langle ?, ? \rangle : C^* \times C \to k$ stands for the evaluation map. We will consider C as a left and right C^*-module with the natural actions:

$$c^* \cdot c = \sum_{(c)} \langle c^*, c_{(2)} \rangle c_{(1)}, \qquad c \cdot c^* = \sum_{(c)} \langle c^*, c_{(1)} \rangle c_{(2)},$$

for $c^* \in C^*$ and $c \in C$. We write C_C (resp. ${}_CC$) to stress that C is viewed as a right (resp. left) comodule.

Co-Frobenius Hopf algebras: The injective hull of a left C-comodule M will be denoted by $E(M)$. Recall from [16] that C is called:

- *left semiperfect* if $E(T)$ is finite dimensional for each simple right C-comodule T.
- *left co-Frobenius* if C, considered as a left C^*-module, embeds in C^*.

In [16, Theorem 3] it is proved that the following assertions are equivalent: (i) H has a non-zero left integral; (ii) H is left semiperfect; (iii) H is left co-Frobenius; (iv) $Rat({}_{H^*}H^*) \neq \{0\}$. Here $Rat({}_{H^*}H^*)$ denotes the maximal rational submodule of H^*, viewed as left H^*-module. All these statements are equivalent to their right versions. A Hopf algebra satisfying any of these statements will be called *co-Frobenius*. There are some other characterizations of co-Frobenius Hopf algebras, see [8, Chapter 5]. We give a new one whose proof captures the essence in the proof of [5, Theorem 2.1].

Proposition 1.1 *The following statements are equivalent:*
(i) H is co-Frobenius.
(ii) H, as a left comodule, has a maximal subcomodule.

Proof: The map $\Phi : H \to k,\ h \mapsto \langle \int_l, h\rangle 1$ is a morphism of left comodules. Hence $Ker(\Phi)$ is maximal. Conversely, let $\mathcal{M}$ be a maximal subcomodule of H, then $H/\mathcal{M}$ is simple. Thus $\mathcal{M}^{\perp(H^*)} \cong (H/\mathcal{M})^*$ is a finite dimensional simple right ideal of H^*. So $Rat(H^*_{H^*})$ is non-zero. ∎

Loewy series: Every left C-comodule M has a filtration

$$\{0\} \subset Soc(M) \subset Soc^2(M) \subset .... \subset Soc^n(M) \subset ...,$$

called the *Loewy series* of M and defined as follows: $Soc(M)$ is the socle of M, and for $n > 1$, $Soc^n(M)$ is the unique subcomodule of M satisfying $Soc^{n-1}(M) \subset Soc^n(M)$ and $Soc(M/Soc^{n-1}(M)) = Soc^n(M)/Soc^{n-1}(M)$, see [13, 1.4]. Let $\{C_n : n \in \mathbb{N}\}$ denote the coradical filtration of the coalgebra C. The coradical filtration of C coincides with the Loewy series of C, viewed as a right or left comodule. There is an alternative description of this series. Let $\rho : M \to C \otimes M$ denote the structure map of M, then $Soc^{n+1}(M) = \rho^{-1}(M \otimes C_n)$. In case $M = Soc^n(M)$ for some n, the *Loewy length* of M is defined to be $ll(M) = min\{n \in \mathbb{N} : M = Soc^n(M)\}$.

2 New proofs of two classical results on co-Frobenius Hopf algebras

In this section we give two alternative proofs of the following result of Radford, [23, Corollary 2].

Theorem 2.1 *Let H be a co-Frobenius Hopf algebra. If the coradical of H is a subalgebra, then H has finite coradical filtration.*

Recall that when the coradical H_0 is a subalgebra of H, the coradical filtration is indeed an algebra filtration, [22, Lemma 5.2.8]. This fact is crucial in all the proofs of this result. Radford's proof goes as follows: it is first proved that a co-Frobenius Hopf algebra H decomposes as $H = E(k1)H_0$. Since $E(k1)$ is finite dimensional, it is contained in some H_n. Then $H = E(k1)H_0 \subseteq H_nH_0 = H_n$.

Radford's result is also proved in [5] by different methods. This new proof provides a way of constructing systematically examples of co-Frobenius Hopf algebras and suggests an strategy for the classification of co-Frobenius Hopf algebras whose coradical is a subalgebra.

Our first proof of Radford's result is very short. It uses properties of injective comodules and of the coefficient space of a comodule. For the properties of the coefficient space we refer to [13].

First proof of Theorem 2.1: Take a family $\{T_i : i \in I\}$ of simple right coideals of H such that $H_H = \oplus_{i\in I}E(T_i)$. Then $H = \sum_{i\in I} cf(E(T_i))$. The simple comodule T_i is contained in $T_i \otimes E(k1)$ and this latter is injective by [10, Corollary 2]. Hence $T_i \otimes E(k1)$ must contain $E(T_i)$. From here, $cf(E(T_i)) \subseteq cf(T_i \otimes E(k1)) = cf(T_i)cf(E(k1))$. Since $E(k1)$ is finite dimensional, there is $n \in \mathbb{N}$ such that $cf(E(k1)) \subseteq H_n$. On the other hand, $cf(T_i) \subseteq H_0$ for all $i \in I$. Then $cf(T_i)cf(E(k1)) \subseteq H_0H_n = H_n$. ∎

Our second proof is longer but it only uses elementary properties of the coradical filtration and the fact that a co-Frobenius Hopf algebra has bijective antipode, [23, Proposition 2]. We record some properties on the coradical filtration to be used in the proof, see [22, Proposition 5.2.9].

