Galois theory for Hopf-Galois extensions
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For a comodule algebra over a Hopf algebra over arbitrary commutative ring we construct a Galois correspondence between the complete lattices of subalgebras and the complete lattice of generalised quotients of the structure Hopf algebra. The construction involves techniques of lattice theory and of Galois connections. Such a 'Galois Theory' generalises the classical Galois Theory for field extensions, and some important results of S. Chase and M. Sweedler, F. van Oystaeyen, Y.H. Zhang and P. Schauenburg.
Galois Theory for Hopf Galois Extensions

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**Definition (Hopf Galois Extension)**

Let $H$ be a Hopf algebra (over a ring $R$), let $A$ be an $H$-comodule algebra, i.e.

- algebra over $R$,
- (left) $H$-comodule: $\delta : A \to H \otimes A$,
- $\delta$ is an algebra homomorphism.

$$\text{co}^H_A := \{ a \in A : \delta(a) = 1_H \otimes a \}$$ the subalgebra of coinvariants.

$A/\text{co}^H_A$ is $H$-Hopf Galois iff the canonical map:

$$\text{can} : A \otimes_{\text{co}^H_A} A \to H \otimes A, \quad a \otimes b \mapsto \delta(a)b$$

is an isomorphism (in $^H\text{Mod}_A$).
Galois Connections

Let \((P, \succeq_P), (Q, \succeq_Q)\) be two posets. A pair of antimonotonic maps:

\[
P \leftrightarrow_Q Q
\]

is called a **Galois connection** if

\[
\phi \circ \psi \succeq id_P \text{ and } \psi \circ \phi \succeq id_Q
\]

An element \(p \in P \) (\(q \in Q\)) is called **closed** if

\[
\phi \psi (p) = p \quad \psi \phi (q) = q
\]

We let \(\cl P, \cl Q\) denote the subsets of **closed elements**.

**Properties:**

1. \(\cl P = \psi (Q)\) and \(\cl Q = \phi (P)\),
2. \(\phi|_{\cl P}\) and \(\psi|_{\cl Q}\) are inverse bijections between \(\cl P\) and \(\cl Q\),
3. \(\phi\) is mono (epi) iff \(\psi\) is epi (mono),
4. \(\phi\) (\(\psi\)) is isomorphism then \(\psi\) (\(\phi\)) is its inverse.
The lattice of generalised quotients

**Definition**

Let $H$ be a Hopf algebra. Then

$$\text{Quot}_{gen}(H) := \{ H/J : J - \text{coideal left ideal} \}$$

is a poset, with partial order: $H/I \geq H/J \iff I \subseteq J$

The poset $\text{Id}_{gen}(H)$ is a lattice, i.e. there exists finite infima and suprema: for $J_1, J_2 \in \text{Id}_{gen}(H)$.

$$J_1 \vee J_2 = J_1 + J_2, \quad J_1 \wedge J_2 = \bigoplus \begin{cases} J \subseteq J_1 \cap J_2 & J \in \text{Id}_{gen}(H) \\ J \subseteq J_1 \cap J_2 & \end{cases}$$

**Proposition**

The lattice $\text{Quot}_{gen}(H)$ is complete, i.e. there exists arbitrary suprema and infima.
Modules with intersection property

Definition

Let $M, N$ be an $R$-modules, and let $(N_\alpha)_{\alpha \in I}$ be a family of submodules of an $R$-module $N$. If $M$ is flat then there exists the canonical map:

$$M \otimes_R \left( \bigcap_{\alpha \in I} N_\alpha \right) \longrightarrow \bigcap_{\alpha \in I} (M \otimes_R N_\alpha)$$

We say that a flat module $M$ has the intersection property with respect to $N$ if the above homomorphism is an isomorphism for any family of submodules $(N_\alpha)_{\alpha \in I}$.

We say that $M$ has intersection property if the above condition holds for any $R$-module $N$. 
Proposition

Every projective module has the intersection property.

Example

Let $p$ be a prime ideal of the ring $\mathbb{Z}$. Then $\bigcap_i p^i = \{0\}$. Let $\mathbb{Z}_p$ denote the localisation of $\mathbb{Z}$ at the prime ideal $p$.

