Suppose that $H$ is a finite dimensional semisimple Hopf algebra over an algebraically closed field whose characteristic does not divide the dimension of $H$. We shall assume that for any positive integer $d > 1$ any two irreducible $H$-modules of dimension $d$ are isomorphic. The category of left $H$-modules $\mathcal{M}_H$ is a monoidal category. In the talk we shall discuss Clebsch-Gordan coefficients in decompositions in $\mathcal{M}_H$ of tensor products of irreducible $H$-modules. Some classifications results are obtained in the case when there exists up to an isomorphism a unique irreducible $H$-module of dimension greater than 1.
Semisimple Hopf algebras

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In the talk we consider a problem of a classification up to an isomorphism of semisimple finite dimensional Hopf algebras $H$ over an algebraically closed field $k$. We shall assume that either $\text{char } k = 0$ or $\text{char } k > \text{dim } H$. 
Dual Hopf algebras

If $H$ has finite dimension then the dual space $H^*$ is again a Hopf algebra with *convolutive* multiplication $l_1 * l_2$, comultiplication $\Delta^*$, counit $\varepsilon^*$ and an antipode $S^*$ which are defined as follows:

$$l_1 * l_2 = \mu \cdot (l_1 \otimes l_2) \cdot \Delta,$$
$$\Delta^*(l)(x \otimes y) = l(xy),$$
$$(S^* l)(x) = l(S(x)), \quad \varepsilon^*(l) = l(1)$$

for all $x, y \in H$. 
Group-like elements

An element $g \in H$ is a group-like element if $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$. The set $G(H)$ of all group-like elements is a multiplicative group in $H$. Elements of $G(H^*)$ of group-like elements in the dual Hopf algebra $H^*$ are just algebra homomorphisms $H \rightarrow k$. 
There are left and right actions $H^* \rightarrow H$, $H \leftarrow H^*$ of $H^*$ on $H$ defined as follows: if $f \in H^*$, $x \in H$ and

$$\Delta(x) = \sum_x x_1 \otimes x_2 \in H \otimes H$$

then

$$f \rightarrow x = \sum_x x_1 \langle f, x_2 \rangle, \quad x \leftarrow f = \sum_x \langle f, x_1 \rangle x_2$$

In particular if $g \in G(H^*)$ then $g \rightarrow x$, $x \leftarrow g$ are algebra automorphisms of $H$. 
Direct decomposition of $H$

In the talk we shall assumed that for any $d > 1$ there exists at most one irreducible $H$-module of dimension $d$. It means that $H$ as a semisimple $k$-algebra has a decomposition

$$H = \left( \bigoplus_{g \in G} ke_g \right) \oplus \text{Mat}(d_1, k) \oplus \cdots \oplus \text{Mat}(d_n, k),$$

(1)

$$1 < d_1 < \cdots < d_n,$$

where $\{e_g \mid g \in G\}$ is a system of central orthogonal idempotents associated with $k$-algebra homomorphisms $g : H \to k$. 


Irreducible $H$-modules

Let $E_g, \ g \in G$, be the one-dimensional $H$-module associated with $g \in G$. It means that $hx = \langle h, g \rangle x$ for any $h \in H$. The number of 1-dimensional non-isomorphic $H$-modules $E_g, \ g \in G$, is equal to the order of $G$. Denote by $M_1, \ldots, M_n$ irreducible $H$-modules of dimensions $1 < d_1 <, \cdots < d_n$, respectively. It can be shown that each module $M_i$ is equipped with a non-degenerated (skew-)symmetric bilinear form $\langle x, y \rangle_i$ such that $\langle hx, y \rangle_i = \langle x, S(h)y \rangle_i$ for all $x, y \in M_i$ and for all $h \in H$. 
Each matrix component $\text{Mat}(d_i, k)$ in $H$ from (1) is invariant under the antipode $S$. Let $U_i$ be the Gram matrix of the bilinear form $\langle x, y \rangle_i$ in some base of $M_i$.