1.1. Consider H^* endowed with the finite topology. The closure of a subspace X of H^* is $\overline{X} = X^{\perp(H)\perp(H^*)}$. Let $J = J(H^*)$ denote the Jacobson radical of H^*. Then $H_n^{\perp(H^*)} = \overline{J^{n+1}}$. Since H_n is a subcoalgebra, $\overline{J^{n+1}}$ is a two-sided ideal of H^*.

1.2. As $H = \cup_{n\in\mathbb{N}}H_n$ we have $\cap_{n\in\mathbb{N}}\overline{J^{n+1}} = \{0\}$.

1.3. Regard H as an H^*-bimodule with the usual actions. Since $\overline{J^n}$ is a two-sided ideal of H^*, the space $\overline{J^n} \cdot H$ is an H^*-subbimodule of H, that is, a subcoalgebra of H.

1.4. Notice that $\overline{J^n} \cdot H_n \subseteq H_0$. Furthermore, if $\overline{J^n} \cdot H_n = \{0\}$, then $\Delta(H_n) \subseteq H \otimes H_{n-1}$. This gives that $H_n \subseteq H_{n-1}$ and then it follows that $H = H_{n-1}$.

Bearing these properties in mind we are ready to broach our second proof.

Second proof of Theorem 2.1 Let $\phi : H \to H^*$ be the monomorphism of left H^*-modules given by hypothesis. We have

$$\phi(\cap_{n\geq 1}(\overline{J^n} \cdot H)) = \cap_{n\geq 1}\phi(\overline{J^n} \cdot H) = \cap_{n\geq 1}\overline{J^n}\phi(H) \subseteq \cap_{n\geq 1}\overline{J^n} = \{0\}.$$

Hence $\cap_{n\geq 1}(\overline{J^n} \cdot H) = \{0\}$. Assume, to get a contradiction, that $H \neq H_n$ for all $n \in \mathbb{N}$. This implies that $\overline{J^n} \cdot H_n \neq \{0\}$. On the other hand, $\overline{J^n} \cdot H_n \subseteq H_0$. Then $(\overline{J^n} \cdot H) \cap H_0$ is a non-zero subcoalgebra of H for all $n \in \mathbb{N}$.

Regard H^* as a left H-module with the action $\langle h \rightharpoonup h^*, h' \rangle = \langle h^*, h'h \rangle$ for $h, h' \in H$ and $h^* \in H^*$. Let $n, m \in \mathbb{N}$ be such that $m \leq n$. Given $x^* \in \overline{J^n}$ and $h \in H_m$ we claim that $h \rightharpoonup x^* \in \overline{J^{n-m}}$. For $h' \in H_{n-m-1}$ we have that $\langle h \rightharpoonup x^*, h' \rangle = \langle x^*, h'h \rangle = 0$ because $h'h \in H_{n-1}$. We have used here that H_n is an algebra filtration. We now prove that $(\overline{J^n} \cdot H)H_m \subseteq \overline{J^{n-m}} \cdot H$. Let $\sum_i x_i^* \cdot h_i \in \overline{J^n} \cdot H$ and $h \in H_m$. Then

$$\begin{aligned}\textstyle\sum_i (x_i^* \cdot h_i)h &= \textstyle\sum_i \langle x_i^*, h_{i(2)}h_{(2)}S(h_{(3)}) \rangle h_{i(1)}h_{(1)} \\ &= \textstyle\sum_i (S(h_{(2)}) \rightharpoonup x_i^*) \cdot (h_i h_{(1)}).\end{aligned}$$

Since the antipode S of H is bijective, $S(H_m) = H_m$. By the preceding claim, $S(h_{(2)}) \rightharpoonup x_i^* \in \overline{J^{n-m}}$. Hence $\sum_i (x_i^* \cdot h_i)h \in \overline{J^{n-m}} \cdot H$.

Finally we are in a position to get the desired contradiction. For each $n \in \mathbb{N}$ we know that $(\overline{J^n} \cdot H) \cap H_0$ is a non-zero subcoalgebra of H. Take $h_n \in (\overline{J^n} \cdot H) \cap H_0$ such that $\varepsilon(h_n) \neq 0$. Then

$$\varepsilon(h_n)1 = \sum h_{n(1)}S(h_{n(2)}) \in (\overline{J^n} \cdot H)H_0 \subseteq \overline{J^n} \cdot H.$$

Hence $1 \in \cap_{n\geq 1}(\overline{J^n} \cdot H) = \{0\}$, a contradiction. Then there is $n \in \mathbb{N}$ such that $H = H_n$. ∎

Remark 2.2 Recall from [3] that the category of right H-comodules has the *Chevalley property* if the tensor product of two simple right H-comodules is semisimple. From the properties of the coefficient space of a comodule, it follows that H_0 is a subalgebra of H if and only if the category of right (or left) H-comodules has the Chevalley property.

