Towards an Operational Interpretation of Fuzzy Measures*

Fernando Reche and Antonio Salmerón
Dpt. Statistics and Applied Mathematics
University of Almería
La Cañada de San Urbano s/n
04120 Almería (SPAIN)
e-mail: freche@stat.ualm.es, asc@stat.ualm.es

Abstract

In this paper we propose an operational interpretation of general fuzzy measures. On the basis of this interpretation, we define the concept of coherence with respect to a partial information, and propose a rule of inference similar to the natural extension [1].

Keywords: Fuzzy measure, partial information, coherence, extension.

1 Introduction

Two main problems arise when applying fuzzy measures in practical applications. One is the lack of a clear understanding about the meaning of the measures; until now, no consensus has been reached about what the numbers mean. The other problem is that typically it is not possible to get a complete specification of the value of the measure for all the subsets in the domain, but for a reduced number of them.

Here we propose an operational interpretation of fuzzy measures in order to give a clear meaning to the numbers. This interpretation leads to a straightforward definition of coherence, and to a natural rule of inference that will allow us to make predictions about the value of the measure in the sets where it is unknown.

We start off proposing an operational interpretation and an example in section 2. In section 3 we define the concept of *partial information* and coherence, which will be the basis of an inference rule called *extension*, analyzed in section 4. An algorithm for computing extension is given in section 5. We have implemented this algorithm to carry out some trials that are described in section 6. The paper ends with conclusions in section 7.

*This work has been supported by CICYT under project TIC97-1135-C04-02.

2 An Operational Interpretation

A fuzzy measure [2] is a mapping $\mu : \Omega \to [0,1]$ verifying the following properties:

1. $\mu(\emptyset) = 0$ and $\mu(\Omega) = 1$.
2. $A \subseteq B \subseteq \Omega \Rightarrow \mu(A) \leq \mu(B)$.

In all of this paper we shall consider Ω to be a finite set of categorical data.

Now, assume an experiment whose possible outcomes are the elements in the power set of Ω, 2^Ω. Assume also that number 1.0 represents the total amount of resources available to the realization of the experiment, and that it coincides with the amount of resources consumed if the result of the experiment is the entire set Ω. In these conditions, for any $A \subseteq \Omega$, $\mu(A)$ can be regarded as the fraction of resources consumed if the result of the experiment is A.

Let us illustrate it with an example. Imagine there is a vehicle covering the connection between the harbor and the railway station in a city. This vehicle has four compartments: one for a car, one for a van, one for a motorbike and another one for a bike. Assume that the gas tank of this vehicle has exactly the capacity necessary to carry the vehicle with the four compartments busy from the harbor to the railway station. Then we can regard this capacity to be equal to 1 unit. In this example, $\Omega = \{c, v, m, b\}$, where c stands for *car compartment busy*, v for *van compartment busy*, m for *motorbike compartment busy* and b for *bike compartment busy*. Assume also that the vehicle does not start the trip unless at least one of the compartments is busy. All the possible transportation situations are then the elements in 2^Ω. In these conditions, for every $A \subseteq \Omega$, $\mu(A)$ can be interpreted as the proportion of gas consumed if A happens. Note that this interpretation avoids ambiguity, since the resources can be exactly and objectively measured.

3 Partial Information and Coherence

As we pointed out before, in many situations it can be difficult to get a complete specification of the measure. For instance, in the very small example in the above section, we would need to specify 14 values. This number grows exponentially in the size of Ω.

However, it can be feasible to obtain the measure for some subsets of Ω. In this case we say that we have a *partial information* over Ω. The formal definition is as follows:

Definition 1 (Partial information) Let Ω be a finite set of categorical data. A *partial information* over Ω is a pair (X, σ), where X is a proper subset of 2^Ω and σ is a mapping $\sigma : X \to [0,1]$.

The following definition imposes a restriction to make a partial information be coherent with the interpretation of a fuzzy measure.

Definition 2 (Coherent partial information) We say that a partial information (X, σ) over Ω is *coherent* if and only if for every $A, B \in X$ such that $A \subseteq B$, it holds that $\sigma(A) \leq \sigma(B)$.

In the transportation vehicle example, the concept of coherence means that the fraction of resources consumed if two compartments are occupied may not be lower than if just one of them is occupied.

