We extend the notion of conjugacy classes and class sums from finite groups to semisimple Hopf algebras and show that the conjugacy classes are obtained from the factorization of $H$ as an irreducible left $D(H)$-module. For quasitriangular semisimple Hopf algebras $H$ we prove that the product of two class sums is an integral combination of the class sums up to $1/d^2$ where $d = \text{dim}(H)$. We show also that in this case the character table is obtained from the $S$-matrix associated to $D(H)$. 
Conjugacy classes and class sums for Hopf algebras

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Based on work of
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Along this lecture the base field \( k \) is assumed to be algebraically closed of characteristic 0. \( H \) is a semisimple algebra of dimension \( d \). We denote by \( S \) and \( s \) the antipodes of \( H \) and \( H^* \) respectively. We denote by \( \Lambda \) the (2-sided) idempotent integral of \( H \).

Let \( \{V_1, \ldots, V_n\} \) be a full set of non-isomorphic irreducible left \( H \)-modules of respective dimension \( d_j \) and corresponding characters \( \chi_j \). We have:

\[
V_i \otimes V_j = \sum_{i,j}^l m_{ij}^l V_l,
\]

where \( m_{ij}^l \) are non-negative integers.

The character algebra \( R(H) \) of \( H \) is the \( k \)-span of all the characters on \( H \). In fact \( \{\chi_1, \ldots, \chi_n\} \) form a basis for \( R(H) \). By Kac (1972) and Zhu (1994) - \( R(H) \) is a semisimple algebra with involution.
By Larson (1971), the bilinear form defined on the ring of characters by

\[(\chi_i, \chi_j) = \dim_k \text{hom}_H(V_i, V_j) = m_{ij}^1 = \langle \chi_is(\chi_j), \Lambda \rangle\]

satisfies -

**The irreducible characters are orthogonal with respect to this form.**

This will be applied in two directions:

(i) \(R(H)\) is a symmetric algebra with a symmetric form \(\beta\) defined by

\[\beta(p, q) = \langle \Lambda, pq \rangle\]

and a Casimir element

\[\sum_{k=1}^{n} \chi_k \otimes s(\chi_k)\]

(ii) Define an inner product on \(R(H)\) as follows: For \(u = \sum \alpha_i \chi_i,\ v = \sum \beta_j \chi_j\), set

\[(u, v) = \sum \alpha_i \beta_i\]
Nichols and Richmond (1998) discussed various properties of that inner product. If we define an involution \(*\) by

\[ \chi^* = s(\chi) \]

and extend it to \( u = \sum \alpha_i \chi_i \in R(H) \) by

\[ u^* = \sum \bar{\alpha}_i \chi_i^* \]

Then

\[ (uv, w) = (v, u^* w) = (u, wv^*) \]

Let \( \{F_1, \ldots, F_m\} \) be the set of central primitive idempotents of \( R(H) \). Then [NR] proved the following:

\[ F_i^* = F_i \]

and for all \( x \in R(H) \), \( \mu \) a character defined on \( R(H) \),

\[ \mu(x^*) = \overline{\mu(x)} \]

These results will be used later to prove that certain matrices are unitary.
For $1 \leq i \leq m$, define the **Class sum**

$$C_i = dF_i \to \Lambda.$$  

**Claim:** The irreducible character $\mu_i$ of $R(H)$ corresponding to $F_i$ can be identified inside $Z(H)$ by

$$\mu_i = q_i C_i, \quad q_i \in \mathbb{Q}.$$  

**Proof** For $x \in R(H)$,

$$\mu_i(x) = \text{Trace}(L_x F_i)$$

Let $\{f_1, \ldots, f_m\}$ be a complete set of primitive orthogonal idempotents in $R(H)$ so that $f_i F_j = \delta_{ij} f_i$.

The Casimir element of $R(H)$ satisfies

$$\sum \chi_i \otimes \chi_i^* = \sum_{k=1}^{n} \chi_k \otimes s(\chi_k) = \sum_j n_j F_j$$

where

$$n_j = \frac{d \dim(C(H) f_j)}{\dim(H^* f_j)}$$
The result follows now from the trace formula for symmetric algebras. The coefficient $q_i$ is given by:

$$q_i = \frac{\dim(C(H)f_i)}{\dim(H^*f_i)}$$

As a result we obtain that:

$$<\chi_i^*, C_j> = <\chi_i, C_j>$$
Recall the left adjoint action of $H$ on itself,

$$h_{ad}x = \sum h_1 x S(h_2)$$

Then

$$\Lambda_{\cdot ad}H = Z(H).$$

(When $H$ is not semisimple then $\Lambda_{\cdot ad}H$ is a proper ideal of $Z(H)$ - the Higman ideal.