There are Hopf algebras which do not have the Chevalley property, for example, Frobenius-Lusztig kernels, see [20]. More examples of Hopf algebras not having the Chevalley property may be obtained from the following result of Molnar, [21, Theorem 2]: Let G be a finite group and k a field of characteristic $p > 0$. Then $(KG)^*$ has the Chevalley property if and only if G has a normal Sylow p-subgroup.

We proceed now to give a new proof of the following result of Sullivan, [25, Theorem 3]. Our proof is inspired by the proof of this result for affine group schemes given in [11]. Another proof appears in [9, Theorem 2].

Theorem 2.3 *Let H be an involutory Hopf algebra such that $char(k)$ does not divide $dim(E(k1))$. Then H is co-Frobenius if and only if H is cosemisimple.*

Proof: Clearly, H cosemisimple implies H co-Frobenius. For the converse write $E = E(k1)$ and fix a basis $\{e_1, ..., e_n\}$ for E. Let $\{e_1^*, ..., e_n^*\} \subseteq E^*$ be a dual basis. For each $j = 1, ..., n$ we write $\rho_E(e_j) = \sum_{i=1}^n e_i \otimes h_{ij}$, where $\rho_E : E \to E \otimes H$ is the comodule structure map of E. Each h_{ij} is uniquely determined, $\Delta(h_{ij}) = \sum_{l=1}^n h_{il} \otimes h_{lj}$ and $\varepsilon(h_{ij}) = \delta_{ij}$. It may be checked that $\rho_{E^*}(e_j^*) = \sum_{i=1}^n e_i^* \otimes S(h_{ji})$. It is routine to verify that the maps

$$\iota : k \to E^* \otimes E,\ 1 \mapsto \tfrac{1}{n} \textstyle\sum_{j=1}^n e_j^* \otimes e_j,$$
$$\pi : E^* \otimes E \to k, e_l^* \otimes e_m \mapsto \langle e_l^*, e_m \rangle = \varepsilon(h_{lm}) = \delta_{lm},$$

are right H-comodule maps. As the reader may check, that S is an involution is only needed to prove that ι is an H-comodule map. Clearly, $\pi\iota = Id_k$. Then k is a direct summand of $E^* \otimes E$. By [10, Corollary 2], $E^* \otimes E$ is injective, so k is injective. Hence H is cosemisimple. ∎

3 Some results on a question of Andruskiewitsch and Dăscălescu

In [5] Andruskiewitsch and Dăscălescu conjectured that the coradical filtration of a co-Frobenius Hopf algebra H is finite. We know that this is true under the additional hypothesis of H_0 being a subalgebra. In this section we show that this conjecture holds for the ring of rational functions $H = k[G]$ of an affine algebraic group G with integral, where k is a perfect field. We start by giving two new sufficient conditions for a co-Frobenius Hopf algebra to have finite coradical filtration.

Proposition 3.1 *Let H be a co-Frobenius Hopf algebra and let $\{T_i : i \in I\}$ be a full set of simple right H-comodules. Let $\{0\} \subset E_1 \subset ... \subset E_n = E(k1)$ be a composition series for $E(k1)$. If either*

(i) $\{dim(T_i) : i \in I\}$ is bounded, or

(ii) $T_i \otimes (E_{j+1}/E_j)$ is semisimple for all $i \in I$ and $j = 1, ..., n$,

then H has finite coradical filtration.

Proof: For a finite dimensional right H-comodule M we have the inequality $ll(M) \leq cl(M) \leq dim(M)$, where $cl(M)$ and $ll(M)$ denote the composition length and the Loewy length of M respectively. Write $H_H = \oplus_{i\in I} E(T_i)^{(n_i)}$ where $\{n_i : i \in I\}$ is a family of finite cardinal numbers. Since the Loewy series commutes with direct sums, to prove that $H = H_n$ it suffices to show that the set $\{ll(E(T_i)) : i \in I\}$ is bounded.

Assume (i). We know from the first proof of Theorem 2.1 that $E(T_i)$ is a subcomodule of $T_i \otimes E(k1)$ for all $i \in I$. Then $ll(E(T_i)) \leq cl(E(T_i)) \leq cl(T_i \otimes E(k1)) \leq dim(T_i \otimes E(k1)) = dim(T_i)dim(E(k1))$. Now apply (i).

Assume (ii). Take $i \in I$ arbitrary. Since $(T_i \otimes E_{j+1})/(T_i \otimes E_j) \cong T_i \otimes (E_{j+1}/E_j)$ is semisimple, we get that $T_i \otimes E_j \subseteq Soc^j(T_i \otimes E(k1))$ for all $j = 1, ..., n$. Then $T_i \otimes E(k1) = Soc^n(T_i \otimes E(k1))$. Considering $E(T_i)$ as embedded in $T_i \otimes E(k1)$, we have $ll(E(T_i)) \leq ll(T_i \otimes E(k1)) \leq n$. ∎

Observe that the hypothesis in (ii) of the preceding proposition is weaker than H_0 being a subalgebra. A similar hypothesis, although a little stronger, would be to ask that the composition factors E_{j+1}/E_j are included in the Hopf socle of H, defined in [5, Definition 4.1]. Let $\mathcal{S}$ be a full set of representatives of simple right H-comodules. Each representative can be taken as a right coideal of H. Let $\hat{\mathcal{S}}$ denote the subset of $\mathcal{S}$ consisting of simple H-comodules V such that $V \otimes W$ and $W \otimes V$ are semisimple for all $W \in \mathcal{S}$. The Hopf socle of H is defined to be $H_{soc} = \sum_{V\in\hat{\mathcal{S}}} cf(V)$ and it is a cosemisimple Hopf subalgebra of H. If H_0 is a subalgebra of H, then $H_{soc} = H_0$.