Example 1 . *Consider again the transportation vehicle case. The following is a coherent partial information over* $\Omega = \{c, v, m, b\}$*:*

$$X = \{\{c\}, \{b\}, \{c, v\}, \{c, v, b\}\} \ ,$$

$$\sigma(\{c\}) = 0.3, \ \sigma(\{b\}) = 0.1, \ \sigma(\{c, v\}) = 0.6, \ \sigma(\{c, v, b\}) = 0.7 \ .$$

4 Extension of a Partial Information

Once we have characterized the coherence of a partial information, it would be desirable to define a rule to make inferences about the measure in the sets for which no information is available, that inference being compatible with the partial information and with the operational interpretation. The key point here is the concept of *compatibility*, that we formally define in this way:

Definition 3 (Compatible fuzzy measure) We say that a fuzzy measure μ over Ω is *compatible* with a coherent partial information (X, σ), if for every $A \in X$, $\mu(A) = \sigma(A)$.

It is clear that many fuzzy measures can be compatible with a given coherent partial information.

The concept of compatibility allows to make inferences about the measure of the sets that are not elements of X. This inference should produce, for each set not in X, an interval where any measure compatible with (X, σ) must lie. To achieve this, we define the next two measures:

Definition 4 (Lower compatible measure) Let (X, σ) be a coherent partial information. We define the *lower compatible measure* with respect to (X, σ) as

$$\mu_*(A) = \begin{cases} \max\limits_{\substack{B \in X \\ B \subseteq A}} \{\sigma(B)\}, & \text{if } \exists B \in X \text{ such that } B \subseteq A \ , \\ \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

for all $A \subseteq \Omega$.

Definition 5 (Upper compatible measure) Let (X,σ) be a coherent partial information. We define the *upper compatible measure* with respect to (X,σ) as

$$\mu^*(A) = \begin{cases} \min\limits_{\substack{B\in X \\ A\subseteq B}} \{\sigma(B)\}, & \text{if } \exists B \in X \text{ such that } A \subseteq B \ , \\ \\ 1 & \text{otherwise.} \end{cases} \quad (2)$$

for all $A \subseteq \Omega$.

Observe that if $A \in X$, then $\mu_*(A) = \mu^*(A) = \sigma(A)$.

With this, we can define the concept of *extension* of a coherent partial information, that will produce the minimum interval for each set where the measure will lie, with the only restriction of coherence. In other words, extension is intended to be the maximum inference we can make from a coherent partial information with the only restriction of coherence.

Definition 6 (Extension) Given a coherent partial information (X,σ), we define its *extension* as the pair of measures (μ_*, μ^*), where μ_* and μ^* are as defined above.

Example 2 *Consider the coherent partial information in example 1. Applying extension for making inference about, say* $\{c, b\}$*, would produce the interval* $[0.3, 0.7]$*. It means that every measure* μ *compatible with* (X,σ) *must verify* $0.3 \leq \mu(\{c,b\}) \leq 0.7$*.*

Proposition 1 *Given a fuzzy measure* μ *compatible with a coherent partial information* (X,σ)*, then for every* $A \in 2^\Omega$*,* $\mu_*(A) \leq \mu(A) \leq \mu^*(A)$*.*

Proof: We shall distinguish two cases:

- If $A \in X$, by definition of compatible measure, we have that $\mu_*(A) = \sigma(A) = \mu(A)$.
- If $A \notin X$ we have two possibilities:

  a) If $\exists B \in X$ such as $B \subseteq A$, by definition of fuzzy measure $\mu(B) \leq \mu(A)$ $\forall B \subseteq A$. By definition of lower compatible measure, $\mu_*(A) = \max\{\sigma(B) \mid B \subseteq A\}$, which is equal to $\max\{\mu(B) \mid B \subseteq A\}$ since μ is compatible with (X,σ). Thus, $\mu_*(A) \leq \mu(A)$.

  b) If $\forall B \subseteq A$, $B \notin X$, $\mu_*(A) = 0 \leq \mu(A)$.

The proof is analogous for upper compatible measures. □

Some interesting cases of fuzzy measures compatible with a coherent partial information are measures based on *averaging operators*.

An averaging operator [3] is a function with the following properties:

- Idempotency: $T(x,x) = x$.

- Monotonicity: If $x \leq x'$ and $y \leq y'$ then $T(x,y) \leq T(x',y')$.
- Commutativity: $T(x,y) = T(y,x)$.

