Double distributive law: (Co)wreath over (Co)ring

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Abstract

Let $\iota : R \to \mathfrak{C}$ be a unital ring extension. That is the three-tuple $(\mathfrak{C}, \mu, \iota)$ form a R-ring (i.e. monoid in the monoidal category of R-bimodules) where $\mu : \mathfrak{C} \otimes_R \mathfrak{C} \to \mathfrak{C}$ denotes the multiplication of $\mathfrak{C}$. Assume that there are two R-bilinear maps $\Delta : \mathfrak{C} \to \mathfrak{C} \otimes_R \mathfrak{C}$ and $\varepsilon : \mathfrak{C} \to R$ such that the three-tuple $(\mathfrak{C}, \Delta, \varepsilon)$ is a R-coring (i.e. comonoid in the monoidal category of R-bimodules). An R-bilinear map $\hbar : \mathfrak{C} \otimes_R \mathfrak{C} \to \mathfrak{C} \otimes_R \mathfrak{C}$ is said to be a *double distributive law* provided the following equalities

$$\begin{aligned}
\hbar \circ (\iota \otimes_R \mathfrak{C}) &= \mathfrak{C} \otimes_R \iota & (1)\\
\hbar \circ (\mu \otimes_R \mathfrak{C}) &= (\mathfrak{C} \otimes_R \mu) \circ (\hbar \otimes_R \mathfrak{C}) \circ (\mathfrak{C} \otimes_R \hbar) & (2)\\
\hbar \circ (\mathfrak{C} \otimes_R \iota) &= \iota \otimes_R \mathfrak{C} & (3)\\
\hbar \circ (\mathfrak{C} \otimes_R \mu) &= (\mu \otimes_R \mathfrak{C}) \circ (\mathfrak{C} \otimes_R \hbar) \circ (\hbar \otimes_R \mathfrak{C}) & (4)\\
(\mathfrak{C} \otimes_R \varepsilon) \circ \hbar &= \varepsilon \otimes_R \mathfrak{C} & (5)\\
(\mathfrak{C} \otimes_R \Delta) \circ \hbar &= (\hbar \otimes_R \mathfrak{C}) \circ (\mathfrak{C} \otimes_R \hbar) \circ (\Delta \otimes_R \mathfrak{C}) & (6)\\
(\varepsilon \otimes_R \mathfrak{C}) \circ \hbar &= \mathfrak{C} \otimes_R \varepsilon & (7)\\
(\Delta \otimes_R \mathfrak{C}) \circ \hbar &= (\mathfrak{C} \otimes_R \hbar) \circ (\hbar \otimes_R \mathfrak{C}) \circ (\mathfrak{C} \otimes_R \Delta). & (8)
\end{aligned}$$

It is easily seen that $\Big(\mathfrak{C} \otimes_R \mathfrak{C}, (\mathfrak{C} \otimes_R \hbar \otimes_R \mathfrak{C}) \circ (\Delta \otimes_R \Delta), \varepsilon \circ (\mathfrak{C} \otimes_R \varepsilon)\Big)$ and $\Big(\mathfrak{C} \otimes_R \mathfrak{C}, (\mu \otimes_R \mu) \circ (\mathfrak{C} \otimes_R \hbar \otimes_R \mathfrak{C}), (\iota \otimes_R \mathfrak{C}) \circ \iota\Big)$ are, respectively, R-coring and R-ring. It is shown that the following conditions are equivalent:

(i) Δ and ε are morphisms of R-rings;

(ii) μ and ι are morphisms of R-corings;

(iii) Δ, ε, μ and ι satisfy:

(a) $\Delta \circ \iota \;=\; \iota \otimes_R \iota$
(b) $(\mu \otimes_R \mu) \circ (\mathfrak{C} \otimes_R \hbar \otimes_R \mathfrak{C}) \circ (\Delta \otimes_R \Delta) \;=\; \Delta \circ \mu$
(c) $\varepsilon \circ \iota \;=\; R$
(d) $\varepsilon \circ \mu \;=\; \varepsilon \otimes_R \varepsilon$.

If one of these conditions is satisfied, we then refere to the 6-tuple $(\mathfrak{C}, \Delta, \varepsilon, \mu, \iota, \hbar)$ a *bimonoid* in the category of R-bimodules. For instance R is in trivial way a bimonoid in the category of R-bimodules, and any bialgebra is in fact a bimonoid in the category of modules taking the flip map as the double distributive law.

We prove the above result in the (strict) monoidal categories framework using for this the general notion of wreath and co-wreath (or Extended distributive laws).

References

[1] J. Beck, *Distributive laws*: In Seminar on Triples and Categorical Homology Theory. Lecture Notes in Math., vol. 80, Springer-Verlag, 1969, pp. 119–140.

[2] Gabriella Böhm and Tomasz Brzeziński, *Pre-torsors and equivalences.* arXiv:math. QA/0607529 (2006).

[3] L. El Kaoutit, *Extended distributive law: Co-wreath over co-rings*, arXiv:math.RA/0612818 (2006).

[4] L. El Kaoutit, *Compatibility condition between ring and coring*, arXiv:math.RA/0701652 (2007).

[5] S. Lack and R. Street, *The formal theory of comonad II*, J. Pure Appl. Algebra **175** (2002), 243-265.