With notation as in Proposition 3.1, an upper bound for the set $\{dim(T_i) : i \in I\}$ is given in [26, Theorem 2.13] in case $H = k[G]$, the ring of rational functions of an affine algebraic group G with integral, where k is algebraically closed of positive characteristic. It is shown there that the number of irreducible components of G is an upper bound for the above set. However, not every co-Frobenius Hopf algebra satisfies that $\{dim(T_i) : i \in I\}$ is bounded. For example, the quantum groups at a root of unity $\mathbb{C}_q[G]$ where G is the simple connected algebraic group associated to a simple finite dimensional Lie algebra, see [19], [5]. We give more examples of Hopf algebras such that the above set is not bounded by means of certain von Neumann regular algebras. See [12] for an account on this important class of algebras.

Proposition 3.2 *Let A be a von Neumann regular algebra. Then the finite dual coalgebra A^0 is cosemisimple.*

Proof: First observe that a coalgebra is cosemisimple if and only if each finite dimensional subcoalgebra is so. Finite dimensional subcoalgebras of the finite dual A^0 are of the form $(A/I)^*$ where I is a cofinite two-sided ideal of A. If

A is von Neumann regular, then A/I is too, but A/I is finite dimensional. Hence A/I is semisimple and thus $(A/I)^*$ is cosemisimple. ∎

Recall that a group G is said to be *locally finite* if every finite subset of G generates a finite subgroup.

Corollary 3.3 *Let G be a locally finite group. Assume that the order of any element of G is not divided by $char(k)$. Then $k[G]^0$ is a cosemisimple Hopf algebra.*

Proof: The group algebra of a group satisfying the hypothesis is known to be von Neumann regular, [24, page 138]. ∎

Example 3.4 Let k be the field of rational numbers and let $G = \oplus_{p\in\mathcal{P}}\mathbb{Z}_p$ where $\mathcal{P}$ is the set of prime numbers and $\mathbb{Z}_p$ denotes the cyclic group of order p. Since G is locally finite, $k[G]^0$ is cosemisimple. On the other hand, the projection of groups $\pi_p : G \to \mathbb{Z}_p$ induces a projection of Hopf algebras $\overline{\pi_p} : k[G] \to k[\mathbb{Z}_p]$. Then $k[\mathbb{Z}_p]^*$ is a subcoalgebra of $k[G]^0$. Observe now that $k[\mathbb{Z}_p]$ (and hence $k[\mathbb{Z}_p]^*$) has a simple module (comodule) whose dimension is $p-1$.

Let K/k be a field extension. If C is a coalgebra (resp. Hopf algebra) over k, then $C_K = C \otimes K$ is a coalgebra (resp. Hopf algebra) over K in the natural way. For a Hopf algebra H over k, Sullivan proved in [26, Proposition 2.1] that H is co-Frobenius if and only if H_K is so. It is natural then to ask how the coradical filtration of H and H_K are related. The answer is given in the following result which is pointed out in Section 7 of the survey paper [4]. We include the proof for the reader's convenience.

Proposition 3.5 *If k is a perfect field, then $(C \otimes K)_n = C_n \otimes K$ for all $n \in \mathbb{N}$.*

Proof: The statement easily follows once the equality $(C \otimes K)_0 = C_0 \otimes K$ is established. Since $\{C_n \otimes K : n \in \mathbb{N}\}$ is a coalgebra filtration of $C \otimes K$, by [22, Proposition 5.3.4], $(C \otimes K)_0 \subseteq C_0 \otimes K$. To show the reverse inclusion, let D be a simple subcoalgebra of C. Since k is perfect, D is a coseparable coalgebra and hence $D \otimes K$ is cosemisimple. So, $D \otimes K \subseteq (C \otimes K)_0$. ∎

We are now in a position to give the proof of the announced result. We first recall Sullivan's result [25, Theorem 2.13] stating that for k algebraically closed of positive characteristic and G an affine algebraic group with integral, the dimension of the injective hull of each simple $k[G]$-comodule is less or equal than the number of irreducible components of G.

Theorem 3.6 *Let G be an affine algebraic group with integral and let $k[G]$ be its ring of rational functions. If k is perfect, then $k[G]$ has finite coradical filtration.*

Proof: In light of Theorem 2.3 we can assume that k has positive characteristic. Let $\overline{k}$ denote the algebraic closure of k. By [26, Proposition 2.1], $\overline{k}[G_{\bar{k}}] \cong k[G]_{\bar{k}}$ is co-Frobenius. Sullivan's result above gives an upper bound for the dimension of the simple $\bar{k}[G_{\bar{k}}]$-comodules. Then Proposition 3.1 yields that $\bar{k}[G_{\bar{k}}]$ has finite coradical filtration. Now the preceding proposition applies. ∎

We provide some other examples of co-Frobenius Hopf algebras satisfying the conjecture. They will be derived from Theorem 2.1, since the coradical will be shown to be a subalgebra.