**Proposition**

$S(x) = U_i^t x U_i^{-1}$ for any $x \in \text{Mat}(d_i, k)$.

**Proposition**

*For any $i$ there exists a faithful projective representation $\Phi_i$ of the group $G$ in $M_i$ such that*

\[
g \leadsto h = \Phi_i(g) h \Phi_i(g)^{-1}, \quad h \longleftarrow g = S(\Phi_i(g)) h S(\Phi_i(g))^{-1}
\]

*for any $h \in \text{Mat}(d_i, k)$. Moreover the group commutator $[\Phi_i(g), S(\Phi_i(f))] = 1$ in $\text{PGL}(M_i)$ for all $f, g \in G$.***
Proposition

If \( g \in G \) then there are \( H \)-module isomorphisms

\[
E_g \otimes M_i \simeq M_i \otimes E_g \simeq M_i,
\]

\[
M_i \otimes M_j \simeq \delta_{ij} \left( \bigoplus_{g \in G} E_g \right) \oplus \left( \bigoplus_{t=1}^n m_{ij}^t M_t \right),
\]

where \( m_{ij}^t \geq 0 \in \mathbb{Z} \). In particular

\[
d_i d_j = \delta_{ij} |G| + \sum_t m_{ij}^t d_t, \quad |G| \leq d_1^2, \quad m_{ij}^s = m_{js}^i = m_{ji}^s.
\]
We can identify the space $M_i \otimes M_i$ with the space of matrices $\text{Mat}(d_i, k)$ using the bilinear form $\langle x, y \rangle_i$. Namely if $a, b, c \in M_i$ then $a \otimes b$ is the linear operator on $M_i$ such that

$$(a \otimes b)c = a\langle b, c \rangle_i \in M_i.$$ 

**Proposition**

*Under this identification the image of the one-dimensional module $E_g$ in $M_i \otimes M_i$ coincides with the linear span of $t^1 \Phi_i(g)^{-1}$. Choosing a special base in $M_i$ we can show that the span is equal to $S(\Phi_i(g)^{-1})$.***
We can associate with $H$ an oriented graph $\Gamma_H$ whose vertices are indices $\{1, \ldots, n\}$ of irreducible $H$-modules $M_1, \ldots, M_n$. Two vertices $i, j$ are connected by an edge $i \to j$ if $m_{tj}^i > 0$ for some $t = 1, \ldots, n$. In other terms the module $M_i$ occurs in $M_t \otimes M_j$ for some index $t$.

**Proposition**

Suppose that there is no edge $i \to j$ in $\Gamma_H$. Then $i = j = 1$ and $|G| = d_1^2$. Moreover $J = \oplus_{j \geq 2} \text{Mat}(d_j, k)$ is a Hopf ideal in $H$ and $H/J$ is the Hopf algebra from Theorems 7 and 8.
Suppose that there exists an index $1 \leq i \leq n$ such that for any index $j \neq i$ there exists a unique edge $i \to j$. If $i = 1$, then $J = \bigoplus_{j \geq 2} \text{Mat}(d_j, k)$ is a Hopf ideal in $H$ and $H/J$ is the Hopf algebra from Theorems 7 and 8. If $i = n$, then $n = 1$. 

**Theorem (V.A. Artamonov, R.B. Mukhatov, R. Wisbauer)**
Theorem

Let $H$ be a semisimple bialgebra with decomposition (1) where $n \geq 2$. Then $m^t_{n-1,n} \geq 2$ for some index $t = 1, \ldots, n$. 
The antipode $S$

Each matrix constituent $\text{Mat}(d_q, k)$ in (1) is stable under the antipode $S$. Moreover $S^2 = 1$ and $S(e_g) = e_{g-1}$ for any central idempotent $e_g$ from (1).