Proposition 2 *Let (X,σ) be a compatible partial information over Ω, and (μ_*,μ^*) its extension. Let μ be a mapping over Ω, defined as*

$$\mu(A) = T(\mu_*(A), \mu^*(A)) \quad A \subseteq \Omega \ , \tag{3}$$

with T an averaging operator. Then μ is a fuzzy measure compatible with (X,σ)

Proof: First we prove that μ is a fuzzy measure.

By idempotency, $\mu(\emptyset) = T(\mu_*(\emptyset),\ \mu^*(\emptyset)) = T(0,0) = 0$ and $\mu(\Omega) = T(\mu_*(\Omega), \mu^*(\Omega)) = T(1,1) = 1$.

If $B \subseteq A$, clearly $\mu_*(B) \leq \mu_*(A)$ and $\mu^*(B) \leq \mu^*(A)$. This, together with monotonicity of the averaging operator, implies that $\mu(B) = T(\mu_*(B), \mu^*(B)) \leq T(\mu_*(A), \mu^*(A)) = \mu(A)$. Thus, μ is a fuzzy measure.

Besides, since T is idempotent, for all $A \in X$, $\mu(A) = T(\mu_*(A), \mu^*(A)) = T(\sigma(A), \sigma(A)) = \sigma(A)$. Thus, μ is compatible with (X,σ). □

As a consequence, this kind of operators can be used to obtain fuzzy measures compatible with a partial information.

5 An Algorithm for Computing the Extension

In this section we present an algorithm for computing the extension for any given set $A \subseteq \Omega$. For a more efficient arrangement of the computations, we shall make use of the lattice representation of 2^Ω. Figure 1 displays the lattice representation corresponding to the transportation vehicle example.

First of all, we must fix some notation. For any $A \subseteq \Omega$, we shall denote by $\Pi(A)$ the set of direct predecessors of A in the lattice, and by $\Lambda(A)$ the set of direct successors of A in the lattice, considering that Ω is the top and $\emptyset$ the bottom. For instance, it can be checked in Fig.1 that $\Pi(\{c,v\}) = \{\{c,v,m\},\{c,v,b\}\}$ and $\Lambda(\{c,v\}) = \{\{c\},\{v\}\}$.

Now assume we want to compute, for instance, $\mu^*(A)$ for $A \subseteq \Omega$. It could be done by asking to each set B in $\Pi(A)$ for its value $\mu^*(B)$ and then take $\mu^*(A) = \min\{\mu^*(B) \mid B \in \Pi(A)\}$. Analogously, to compute $\mu_*(A)$ it would be enough to know $\mu_*(B)$ for every $B \in \Lambda(A)$ and then taking $\mu_*(A) = \max\{\mu_*(B) \mid B \in \Lambda(A)\}$. This facts allow the specification of a single algorithm to compute the extension of a set $A \subseteq \Omega$ based on two recursive procedures. More precisely, the algorithm can be written as follows, where A is a subset of Ω and (X,σ) a coherent partial information over Ω:

EXTENSION(A,Ω,X,σ**)**
$\pi_*(A) =$**LOWER(**A,Ω,X,σ**)**;
$\pi^*(A) =$**UPPER(**A,Ω,X,σ**)**;
Give $[\pi_*(A), \pi^*(A)]$ as the extension of A.

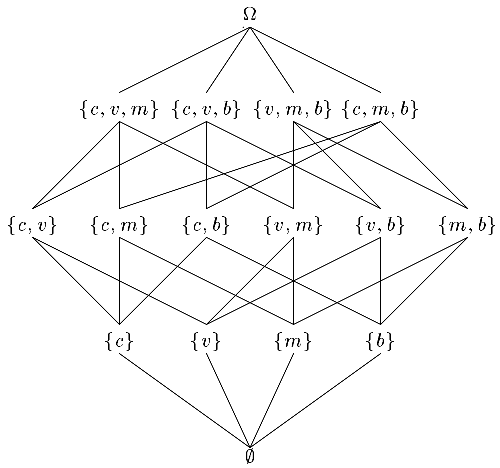

Figure 1: Lattice representation of the power set of $\Omega = \{c, v, m, b\}$.

In the algorithm above, **LOWER** and **UPPER** are the next two recursive procedures:

LOWER(A,Ω,X,σ**)**
if $A = \emptyset$ **then return** 0;
else
if $A \in X$ **then return** $\sigma(A)$;
else
$M := 0$;
for each B in $\Lambda(A)$ **do**
$N :=$**LOWER(**B,Ω,X,σ**)**;
if $N > M$ **then** $M := N$;
return M;

which returns the value of the lower compatible measure of A, and

UPPER(A,Ω,X,σ**)**
if $A = \Omega$ **then return** 0;
else
if $A \in X$ **then return** $\sigma(A)$;
else
$M := 1$;
for each B in $\Pi(A)$ **do**
$N :=$**UPPER(**B,Ω,X,σ**)**;
if $N < M$ **then** $M := N$;
return M;

which returns the upper compatible measure of A.

