Computing of the number of right coideal subalgebras of quantum groups

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This talk is based on the joint work with A.V. Lara Sagahon and J.L. Garza Rivera. We use a super computer KanBalam of the UNAM in order to find the total number r_n of the homogeneous right coideal subalgebras containing all group-like elements for the multiparameter versions of the quantum groups $U_q(\mathfrak{so}_{2n+1})$, $q^t \neq 1$ and $u_q(\mathfrak{so}_{2n+1})$, $q^t = 1$, $t > 4$ for small n :

$$r_2 = 38;\ r_3 = 546;\ r_4 = 10696;\ r_5 = 233216;$$

$$r_6 = 6257254;\ r_7 = 178413634.$$

The numerical experiments allow us to conjecture that $n!4^n < r_n < n!n4^n$ for big n. The similar numbers for $U_q(\mathfrak{sl}_{n+1})$ was found in [1]:

$$r_2 = 26;\ r_3 = 252;\ r_4 = 3368;\ r_5 = 58810;$$

$$r_6 = 1290930;\ r_7 = 34604844.$$

Additionally, in the present work, we get $r_8 = 1,107,490,596$. Recall that, in the G_2 case we have $r_2 = 60$; see [2]. For the other types, C, D, E, F, it is already known from a theorem of Heckenberger and Schneider [3] that the similar numbers $r_n^{\rm Borel}$ related to the Borel subalgebras coincide with the order of the corresponding Weyl group W. This implies $r_n < |W|^2$, see [4].

Bibliography

[1] V.K. Kharchenko, A.V. Lara Sagahon, *Right coideal subalgebras in $U_q^+(\mathfrak{sl}_{n+1})$* J. Algebra **319** (2008), 2571–2625.

[2] B. Pogorelsky, *Right coideal subalgebras of the quantized universal enveloping algebra of type G_2*. Comm. Algebra **39** No. 4 (2011), 1181–1207.

[3] I. Heckenberger, H.-J. Schneider, *Right coideal subalgebras of Nichols algebras and the Duflo order on the Weyl groupoid.* Preprint arXiv: 0909.0293, 43 pp.

[4] V.K. Kharchenko, *Triangular decomposition of right coideal subalgebras.* J. Algebra **324** (2010), 3048–3089.

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V.K. Kharchenko
(with A.V.Lara Sagahón and J.L.Garza Rivera)

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Actions of groups

- Kh. (1977). *Let G be a finite group of homogeneous automorphisms of a free algebra $\mathbf{k}\langle X\rangle$. The Galois correspondence*

$$A \longrightarrow \mathbf{k}\langle X\rangle^A = \{f \in \mathbf{k}\langle X\rangle \mid f^a = f,\ a \in A\}$$

is a one-to-one correspondence between all subgroups A of G and all intermediate free subalgebras of $\mathbf{k}\langle X\rangle$.

Actions of Hopf algebras

- Ferreira, Murakami, Paques (2004). *Let H be a finite dimensional Hopf algebra acting homogeneously on a free algebra* $\mathbf{k}\langle X\rangle : \Delta(h) = \sum h_1^i \otimes h_2^i; \quad (xy)^h = \sum x^{h_1^i} y^{h_2^i}$. *The correspondence*

$$U \longrightarrow \{f \in \mathbf{k}\langle X\rangle \mid f^h = \varepsilon(h)f, \ h \in U\}$$

is a one-to-one correspondence between all **right coideal subalgebras** *of H and all intermediate free subalgebras.*

- A. Milinski (1995, 1996), S. Westreich (1999, 2000, 2001), A. Masuoka (2003), D.Fichman, T.Yanai (1997, 2001, 2005).

PBW-bases

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- Kh. (2006). *Let H be a Hopf algebra generated by skew primitive semi invariants. Every right coideal subalgebra that contains all group-like elements has a PBW-basis which can be extended up to a PBW-basis of H.*
- I. Heckenberger, H.-J. Schneider (2009). *The Drinfeld–Jimbo quantum universal enveloping algebra $U_q(\mathfrak{g}^+)$ of a Borel algebra $\mathfrak{g}^+$ has precisely $|W|$ right coideal subalgebras over the coradical, where W is the Weyl group of the semisimple Lie algebra $\mathfrak{g}$.*
- Kh., A.V. Lara Sagahon (2007) case A_n;
  Kh. (2008) case B_n; B. Pogorelsky (2008) case G_2.

Triangular decomposition

- Kh. (2010). *Every right coideal subalgebra U of $U_q(\mathfrak{g})$ has a triangular decomposition $U = U^- \mathbf{k}[G] U^+$, here $\mathfrak{g}$ is a Kac-Moody algebra.*

Triangular decomposition

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- In particular, due to the I. Heckenberger, H.-J. Schneider theorem, $U_q(\mathfrak{g})$ has at most $|W|^2$ r.c.s., provided that $\mathfrak{g}$ is a semisimple Lie algebra.

Probabilities

- If U^+, U^- are right coideal subalgebras of the quantum Borel subalgebras, then $U^-\mathbf{k}[G]U^+$, is a right coideal but not always a subalgebra.