The $\mathbb{Z}$-module $\mathbb{Z}_p$ is flat. Then $\mathbb{Z}_p \otimes_{\mathbb{Z}} \bigcap_i p^i = \{0\}$. From the other side $\bigcap_i \mathbb{Z}_p \otimes_{\mathbb{Z}} p^i = \bigcap_i \mathbb{Z}_p = \mathbb{Z}_p$.

By similar argument $\mathbb{Q} \otimes_{\mathbb{Z}}$ — doesn’t posses the intersection property.

The reason for this is that, the intersection property is stable under arbitrary sums but not under cokernels.
Theorem
Every flat $R$-module has the intersection property if and only if for any exact sequence:

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

with $M'$, $M$ projective, $M''$ flat and any family of submodules $(N_\alpha)_{\alpha \in I}$ of an $R$-module $N$ the sequence:

$$0 \rightarrow \bigcap_{\alpha} (M' \otimes_R N_\alpha) \rightarrow \bigcap_{\alpha} (M \otimes_R N_\alpha) \rightarrow \bigcap_{\alpha} (M'' \otimes_R N_\alpha) \rightarrow 0$$

is exact.
**Theorem**

Let $A/B$ be an $H$-comodule algebra over a ring $R$, such that one of the following conditions is satisfied:

- every finitely generated $R$-submodule of $A$ is projective and $A$ has the intersection property with respect to $H$, or
- $A$ is projective as an $R$-module.

Then there exists a **Galois connection**:

\[ \begin{array}{ccc}
\text{Sub}_{\text{Alg}}(A) & \overset{\psi}{\leftarrow} & \text{Quot}_{\text{gen}}(H) \\
& \text{co } Q_A & \text{by } Q
\end{array} \]

where

\[ \psi(A') = \bigvee \{ Q \in \text{Quot}_{\text{gen}}(H) : A' \subseteq A^{\text{co } Q} \} \]
**Theorem** (F. van Oystaeyen and Y. Zhang)

Let $k \subseteq \mathbb{F}$ be a field extension and let $H$ be a commutative and cocommutative $k$-Hopf algebra. Let $\mathbb{F} \subseteq \mathbb{E}$ be a field extension and an $H$-Hopf–Galois extension.

Then there is a **one-to-one correspondence**:

\[
\left\{ \begin{array}{c}
H\text{-subcomodule subfields of } \mathbb{E} \\
\text{Hopf ideals of } \mathbb{F} \otimes_k H
\end{array} \right\} \cong \left\{ \begin{array}{c}
\text{Hopf ideals of } \mathbb{F} \otimes_k H
\end{array} \right\}
\]

If $I$ – a Hopf ideal, and $\mathbb{M}$ – an intermediate field extension of $\mathbb{E}/\mathbb{F}$, which corresponds to each other then

\[
\mathbb{E}/\mathbb{M} \text{ is } \mathbb{F} \otimes_k (H/I)\text{-Hopf–Galois.}
\]
For a right $H$-Hopf Galois extensions $A$ of the base ring $R$ there exists a Hopf algebra $L(H, A)$ such that $A$ becomes $L(H, A)$-$H$-biGalois extension. As an algebra $L(H, A)$ is a subalgebra of $A \otimes A^{op}$ given by $(A \otimes A)^{coH}$.

**Theorem**

Let $R \subseteq A$ be a faithfully flat $H$-Hopf–Galois extension of a ring $R$. Then there is the following **Galois connection**:

\[
\begin{array}{ccc}
\text{Sub}_{\text{Alg}}(A) & \xrightarrow{\psi} & \text{Quot}_{\text{gen}}(L(H, A)) \\
\end{array}
\]

\[
\begin{array}{ccc}
Q & \xleftarrow{\text{co} Q_A} & \text{Quot}_{\text{gen}}(L(H, A)) \\
\end{array}
\]

\[
\psi(B) = (A \otimes_B A)^{coH}
\]

It restricts to bijection between:

- $Q \in \text{Quot}_{\text{gen}}(L(H, A))$ such that $L(H, A)$ is (right, left) faithfully flat over $Q$, and $Q$ is flat $R$-module;
- $B \in \text{Sub}_{\text{Alg}}(A)$ such that $A$ is (right, left) faithfully flat $B$ module.
Proposition