Theorem 3.7 *Let H be a Hopf algebra over a perfect field k. Assume that H_0 is cocommutative. Then H_0 is a subalgebra. As a consequence, if H is co-Frobenius, then H has finite coradical filtration.*

Proof: Let $\bar{k}$ be the algebraic closure of k. We know from Proposition 3.5 that $(H\otimes\bar{k})_0 = H_0\otimes\bar{k}$. Hence $(H\otimes\bar{k})_0$ is cocommutative and consequently pointed. So $(H\otimes\bar{k})_0$ is a subalgebra of $H\otimes\bar{k}$. Then

$$H_0\otimes\bar{k} = (H\otimes\bar{k})_0 = (H\otimes\bar{k})_0(H\otimes\bar{k})_0 = (H_0\otimes\bar{k})(H_0\otimes\bar{k}) = H_0H_0\otimes\bar{k}.$$

Since $H_0 \subseteq H_0H_0$, it follows that $H_0H_0 = H_0$ and so H_0 is a subalgebra of H. ∎

Proposition 3.8 *Let H be a co-Frobenius Hopf algebra such that either $char(k) = 0$ or $char(k) > dim(E(k1))$. Assume that $S|_{H_0}$ is an involution. Then H_0 is a subalgebra. In particular H has finite coradical filtration.*

Proof: The argument is analogous to [3, Proposition 4.2, 5. $\Rightarrow$ 3.]. Let L be the Hopf subalgebra of H generated by H_0. By [26, Theorem 2.15], L is co-Frobenius. Since $S|_{H_0}$ is an involution, the antipode of L is an involution. Let $E_L(k1)$ denote the injective hull of $k1$ as an L-comodule. Then $E_L(k1)$ may be considered as a subcomodule of $E(k1)$. So $dim(E_L(k1)) \leq dim(E(k1))$. The hypothesis on k together with Theorem 2.3 give that L is cosemisimple. Hence $L = H_0$. ∎

4 Characterizing non-cosemisimple co-Frobenius Hopf algebras

In this section we give several characterizations of co-Frobenius Hopf algebras which are not cosemisimple. Two of these characterizations involve the radical, as a coalgebra, of the Hopf algebra. We start by presenting several properties of the radical of a coalgebra and a comodule.

Let C be a coalgebra and M a right C-comodule. The *radical of* M, denoted by $Rad(M)$, is the intersection of all maximal subcomodules of M. Notice that $Rad(M)$ is equal to the radical of M considered as a left C^*-module. Then $Rad(M)$ enjoys the following properties:

1.- $Rad(\oplus_{i\in I} M_i) = \oplus_{i\in I} Rad(M_i)$ for a family $\{M_i\}_{i\in I}$ of right C-comodules.

2.- $Rad(N) = N \cap Rad(M)$ for a subcomodule N of M.

3.- $Rad(M/Rad(M)) = \{0\}$.

4.- Consider C as a left C^*-module. Then $Rad(C_C) = Rad({}_{C^*}C)$ is an $(End({}_{C^*}C), C^*)$-bimodule. Using the isomorphism of algebras $End({}_{C^*}C) \cong C^*$, we get that $Rad(C_C)$ is a C^*-bimodule (hence a subcoalgebra) of C.

The finiteness conditions of comodules allow the following characterization of semisimple comodules which, in general, does not hold for modules.

5.- *A right C-comodule M is semisimple if and only if $Rad(M) = \{0\}$.* It is known that any semisimple module has zero radical. For the converse, note that M is semisimple if and only if each finite dimensional subcomodule N of M is semisimple. The latter is equivalent to that $Rad(N) = N \cap Rad(M) = \{0\}$. In particular, $M/Rad(M)$ is semisimple.

We will next prove that for a co-Frobenius coalgebra the radical of the regular right comodule coincides with the radical of the regular left comodule. First we need the following lemma.

Lemma 4.1 *Let C be a coalgebra. Then $Rad(C_C)^{\perp(C^*)} = Soc(Rat(C^*_{C^*}))$.*

Proof: If $\mathcal{M}$ is a maximal subcomodule of C_C, then $\mathcal{M}^{\perp(C^*)} \cong (C/\mathcal{M})^*$ is a finite dimensional simple right ideal of C^*. So it is contained in $Soc(Rat(C^*_{C^*}))$. From here, $Rad(C_C)^{\perp(C^*)} = (\cap \mathcal{M})^{\perp(C^*)} = \sum \mathcal{M}^{\perp(C^*)} \subseteq Soc(Rat(C^*_{C^*}))$ where the intersection and the sum runs over all maximal subcomodules of C_C. To show the reverse inclusion, let T be a simple rational right ideal of C^*. Then T is finite dimensional and thus it is closed in the finite topology of C^*. It easily follows that $T^{\perp(C)}$ is a maximal right coideal of C and $T = \overline{T} = T^{\perp(C)\perp(C^*)}$. ∎

Theorem 4.2 *Let C be a co-Frobenius coalgebra (i.e. left and right co-Frobenius). Then $Soc(Rat(C^*_{C^*})) = Soc(Rat({}_{C^*}C^*))$. Hence $Rad(C_C) = Rad({}_CC)$.*