6 Experimental Evaluation

In this section we present the results of an experimental evaluation of the algorithm. The aim of this experimentation is to show the amplitudes of the intervals produced by the extension algorithm for some randomly generated partial informations.

We have considered four experiments: the first one with 7 elements in Ω and the other ones with 8, 9 and 10 respectively. In each experiment, we have randomly generated 500 partial informations with $|X| = 0.1 \times |2^\Omega|$ (i.e. the cardinal of X being a 10 percent of the cardinal of 2^Ω), 500 with $|X| = 0.2 \times |2^\Omega|$, 500 with $|X| = 0.3 \times |2^\Omega|$ and 500 with $|X| = 0.4 \times |2^\Omega|$. For each partial information, we have computed its extension and the average amplitude of the intervals produced.

The results of the experiments are displayed in table 1 and in figures 2 and 3.

	Percentage of subsets in X			
$\lvert\Omega\rvert$	10%	20%	30%	40%
10	0.2607108	0.1600303	0.1121037	0.08076393
9	0.2940818	0.1795447	0.1250788	0.08986469
8	0.3372416	0.2080419	0.1397554	0.1008586
7	0.3900888	0.2403648	0.1649506	0.118424

Table 1: Experimental results.

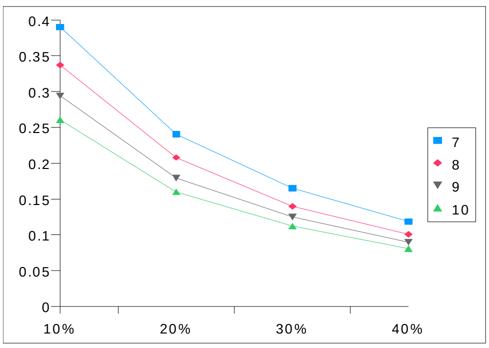

Figure 2: Average amplitude vs. percentage of subsets in X.

We can see in figure 2 how the average amplitude quickly decreases as the percentage of sets for which some information is provided grows. Also, the

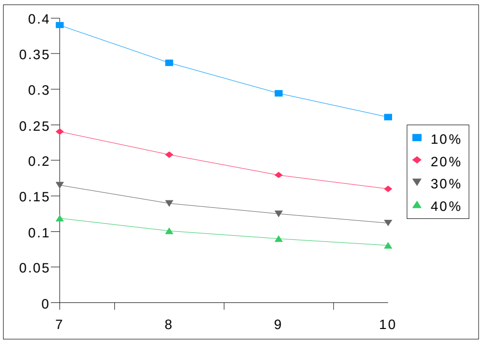

Figure 3: Average amplitude vs. $|\Omega|$.

average amplitude decreases as the number of elements in Ω increases (see figure 3) for a fixed percentage of sets in X, but not so quickly as in the previous case.

7 Conclusions

In this paper we have proposed an operational interpretation of general fuzzy measures. The aim is to provide a clear meaning to the numbers, being this meaning completely objective. We think that this interpretation can avoid misunderstandings that are quite frequent in the use of fuzzy measures.

On the basis of this interpretation, a concept of coherence can be defined. By coherence we understand the minimum restriction that one must impose to every partial information in such a way that it does not violate the interpretation we formulate. This comes up to match with the concept of monotonicity of a fuzzy measure.

About the extension of a partial information, it is the maximum inference we can do based only in the restriction of coherence. In that sense, it is similar to the concept of natural extension [1].

Many more concepts are to be studied in further works. For instance, how the combination of some measures can be performed under the restriction of coherence.

Acknowledgments

We want to thank Professors M. Jorge Bolaños and Serafín Moral for their valuable comments on this paper.

References

[1] Walley, P. (1991) *Statistical reasoning with imprecise probabilities.* Chapman and Hall.

[2] Wang, Z, and Klir, G. (1992) *Fuzzy measure theory.* Plenum Press.

[3] Yager, R. (1996) On mean type aggregation. *IEEE Transactions on Systems, Man, and Cybernetics* **26**, 209-221.