Let $A$ be an $H$-comodule algebra with surjective canonical map:

$$A \otimes_{co H_A} A \xrightarrow{can} H \otimes A$$

Let $Q_1, Q_2 \in \text{Quot}_{gen}(H)$ such that $A/\text{co}Q_1A$ and $A/\text{co}Q_2A$ are $Q$-Galois extensions, then:

$$\text{co}Q_1A = \text{co}Q_2A \iff Q_1 = Q_2$$

Proof.
**Corollary**

Let $A$ be an $H$-comodule algebra with surjective canonical map. Let $Q \in \text{Quot}_{\text{gen}}(H)$ be such that $A/\text{co}Q\,A$ is $Q$-Galois extension. Then $Q$ is a closed element in:

$$\text{Quot}_{\text{gen}}(H) \leftrightarrow \text{Sub}_{\text{alg}}(A/\text{co}H\,A)$$

**Note:** if $H$ is finite dimensional then every $Q \in \text{Quot}_{\text{gen}}(H)$ is closed and thus:

$$\varphi : \text{Quot}_{\text{gen}}(H) \rightarrow \text{Sub}_{\text{Alg}}(A)$$

is a monomorphism.
Definition

Let $C$ be a $k$-coalgebra and let $C \rightarrow \tilde{C}$ be a quotient coalgebra. Then $\tilde{C}$ is called left (right) admissible if it is $k$-flat and $C$ is left (right) faithfully coflat over $\tilde{C}$.

Let $S$ belong to $\text{Sub}_{\text{Alg}}(A/B)$ for an $H$-Hopf Galois extension. We call $S$ left (right) $H$-admissible if the following conditions are satisfied:

1. $A$ is left (right) faithfully flat over $S$,

2. $\text{can}_S : A \otimes_S A \rightarrow A \otimes_{\text{co } \psi(S)} A \xrightarrow{\text{can}_{\psi(S)}} \psi(S) \otimes A$ which is well defined since $S \subseteq \text{co } \psi(S)A$, is a bijection, and

3. $\psi(S)$ is flat over $R$.

An element is called admissible if it is both left and right admissible.
**Theorem (P. Schauenburg)**

Let $A/B$ be a faithfully flat $H$-Galois extension over a ring $R$, such that $A$ is projective as an $R$-module.

Then the Galois connection:

\[
\begin{array}{ccc}
\text{Sub}_{\text{Alg}}(A) & \leftrightarrow & \text{Quot}_{\text{gen}}(H) \\
\end{array}
\]

\[\text{co} \mathcal{Q}_A \leftrightarrow \mathcal{Q}\]

gives rise to a bijection between (left, right) admissible objects (thus (left, right) admissible objects are closed).

**Note**: the proof of P. Schauenburg for Hopf Galois extensions of the base ring $R$ works without a change.
The base ring $R$ is now assumed to be a field.

**Definition**

Let $C$ be a $H$-module coalgebra. (the $H$-module structure map is a coalgebra map.) Then $C^H := C/CH^+$ is a quotient coalgebra. We call $C \to C^H$ a **Galois coextension** if the canonical map:

$$ \text{can}_H : C \otimes H \longrightarrow C \Box_{CH} C \quad \text{can}_H(c \otimes h) = c^{(1)} \otimes c^{(2)} h $$

is an isomorphism.

Let $K$ be a right coideal of $H$. Then $C \to C^K := C/CK^+$ is a quotient coalgebra. We call $C/CK$ a **Galois coextension** if the canonical map:

$$ \text{can}_K : C \otimes K \longrightarrow C \Box_{CK} C \quad \text{can}_K(c \otimes k) = c^{(1)} \otimes c^{(2)} k $$

is an isomorphism.
Galois coextensions

**Theorem**

Let $C$ be an $H$-module coalgebra over a field $k$. Then there exists a Galois connection:

$$\begin{align*}
\text{Quot}(C/C^H) & \quad \leftrightarrow \quad \text{cold}_r(H) \\
C/C^I^+ & \quad \leftrightarrow \quad H/I
\end{align*}$$

where $\text{cold}_r(H)$ is a complete lattice of right coideals of $H$, and

$$\text{Quot}(C/C^H) := \left\{ \tilde{C} \in \text{Quot}(C) : C \longrightarrow \tilde{C} \longrightarrow C^H \right\}$$
Proposition

Let $C$ be an $H$-Galois coextension over a field $k$ and $K_1, K_2 \in \text{cold}_r(H)$ such that both:

$$\text{can}_{K_i} : C \otimes K_i \rightarrow C \square_{C^{K_i}} C \quad i = 1, 2$$

are bijections. Then $K_1 = K_2$, whenever $C^{K_1} = C^{K_2}$.