Proof: Let $\phi : {}_{C^*}C \to Rat({}_{C^*}C^*)$ and $\psi : C_{C^*} \to Rat(C^*_{C^*})$ be the isomorphism, of left and right C^*-modules respectively, given by hypothesis. Take a set $\{e_i\}_{i\in I}$ of orthogonal primitive idempotents in C^* such that $C_C = \oplus_{i\in I} C \cdot e_i$ and ${}_CC = \oplus_{i\in I} e_i \cdot C$. Each $C \cdot e_i$ (resp. $e_i \cdot C$) is an injective indecomposable right (resp. left) C-comodule with simple socle $C_0 \cdot e_i$ (resp. $e_i \cdot C_0$). Since C is left and right semiperfect, $C \cdot e_i$ and $e_i \cdot C$ are finite dimensional. From the isomorphism of left C^*-modules $(e_i \cdot C)^* \cong C^* e_i$ we obtain that C^*e_i is finite dimensional. Then $C^*e_i \subseteq Rat({}_{C^*}C^*)$ and so $C^*e_i = Rat({}_{C^*}C^*)e_i$.

We know that the subspaces $Rat(C^*_{C^*})$ and $Rat({}_{C^*}C^*)$ are equal as C is semiperfect, and $Rat(C^*_{C^*})$ is a two-sided ideal. To avoid confusion we write $Soc^r(Rat(C^*_{C^*}))$ for the socle of $Rat(C^*_{C^*})$ when viewed as a right C^*-module. Analogously, we write $Soc^l(Rat({}_{C^*}C^*))$. Since $Soc(C^*_{C^*})$ is a two-sided ideal and $Soc^r(Rat(C^*_{C^*})) = Rat(C^*_{C^*}) \cap Soc(C^*_{C^*})$, we get that $Soc^r(Rat(C^*_{C^*}))$ is a two-sided ideal. We claim that the left C^*-module $Soc^r(Rat(C^*_{C^*}))e_i$ is non-zero. Let $x \in C_0 \cdot e_i$ be non-zero. Then $0 \neq \psi(x) = \psi(x \cdot e_i) = \psi(x)e_i$ and $\psi(x)e_i \in Soc^r(Rat(C^*_{C^*}))e_i$ because $x \in C_0$ and $\psi(C_0) = Soc^r(Rat(C^*_{C^*}))$. Using now that $Soc(C^*e_i)$ is simple and essential in C^*e_i, we find that $Soc(C^*e_i) = Soc(C^*e_i) \cap Soc^r(Rat(C^*_{C^*}))e_i$. Hence $Soc(C^*e_i) = Soc(Rat(C^*_{C^*})e_i) \subseteq Soc^r(Rat(C^*_{C^*}))$.

The equality of subspaces of C^*,

$$\begin{aligned} Rat(C^*_{C^*}) &= \psi(C) = \psi(\oplus_{i\in I} C \cdot e_i) = \oplus_{i\in I}\psi(C \cdot e_i) = \oplus_{i\in I}\psi(C)e_i \\ &= \oplus_{i\in I} Rat(C^*_{C^*})e_i, \end{aligned}$$

gives the equality $Rat({}_{C^*}C^*) = \oplus_{i\in I} Rat(C^*_{C^*})e_i$ of left C^*-modules. Then

$$Soc^l(Rat({}_{C^*}C^*)) = Soc(\oplus_{i\in I} Rat(C^*_{C^*})e_i) = \oplus_{i\in I} Soc(Rat(C^*_{C^*})e_i),$$

and the latter is contained in $Soc^r(Rat(C^*_{C^*}))$. A symmetric argument shows that $Soc^r(Rat(C^*_{C^*})) \subseteq Soc^l(Rat({}_{C^*}C^*))$. Finally, using the preceding lemma and the first statement,

$$Rad(C_C) = Soc^r(Rat(C^*_{C^*}))^{\perp(C)} = Soc^l(Rat({}_{C^*}C^*))^{\perp(C)} = Rad({}_CC).$$

∎

We give a characterization in terms of the radical of when the coradical of a co-Frobenius Hopf algebra is a subalgebra.

Proposition 4.3 *Let H be a co-Frobenius Hopf algebra. The following assertions are equivalent:*

(i) $Rad(H)H_0 = Rad(H)$.

(ii) H_0 *is a subalgebra of* H.

Proof: (i) $\Rightarrow$ (ii) Let $\phi : H \to H^*$ be the monomorphism of left H^*-modules given by hypothesis. It is defined by $\langle \phi(h), h' \rangle = \langle \int_l, h'S(h) \rangle$ for $h, h' \in H$. We know from Lemma 4.1 that $Rad(H_H)^{\perp(H^*)} = Soc^l(Rat({}_{H^*}H^*)) = \phi(H_0)$. For $g, g' \in H_0$ and $h \in Rad(H_H)$ we have:

$$\langle \phi(gg'), h \rangle = \langle \int_l, hS(g')S(g) \rangle = \langle \phi(g), hS(g') \rangle.$$

Since $g \in H_0$, $\phi(g) \in Soc^l(Rat({}_{H^*}H^*)) = Rad(H_H)^{\perp(H^*)} = (Rad(H)H_0)^{\perp(H^*)}$. Then $\langle \phi(g), hS(g') \rangle = 0$ for all $h \in Rad(H_H)$ (we use here that H has bijective antipode to assure $S(g') \in H_0$). From this it follows that $\phi(gg') \in Rad(H_H)^{\perp(H^*)} = \phi(H_0)$. Then $gg' \in H_0$.