We have also left coadjoint action of $H$ on $H^*$, $\triangleright$ given by:

$$\triangleright p = \sum h_2 \triangleright p \leftarrow Sh_1$$

When $H$ is semisimple then

$$\Lambda \triangleright H^* = R(H).$$

Recall the Frobenius map $\Psi : H \to H^*$ defined as:

$$\Psi(h) = \lambda \leftarrow S(h).$$
We show that $\Psi$ an $H$-module map from $(H, \dot{ad})$ to $(H^*, \triangleright)$ in the sense that

$$\Psi(h \dot{ad} a) = h \triangleright \Psi(a).$$

Note, $\dot{ad}$ makes $H$ into an $H$-module algebra, while $\triangleright$ does not make $H^*$ into an $H$-module algebra. However, it has some nice properties.

(i) For all $h \in H$, $p \in R(H)$, $x \in H^*$,

$$h \triangleright (px) = p(h \triangleright x)$$

If moreover $h \in \text{Coc}(H)$, then

$$h \triangleright (xp) = (h \triangleright x)p.$$

Define the **conjugacy class** $C_i$ as:

$$C_i = \Lambda \leftarrow f_i H^*.$$

Then by the properties of $\Psi$ mentioned above it is not hard to see that:

* $C_i$ is stable under the adjoint action of $H$.*

By definition, $C_i$ is stable under the right hit action of $H^*$ on $H$. 
It is known that $H$ is a left module over $D(H)$ where the $H^*$ part acts by right hit and the $H$ part acts by the left adjoint action. We can show that:

$C_i$ is a $D(H)$-submodule of $H$.

But more is true,

**Theorem [CW]:** Let $H$ be a semisimple Hopf algebra and let $\{f_1, \ldots, f_m\}$ be idempotents in $R(H)$ so that $\{f_i R(H)\}$ is the complete set of non-isomorphic irreducible $R(H)$-modules. Assume $\dim(f_i R(H)) = m_i$. Then $C_i$ is an irreducible $D(H)$-module and moreover,

$$H \cong \bigoplus_{i=1}^n C_i^{m_i}$$

as $D(H)$-modules.

The proof is based on the properties of the Frobenius maps $\Psi$. 
When $R(H)$ is commutative the central primitive idempotents $\{F_j\}$ form a basis of $R(H)$, hence the class sums $\{C_j\}$ where $C_j = \Lambda \leftarrow dF_j$, form a basis for $Z(H)$.

One can check that

$$< F_j, \Lambda > = \frac{\dim(F_jH^*)}{d},$$

hence by definition

$$\{F_i\} \text{ and } \left\{ \frac{C_j}{\dim(F_jH^*)} \right\} \text{ are dual bases.}$$

We can define a character table for $H$ as follows:

$$\xi_{ij} = \frac{1}{\dim(F_jH^*)} < \chi_i, C_j >$$

The dual bases imply that the character table is actually the change of bases matrix $A$ from $\{\chi_i\}$ to $\{F_j\}$. 
Recall that for groups the character table is defined by $\xi_{ij} = \chi_i(g)$ for some $g$ in the conjugacy class $C_j$. Hence $\chi_i(g) = \chi_i(C_j^{j/|C_j|})$. Thus the definition extends the definition of character tables for groups.

In [CW,3.1] we proved that the inverse change of bases matrix $(\beta_{jk}) = A^{-1}$ satisfies

$$\beta_{jk} = \frac{\dim(F_jH^*)}{d} \alpha_{k^*j}^*,$$

By using this we can show first and second orthogonality relations (as for groups). That is,

(a) $\sum_j \dim(F_jH^*) \xi_{mj} \xi_{nj}^* = \delta_{mn}d.$

(b) $\sum_m \xi_{mi} \xi_{mj}^* = \delta_{ij} \frac{d}{\dim(F_iH^*)}$

By [NR], $\xi_{ij}^* = \overline{\xi_{ij}}$, thus the character table is “almost” unitary.
When $H$ is a factorizable Hopf algebra we have the Drinfeld map $f_Q : H^* \to H$, which is an algebra isomorphism between $R(H)$ and $Z(H)$. In particular, for any primitive idempotent $F$ of $R(H)$, $f_Q(F) = E$ is a primitive central idempotents of $H$.

Reorder the set $\{F_j\}$ so that for all $1 \leq j \leq m$,

$$f_Q(F_j) = E_j$$

Recall [CW] that for semisimple factorizable Hopf algebra we have:

$$f_Q(\chi_j) = \frac{1}{d_j} C_j.$$

It follows that the $S$-matrix satisfies

$$s_{ij} = \langle \chi_i, f_Q(\chi_j) \rangle = \frac{1}{d_j} \langle \chi_i, C_j \rangle = \frac{\dim(F_j H^*)}{d_j} \xi_{ij}$$

Since $\dim(F_j H^*) = \dim(E_j H) = d_j^2$, we obtain

$$s_{ij} = d_j \xi_{ij}$$
Hence

\[ s_{i^*j} = \overline{s_{ij}} \]

Thus we obtain the result of [ENO, 2005]

For a factorizable semisimple Hopf algebra, the $S$-matrix (multiplied by $\frac{1}{\sqrt{d}}$) is unitary.