Proof.
Corollary

Let $C$ be an $H$-Galois coextension. Then $K$ - a right coideal of $H$ is a closed element of the Galois connection:

$$\text{Quot}(C/c^H) \iff \text{cold}_r(H)$$

if $C$ is $K$-Galois.
Let $H$ be a Hopf algebra. Then there exists a *Galois connection*:

$$\{K \subseteq H : K - \text{right coideal subalgebra}\} \leftrightarrow_{\psi \phi} \{H/I : I - \text{left ideal coideal}\}$$

$$=: \text{Sub}_{gen}(H)$$

$$=: \text{Quot}_{gen}(H)$$

where $(\varphi(Q) = ^{co}QH, \psi(K) = H/HK^+)$ is the considered Galois connection. This Galois correspondence restricts to normal elements:

$$\{K \subseteq H : K - \text{normal Hopf subalgebra}\} \leftrightarrow_{\psi \phi} \{H/I : I - \text{normal Hopf ideal}\}$$

$$=: \text{Sub}_{nHopf}(H)$$

$$=: \text{Quot}_{normal}(H)$$

1. $K \in \text{Sub}_{gen}(H)$ such, that $H$ is *faithfully flat* over $K$, is a **closed element**.
2. $Q \in \text{Quot}_{gen}(H)$ such, that $H$ is *faithfully coflat* over $Q$, is a **closed element**.
3. if $H$ is *finite dimensional* then $\varphi$ and $\psi$ are *inverse bijections*. 
Whenever $H$ is finite dimensional, for every $Q$ the extension $\text{co}^Q H \subseteq H$ is $Q$-Galois. Hence every $Q$ is closed and the map

$$\Phi : \text{Quot}_{\text{gen}}(H) \longrightarrow \text{Sub}_{\text{gen}}(H), \quad \Phi(Q) = \text{co}^Q H$$

is a monomorphism.

To show that we have a pair of isomorphisms it is enough to prove that

$$\Psi : \text{Sub}_{\text{gen}}(H) \longrightarrow \text{Quot}_{\text{gen}}(H), \quad \Psi(K) = H / HK^+$$

is a monomorphism.

Let us consider $H^*$. To distinguish $\Phi$ and $\Psi$ for $H$ and $H^*$ we will write $\Phi_H$ and $\Psi_H$ for $H$ and $\Phi_{H^*}$ and $\Psi_{H^*}$ considering $H^*$.

It turns out that:

- $(\Psi_H(K))^* = \Phi_{H^*}(K^*)$, i.e. $(H / HK^+)^* = \text{co}^{K^*} H^*$,
- $\text{can}_{K^*} = (\text{can}_K)^* : H^* \otimes_{\text{co}^{K^*} H^*} H^* \rightarrow H^* \otimes K^*$.

But $\text{can}_{K^*}$ is an isomorphism, hence $\text{can}_K$ is an isomorphism for every right coideal subalgebra $K$ of $H$. 

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Theorem
Let $H$ be a Hopf algebra. Then the correspondence:

$$\{ K \subseteq H : K \text{ - right coideal subalgebra} \} \overset{\sim}{\longleftrightarrow} \{ H/I : I \text{ - left ideal coideal} \}$$

is a bijection if and only if

1. for every $Q \in \text{Quot}_{\text{gen}}(H)$, the extension $\text{co}^Q H \subseteq H$ is $Q$-Galois.
2. $\text{co}^{H,K^+} H \subseteq K$ for every right coideal subalgebra $K$ of $H$. 
Cleft Extensions

Theorem (Doi and Takeuchi)
Let $A^{coHA}$ be an $H$-extension. Then

$$A^{coHA} \text{ is cleft} \iff A^{coHA} \text{ is } H\text{-Galois and has the normal basis property}$$

Theorem
An element $Q \in \text{Quot}_{gen}(H)$ is closed in the Galois connection:

$$Q \mapsto ^{coQA}$$

$$\text{Quot}_{gen}(H) \iff \text{Sub}_{alg}(A^{coHA})$$

if and only if the extension $A^{coQA}$ is $Q\text{-Galois}$. 
Thank you.