(ii) $\Rightarrow$ (i) Clearly $Rad(H) \subseteq Rad(H)H_0$. Since H is co-Frobenius, H is projective as a left H^*-module, [16, Theorem 23]. Then, by [2, Proposition 17.10], $Rad(H_H) = Rad({}_{H^*}H) = J \cdot H$ where $J = J(H^*)$ is the Jacobson radical of H^*. We have seen in the proof of Theorem 2.1 that $(J \cdot H)H_0 \subseteq J \cdot H$. ∎

We finally arrive to the main theorem of this section.

Theorem 4.4 *Let H be a co-Frobenius Hopf algebra whose coradical is a subalgebra. The following statements are equivalent:*

(i) H *is not cosemisimple.*

(ii) $1 \in Rad(H)$.

(iii) $H_0 \subseteq Rad(H)$.

(iv) No simple left (or right) H-comodule is injective.

Proof: (i) $\Rightarrow$ (ii) Let $\int_l$ be a left integral of H and let $\phi : H \to H^*$ denote the monomorphism of left H^*-modules afforded by $\int_l$. We claim that $\int_l$ vanishes on H_0. Since H is not cosemisimple, $\langle \int_l, 1 \rangle = 0$. Write $H_0 = k1 \oplus M$ for some subcoalgebra M of H_0. For $h \in H$ we have the equality $\langle \int_l, h \rangle 1 = \langle \int_l, h_{(2)} \rangle h_{(1)}$. If $h \in M$, then $\langle \int_l, h \rangle 1 \in M \cap k1 = \{0\}$. Hence $\langle \int_l, M \rangle = \{0\}$.

Since $\int_l$ vanishes on H_0, we have that $\phi(1) \in H_0^{\perp(H^*)} = J(H^*)$. Indeed, $\phi(1) \in Rat({}_{H^*}H^*) \cap J(H^*)$ and the latter is $Rad(Rat({}_{H^*}H^*))$. But ϕ establishes an isomorphism between ${}_{H^*}H$ and $Rat({}_{H^*}H^*)$. Then $\phi(Rad({}_{H^*}H)) = Rad(Rat({}_{H^*}H^*))$. Hence $1 \in Rad(H)$.

(ii) $\Rightarrow$ *(iii)* If $1 \in Rad(H)$, then $H_0 \subseteq Rad(H)H_0$. By Proposition 4.3, $Rad(H)H_0 = Rad(H)$.

(iii) $\Rightarrow$ *(iv)* Let $\{T_i : i \in I\}$ be a full set of representative of simple right H-comodules. As a right H-comodule, H may be decomposed as $H_H = \oplus_{i\in I} E(T_i)^{(n_i)}$ where $\{n_i : i \in I\}$ is a family of finite cardinal numbers. Then $Rad(H) = \oplus_{i\in I} Rad(E(T_i))^{(n_i)}$. If T_i is injective for some $i \in I$, then $T_i = E(T_i)$ and $Rad(E(T_i)) = \{0\}$. Hence T_i is contained in H_0 but not in $Rad(H)$.

(iv) $\Rightarrow$ *(i)* If no simple is injective, then $k1$ is not injective and so H is not cosemisimple. ∎

Remark 4.5 The equivalence *(iii)* $\Leftrightarrow$ *(iv)* holds for any coalgebra H. Keeping notation, assume that no simple right H-comodule is injective. Then $Rad(E(T_i)) \neq \{0\}$ for every $i \in I$. If $Rad(E(T_i)) = \{0\}$ for some i, then $E(T_i)$ is semisimple. Since it is indecomposable, it is simple. Then $T_i = E(T_i)$ is injective, a contradiction. As $Rad(E(T_i)) \neq \{0\}$ it contains T_i. Hence $H_0 \subseteq Rad(H_H)$.

5 The head of an injective comodule

In this section we describe the head of an injective indecomposable comodule over a co-Frobenius Hopf algebra. Recall that the *head* of a right H-comodule M is the semisimple H-comodule $M/Rad(M)$. The main result of this section is obtained in [11, Proposition 1] for the Hopf algebra of rational functions of a virtually linearly reductive affine group scheme. This result is in turn inspired by the finite group scheme case, see [17, I, Chapter 8]. The arguments used in the aforementioned result easily extend to co-Frobenius Hopf algebras as we show next. The fact that a co-Frobenius Hopf algebra has bijective antipode is what makes the proof also works in this case. We will closely follow the proof of [11, Proposition 1] up to some modifications.

Lemma 5.1 *Let H be a co-Frobenius Hopf algebra. Let $[?,?] : H \times H \to k$ be the bilinear form defined by $[h, h'] = \langle \int_l, hh' \rangle$ for all $h, h' \in H$. Then:*

(i) $[h \cdot S^*(\varphi), h'] = [h, h' \cdot \varphi]$ *for all* $\varphi \in H^*$.