Unlike for groups, the structure constants for the product of two class sums are not necessarily integers. We can prove integrability up to $d^2$ in case $H$ is quasitriangular. In this case, $H$ is a Hopf image of $D(H)$ which is a factorizable Hopf algebra. Denote this map by $\Phi$.

The images of the $F_i$'s under $\Phi^* : H^* \to D(H)^*$, are sums of primitive idempotents in $R(D(H))$, and thus induce a partition $\{I_s\}$ on their indexes. All class sums of $D(H)$ belonging to the same $I_s$ are mapped under $\Phi$ to the corresponding class sum of $H$ with a certain coefficient.
On the other hand, if \( \{ \hat{E}_i \}_{i=1}^m \) is the set of central primitive idempotents of \( D(H) \) then

\[
\Phi(\hat{E}_i) = \begin{cases} 
E^i & 1 \leq i \leq n, \\
0 & n + 1 \leq i \leq m 
\end{cases}
\]

We use these and the fact that \( D(H) \) is factorizable to prove:

**Let \( H \) be a quasitriangular Hopf algebra. Then the product of two class sums is an integral combination up to a factor of \( d^{-2} \) of the class sums of \( H \).**

The character table of a quasitriangular Hopf algebra \( H \) is strongly related to the \( S \)-matrix of \( D(H) \). We show:

If \( (H, R) \) is quasitriangular and \( (\xi_{si}) \) is its character table, then \( \xi_{si} = d_i^{-1} s_{ij} \) for all \( i \in I_s \), where \( s_{it} \) arise from the \( S \)-matrix of \( D(H) \).

The factor \( d^{-2} \) can not be avoided as will be demonstrated in the next example - the character table of \( D(kS_3) \).
Conjugacy classes of $S_3$ are given by:

$$C_1 = \{1\} \quad C_{(12)} = \{(12), (13), (23)\}$$

$$C_{(123)} = \{(123), (132)\}$$

The centralizers are given by:

$$C_G(1) = S_3 \quad C_G(12) = \{1, (12)\} \cong \mathbb{Z}_2$$

$$C_G(123) = \{1, (123), (132)\} \cong \mathbb{Z}_3$$

For $\sigma = 1$ we have 3 irreducible representations of $S_3$.

- $M_1$ is the trivial representation of $S_3$.

- $M_2$ is the sign representation of $S_3$.

- $M_3$ is the 2 irreducible dimension of $S_3$ with $\chi_{M_3}(123) = -1$, $\chi_{M_3}(12) = 0$. 
For $\sigma = (12)$ we have two representations:

- $M_4$ is the trivial representation of $\mathbb{Z}_2$.

- $M_5$ the unique non-trivial representation of $\mathbb{Z}_2$.

For $\sigma = (123)$ we have 3 representations:

- $M_6$ is the trivial representation of $\mathbb{Z}_3$.

- $M_7$ is the representation with $\chi_{M_7}(123) = \omega$, $\chi_{M_7}(132) = \omega^2$, $\omega$ a third root of unity.

- $M_8$ is the representation with $\chi_{M_8}(123) = \omega^2$, $\chi_{M_8}(132) = \omega$. 
We can compute now the $S$-matrix which is actually well known known (e.g [BK]). Finally, Denote $\frac{1}{d_i}C_i$ by $\eta_i$. Then the generalized character table of $D(kS_3)$ is given by

\[
\begin{pmatrix}
\eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 & \eta_6 & \eta_7 & \eta_8 \\
\chi_1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\chi_2 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\
\chi_3 & 2 & 2 & 2 & 0 & 0 & -1 & -1 & -1 \\
\chi_4 & 3 & -3 & 0 & 1 & -1 & 0 & 0 & 0 \\
\chi_5 & 3 & -3 & 0 & -1 & 1 & 0 & 0 & 0 \\
\chi_6 & 2 & 2 & -1 & 0 & 0 & 2 & -1 & -1 \\
\chi_7 & 2 & 2 & -1 & 0 & 0 & -1 & -1 & 2 \\
\chi_8 & 2 & 2 & -1 & 0 & 0 & -1 & 2 & -1 \\
\end{pmatrix}
\]

We can check that:

$$\chi_4\chi_5 = \chi_2 + \chi_3 + \chi_6 + \chi_7 + \chi_8$$

Hence

$$C_4C_5 = 9f_Q(\chi_4\chi_5) = 9C_2 + \frac{9}{2}C_3 + \ldots$$