**Theorem**

*If the group $G$ is nilpotent then taking an isomorphic copy of each matrix component in (1) we can assume that the matrices $\Phi_i(g), S(\Phi_i(g))$ are monomial.*
Theorem

Let $H$ be a semisimple Hopf algebra with semisimple decomposition (1). Suppose that there exists a matrix constituent $\text{Mat}(d_i, k)$ which is a Hopf ideal in $H$. Then $n = 1$. 
Elements $\mathcal{R}_q$

Denote by $\mathcal{R}_q$ the element

$$\mathcal{R}_q = \frac{1}{d_q} \sum_{i,j=1}^{d_q} E_{ij}^{(q)} \otimes E_{ji}^{(q)}$$

in $\text{Mat}(d_q, k)^{\otimes 2}$. Here $E^{(q)}_{**}$ are matrix units from $\text{Mat}(d_q, k)$. The element $\mathcal{R}_q$ is the unique element in $\text{Mat}(d_q, k)^{\otimes 2}$ up scalar multiple such that

$$(A \otimes B)\mathcal{R}_q = \mathcal{R}_q(B \otimes A)$$

for all $A, B \in \text{Mat}(d_q, k)$. 
Theorem

Let $G$ be a finite group whose order is coprime with $\text{char } k$. A projective representation $\Omega : G \rightarrow \text{PGL}(d, k)$ such that

$$\Omega(g^{-1}) = \Omega(g)^{-1}, \quad \Omega(E) = E,$$

is irreducible if and only if

$$R_d = \frac{1}{|G|} \sum_{g \in G} \Omega(g^{-1}) \otimes \Omega(g).$$
Let $g \in G$ and $x \in \text{Mat}(d_r, k)$. Put $\Delta_q = (1 \otimes S)\mathcal{R}_q$. Then $\varepsilon(e_g) = \delta_{1,g}$, $\varepsilon(x) = 0$ and

$$
\Delta(e_g) = \sum_{f \in G} e_f \otimes e_{f^{-1}g} + \sum_{t=1,\ldots,n} (1 \otimes (g \rightarrow )) \Delta_t,
$$

$$
\Delta(x) = \sum_{g \in G} [(g \rightarrow x) \otimes e_g + e_g \otimes (x \leftarrow g)] + \sum_{i,j=1}^n \Delta^r_{ij}(x),
$$

where $\Delta^r_{ij}(x) \in \text{Mat}(d_i, k) \otimes \text{Mat}(d_j, k)$.
Proposition

For indices $i, j$ the following are equivalent:

1. there exists an edge $i \to j$;
2. $\Delta^i_{tj} \neq 0$ for some $t$;
3. $\Delta^t_{ij} \neq 0$ for some $t$. 
Hopf algebras with $n = 1$ were considered by several authors. If the order of $G$ has maximal possible value $d_1^2$ then the group $G$ is Abelian. In the paper


Hopf algebra $H$ is classified using monoidal category of its representations in terms of bicharacters of the group $G$. 
If $d_1 = 2$ then there exist up to equivalence four classes of Hopf algebras $H$, namely group algebras of Abelian groups of order 8, group algebras of dihedral group $D_4$ and of quaternions $Q_8$, and G. Kac Hopf algebra $H$ generated by elements $x, y, z$ with defining relations

\[ x^2 = y^2 = 1, \ xy = yx, \ zx = yz, \ zy = xz, \]

\[ z^2 = \frac{1}{2}(1 + x + y - xy), \]

\[ \varepsilon(z) = 1, \ S(z) = z^{-1}, \]

\[ \Delta(z) = \frac{1}{2} \ ((1 + y) \otimes 1 + (1 - y) \otimes x) \ (z \otimes z), \]

and $x, y$ are group-like elements.
Interesting results were obtained by


Let $H$ be a semisimple Hopf algebra of dimension $2p^2$, where $p$ is an odd integer. Then either $H$ has a semisimple decomposition (1) with $n = 1$, $d_1 = p$ and $|G| = p^2$ or $H$ is its dual and it has a semisimple decomposition with $2p$ one-dimensional components and $\frac{p(p-1)}{2}$ components isomorphic to Mat(2, $k$).
Theorem (Artamonov V.A., 2009 — 2010)

Let \( H \) be from (1) with \( n = 1 \) and \( G = G(H^*) \). The order of \( G \) is divisible by \( d_1 \) and is a divisor of \( d_1^2 \).