(ii) $[?,?] : H \times H \to k$ *is non-singular.*

Proof: *(i)* Since $\int_l$ is a left integral, $\langle \int_l, hh' \rangle 1 = \sum \langle \int_l, h_{(2)} h'_{(2)} \rangle h_{(1)} h'_{(1)}$ for all $h, h' \in H$. Now we have:

$$\begin{aligned}
[h \cdot S^*(\varphi), h'] &= \sum \langle \textstyle\int_l, h_{(2)} h' \rangle \langle \varphi, S(h_{(1)}) \rangle \\
&= \sum \langle \textstyle\int_l, h_{(2)} h'_{(3)} \rangle \langle \varphi, h'_{(1)} S(h_{(1)} h'_{(2)}) \rangle \\
&= \sum \langle \textstyle\int_l, h h'_{(2)} \rangle \langle \varphi, h'_{(1)} \rangle \\
&= [h, h' \cdot \varphi].
\end{aligned}$$

(ii) Let $B = \{b \in H : [b, l] = 0 \ \forall l \in H\}$. Let $\psi \in H^*$ and $b \in B$. Since S is bijective, $\psi = S^*(\varphi)$ for some $\varphi \in H^*$. Using (i), $[b \cdot \psi, h'] = [b, h' \cdot \varphi] = 0$. Hence $b \cdot \psi \in B$. So B is a right H^*-submodule of H and thus a left coideal of H. Take $h \in H$ such that $[1, h] \neq 0$. For $b \in B$ arbitrary,

$$[1, h]b = \sum [\varepsilon(b_{(2)})1, h] b_{(1)} = \sum [b_{(2)} S(b_{(3)}), h] b_{(1)} = \sum [b_{(2)}, S(b_{(3)}) h] b_{(1)} = 0.$$

Hence $b = 0$. ∎

In [23, Proposition 3] it is proved that there is a group-like element $g \in H$ such that $\int_l h^* = \langle h^*, g \rangle \int_l$ for all $h^* \in H^*$. This means that the map $\varphi : H \to kg, h \mapsto \langle \int_l, h \rangle g$ is right H-colinear. Such an element g is called the *distinguished group-like element* of H.

Theorem 5.2 *Let H be a co-Frobenius Hopf algebra. For a simple right H-comodule T_i the head of $E(T_i)$ is isomorphic to $kg \otimes T_i^{**}$.*

Proof: Let $\{e_i : i \in I\}$ be a complete set of primitive orthogonal idempotents of H^* such that $H_H = \oplus_{i \in I} H \cdot e_i$ and ${}_H H = \oplus_{i \in I} e_i \cdot H$. Applying S to the latter decomposition we obtain a decomposition of right H-comodules $H = \oplus_{i \in I} S(e_i \cdot H) = \oplus_{i \in I} H \cdot (S^{-1})^*(e_i)$. For $i \in I$ we write $E_i = H \cdot e_i$ and $E'_i = H \cdot (S^{-1})^*(e_i)$. Take $i, j \in I$ with $i \neq j$ and $h, h' \in H$. By the foregoing lemma, $0 = [(h \cdot e_i) \cdot e_j, h'] = [h \cdot e_i, h' \cdot (S^{-1})^*(e_j)]$. Thus, by restriction, we have a non-singular pairing $[?, ?] : E_i \times E'_i \to k$ for each $i \in I$. Using that $\int_l h^* = \langle h^*, g \rangle \int_l$ for all $h^* \in H^*$, it is not difficult to verify that the map $\Psi : E_i \to kg \otimes E_i'^*, h_i \mapsto g \otimes \Phi(h_i)$ where $\Phi(h_i)(h) = [h_i, h]$ for $h \in E'_i$, is a morphism of right H-comodules. Since $[?, ?]$ is non-singular, Ψ is an isomorphism.

Let $cf(T_i)$ be the coefficient space of the simple right H-comodule $T_i = H_0 \cdot e_i$. Then $T'_i = e_i \cdot H_0$ is a simple left $cf(T_i)$-comodule. Since $Soc(E'_i) = S(e_i \cdot H_0) = S(T'_i)$, the coefficient space of T_i^* and $S(T'_i)$ is $S(cf(T_i))$. Hence $S(T'_i) \cong T_i^*$ and so $Soc(E'_i) \cong T_i^*$. Then $E'_i \cong E(T_i^*)$. From the isomorphism in the preceding paragraph, $E_i \cong kg \otimes E(T_i^*)^*$. Let H_i be the head of E_i. Then $H_i \cong kg \otimes Soc(E(T_i^*))^* \cong kg \otimes T_i^{**}$. ∎

Corollary 5.3 *Let H be a co-Frobenius Hopf algebra. Then $H/Rad(H) \cong H_0$ as a right (or left) H-comodule.*

Proof: Let $H_0 = \oplus_{j\in J} T_j$ be a decomposition of the coradical into simple right comodules. We know that $H = \oplus_{j\in J} E(T_j)$. Then $Rad(H) = \oplus_{j\in J} Rad(E(T_j))$ and we have:

$$H/Rad(H) \cong \oplus_{j\in J} E(T_j)/Rad(E(T_j)) \cong \oplus_{j\in J} kg \otimes T_j^{**} \cong kg \otimes (\oplus_{j\in J} T_j^{**}).$$

Since the antipode of H is bijective, $H_0 \cong \oplus_{j\in J} T_j^{**}$. Finally, it is easy to check that the map $\mu : kg \otimes H_0 \to H_0,\ g \otimes h \mapsto gh$ is an isomorphism of right H-comodules. ∎

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