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- Kh., A.V. Lara Sagahon (2007). *The probabilities p_n for a pair U^-, U^+ to define a right coideal subalgebra of $U_q(\mathfrak{g})$, $\mathfrak{g} = \mathfrak{s}l_{n+1}$ are:*

$$p_2 = 72.3\%;\ \ p_3 = 43.8\%;\ \ p_4 = 23.4\%;$$

$$p_5 = 11.4\%;\ \ p_6 = 5.1\%;\ \ p_7 = 2.2\%; p_8 = 0.841\%.$$

Probabilities

- Kh., A.V. Lara Sagahon, J.L. Garza Rivera (2011). *The probabilities p_n for a pair U^-, U^+ to define a right coideal subalgebra of $U_q(\mathfrak{g})$, $\mathfrak{g} = \mathfrak{so}_{2n+1}$ are:*

$$p_2 = 59.4\%;\ \ p_3 = 23.7\%;\ \ p_4 = 7.3\%;$$

$$p_5 = 1.6\%;\ \ p_6 = 0.295\%;\ \ p_7 = 0.043\%.$$

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- B. Pogorelsky (2011). *If $\mathfrak{g}$ is the simple Lie algebra of type G_2, then the probability equals* $60/144 = 41.7\%$.

PBW-generators of the Borel component

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- $\Delta(x_i) = x_i \otimes 1 + g_i \otimes x_i;\ \Delta(g_i) = g_i \otimes g_i;\ x_i g_j = p_{ij} g_j x_i$, where p_{ij} are arbitrary parameters satisfying:

$$p_{nn} = q,\ \ p_{ii} = q^2,\ \ p_{i\,i+1} p_{i+1\,i} = q^{-2},\ \ 1 \leq i < n;$$

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- PBW-genearators are $[u_{km}], k \leq m \leq 2n-k$:
$[\dots[[[[\dots[x_k, x_{k+1}]\cdots x_n,]x_n,]x_{n-1},]x_{n-2},]\cdots x_{2n-m+1}]$,
here $[u,v] = uv - p(u,v)vu$, while the bimultiplicative map $p(u,v)$ is so that $p(x_i, x_j) = p_{ij}$.

PBW-generators of right coideal subalgebras

- $S = \{s_1 < s_2 < \ldots < s_r\}, \quad S \subseteq [1, 2n].$

$$\Phi^S(k,m) = u[k,m] - (1-q^{-2}) \sum_{i=1}^{r} \alpha_{km}^{s_i} \Phi^S(1+s_i, m) u[k, s_i],$$

where $\alpha_{km}^{s} = \tau_s p(u(1+s,m), u(k,s))^{-1}$, while $\tau_s = 1$ for all s except that $\tau_n = q$.

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- We display the element $\Phi^S(k,m)$ schematically:

$k-1$	k	$k+1$	$k+2$	$k+3$		$m-2$	$m-1$	m
○	○	○	●	○	$\ldots$	●	○	●

The main theoretical result

- Let $\Phi^S(k,m)$, $\Phi^T_-(i,j)$ be PBW-generators.

$$\begin{array}{lccccccccc}
 & k-1 & & i-1 & i & i+1 & & m & & j \\
S & \circ & \cdots & \bullet & \bullet & \circ & \cdots & \bullet & & \\
T & & & \circ & \circ & \bullet & \cdots & \bullet & \cdots & \bullet
\end{array} .$$

It is *balanced* if it has no fragments of the form

$$\begin{array}{ccc}
t & & s \\
\circ & \cdots & \bullet \\
\circ & \cdots & \bullet
\end{array}$$

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It is *balanced* if it has no fragments of the form

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- THEOREM. *A triangular composition* $U^-\mathbf{k}[G]U^+$ *is a subalgebra if and only if, for each pair* $\Phi^S(k,m)$, $\Phi^T_-(i,j)$ *all four schemes are balanced or one of them has the form*

$$\begin{array}{cccccccc} t & & & & & & s & \\ \circ & \cdots & \circ & \cdots & \bullet & \cdots & \bullet & \\ \circ & \cdots & \bullet & \cdots & \circ & \cdots & \bullet & . \end{array}$$

Problems and hypothesis

$$B_n: \quad r_2 = 38; \; r_3 = 546; \; r_4 = 10,696; \; r_5 = 233,216;$$

$$r_6 = 6,257,254; \; r_7 = 178,413,634.$$

- The C++ -program: A.V. Lara Sagahón r_2, r_3, r_4, r_5;
- A parallelization: J.L.Garza Rivera r_6 (6 min 128 processors), r_7(22 hours 128 processors) and $r_8(A_n)$(22.3 hours 128 proc.)

$$A_n: \quad r_2 = 26; \; r_3 = 252; \; r_4 = 3,368; \; r_5 = 58,810;$$

$$r_6 = 1,290,930; \; r_7 = 34,604,844; \; r_8 = 1,107,490,596.$$

Problems and hypothesis

- Find the number of right coideal subalgebras of $U_q(\mathfrak{g})$ when $\mathfrak{g}$ is a simple Lie algebra of types F_4, E_6, E_7, E_8.

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$$\lim_{n\to\infty} n!p_n = \infty, \quad \lim_{n\to\infty} (n-1)!p_n = 0.$$

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- A_n :

$$\lim_{n\to\infty} n2^n p_n = \infty, \quad \lim_{n\to\infty} 2^n p_n = 0.$$

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- Due to H-Sch. theorem it is to be expected that there exists a description of right coideal subalgebras of $U_q(\mathfrak{g})$ in terms of Weyl group combinatorics for $\mathfrak{g}$ of arbitrary types A–G.

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- Due to H-Sch. theorem it is to be expected that there exists a description of right coideal subalgebras of $U_q(\mathfrak{g})$ in terms of Weyl group combinatorics for $\mathfrak{g}$ of arbitrary types A–G.
- The given here computational determination of the probabilities p_n (and the numbers r_n) for the types A, B, G will provide a valuable test as soon as a conjecture for such a description is formulated.