The following conditions are equivalent.

1. The order of \( G \) is equal to \( d_1^2 \).
2. \( \Delta_{11}^1 = 0 \) in Theorem 6.
3. \( \Phi_1 \) is an irreducible projective representation of \( G \) in \( M_1 \).
Under these restrictions $H = \left( \bigoplus_{g \in G} k e_g \right) \oplus \text{Mat}(d_1, k)$ and $\varepsilon(e_g) = \delta_{1,g}$, $\varepsilon(x) = 0$ where $x \in \text{Mat}(d_1, k)$. Moreover

$$
\Delta(e_g) = \sum_{f \in G} e_f \otimes e_{f^{-1}g} + \frac{1}{d_1} \sum_{i,j=1}^{d_1} E_{ij} \otimes \left( g^{-1} \mapsto \text{S}(E_{ji}) \right),
$$

$$
\Delta(x) = \sum_{g \in G} \left[ \left( \Phi_1(g)x\Phi_1(g)^{-1} \right) \otimes e_g \right.
$$

$$
+ e_g \otimes \left( \text{S}(\Phi_i(g)) x\text{S}(\Phi_i(g)^{-1}) \right). \]
$$

Let $H$ be from (1), $n = 1$ and $G = G(H^*)$. If $\Delta_{11}^1 = 0$ then $G = A \times A$ for some Abelian group $A$ of order $d_1$. 

Suppose that $G$ is Abelian group of order $d^2$ with direct decomposition $G \simeq A \times A$ for some Abelian group $A$ of order $d$. The group $G$ has a faithful irreducible projective representation $\Phi$ of degree $d$. There exists a (skew-)symmetric matrix $U \in \text{GL}(d, k)$ such that $[\Phi(g), S(\Phi(f))] = 1$ in $\text{PGL}(d, k)$ for all $f, g \in G$. Here $S(x) = U^txU^{-1}$ for any $x \in \text{Mat}(d, k)$. Then an algebra $H$ with direct decomposition (1) admits Hopf algebra structure defined in Theorem 6.

There is a group isomorphism $G \simeq G(H^*)$.  

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There is a group isomorphism $G \simeq G(H^*)$.  


Theorem (Puninsky E., 2009)

*Under the assumption of Theorem 8 G(H) is a cyclic group of order $2d_1$, provided $d_1$ is an odd prime.*
Theorem (Artamonov V.A., Chubarov I.A., 2008)

Let $n = 1$, $d_1 > 2$ and $H$ from Theorem 8. Then $H^*$ is not isomorphic to any Hopf algebra belonging to the class of Hopf algebras from Theorem 8.
Previous results use

**Theorem (R. Frucht, J. Reine Angew. Math. 166 (1932), 16-29)**

Let $G$ be a finite Abelian group of and let $k$ be an algebraically closed field such that $\text{char } k$ does not divide the order of $G$. The group $G$ admits a faithful irreducible projective representations of dimension $d$ over $k$ if and only if $G$ is a direct product of two isomorphic groups of order $d$. Dimensions of any irreducible projective representations of the group $G$ are equal either to $d$ or to 1.
Theorem (E. M. Jmud, 1972)

A finite abelian group $G$ of order $d^2$ has decomposition $G \cong A \times A$ if and only if it admits a non-degenerate bilinear symmetric form. Any irreducible projective representation of $G$ of degree $d$ is obtained from another one by an automorphism of $G$. 