Deformations of a class of graded Hopf algebras with quadratic relations
Jiwei He (Shaoxing University, China)
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We consider a special class of graded Hopf algebras, which are finitely generated quadratic algebras with anti-symmetric generating relations. We discuss the automorphism group and Calabi-Yau property of a PBW-deformation of such a Hopf algebra. We show that the Calabi-Yau property of a PBW-deformation of such a Hopf algebra is equivalent to that of the corresponding augmented PBW-deformation under some mild conditions.
Deformations of graded Hopf algebras with quadratic relations

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Hopf algebras and tensor categories
July 4–8, 2011, Almeria
(I) Hopf algebras with quadratic relations

(II) Poincaré-Birkhoff-Witt (PBW) deformation

(III) Calabi-Yau algebras

(IV) Main results
(1) Hopf algebras with quadratic relations
Notions

- We work over an algebraically closed field $\mathbb{k}$ of characteristic zero.
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$$U = T(V)/(R),$$

where $R \subseteq V \otimes V$.
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A **quadratic algebra** is a positively graded algebra $U$ defined as

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The **quadratic dual** of $U$ is defined to be the algebra $U^! = T(V^*)/(R^\perp)$, where $R^\perp$ is the orthogonal complement of $R$ in $V^* \otimes V^*$. 
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Example. The polynomial algebra $U = \mathbb{k}[x_1, \ldots, x_n]$ is a quadratic algebra, its quadratic dual is the exterior algebra $U^! = \bigwedge\{y_1, \ldots, y_n\}$. 

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Deformations of quadratic Hopf algebras
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An antisymmetric element may be written as \( r = x^t M x \), where \( x^t = (x_1, \ldots, x_n) \) and \( M \) is an antisymmetric \( n \times n \)-matrix.

Let \( U = T(V)/(r_1, \ldots, r_m) \) be a quadratic algebra with antisymmetric generating relations \( r_i \in V \otimes V \) for \( 1 \leq i \leq m \). We call such a quadratic algebra \( U \) as a **weakly symmetric** algebra.
An weakly symmetric algebra $U$ is a graded Hopf algebra with coproducts and antipode

$$\Delta(x) = x \otimes 1 + 1 \otimes x,$$

for $x \in V$. 

Example. Let $M$ be an $n \times n$ antisymmetric invertible matrix, and let $r = x^t M x$ where $x^t = (x_1, \ldots, x_n)$. Let $U = I_k \langle x_1, \ldots, x_n \rangle / (r)$. Then

(i) [Dubois-Violette, 2007] $U$ is a Koszul algebra.

(ii) [Berger, 2009] $U$ is a Calabi-Yau algebra of dimension 2.

(iii) [Berger, Bocklandt] Any (connected graded) Calabi-Yau algebra of dimension 2 is obtained in this way.
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(II) PBW-deformations
Let $U = \bigoplus_{n \geq 0} U_n$ be a positively graded algebra. A \textbf{PBW-deformation} of $U$ is a filtered algebra $A$ with filtration $0 \subseteq F_0 A \subseteq F_1 A \subseteq \cdots \subseteq F_n A \subseteq \cdots$, together with a graded algebra isomorphism $p : U \to gr(A)$. 
A PBW-deformation $A$ of a quadratic algebra $U = T(V)/(R)$ is determined by two linear maps:

$$\varphi : R \to V \text{ and } \theta : R \to \mathbb{K},$$

so that

$$A = T(V)/(I_2), \text{ where } I_2 = \{ r - \varphi(r) - \theta(r) | r \in R \}.$$  

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It is more convenient to consider the augmented PBW-deformations than the nonaugmented cases. Especially, when we consider the PBW-deformations of a graded Hopf algebra, we have the tool homological integrals to do with the homological properties of augmented PBW-deformations.
Examples.  (i) A universal enveloping algebra a finite dimensional algebra is an augmented PBW-deformation of a polynomial algebra.

(ii) Weyl algebra $A_1$ is a PBW-deformation of the polynomial algebra $\mathbb{k}[x_1, x_2]$.

(iii) Sridharan enveloping algebras: $\mathfrak{g}$ is a finite dimensional algebra, $f : \mathfrak{g} \times \mathfrak{g} \to \mathbb{k}$ is a 2-cocycle of $\mathfrak{g}$, then

$$U_f(\mathfrak{g}) = T(\mathfrak{g})/I,$$

where the ideal $I$ is generated by

$$x \otimes y - y \otimes x - [x, y] - f(x, y), \text{ for all } x, y \in \mathfrak{g}.$$
Let $U = T(V)/(R)$ be a quadratic algebra, and let $\phi : R \to V$ be a linear map that provides an augment PBW-deformation of $U$. 

Theorem (Polishchuk-Positselski) The dual map $\phi^* : V^* \to R^*$ induces a differential $d$ on the quadratic dual $U^!$ of $U$ so that $(U^!, d)$ is a differential graded algebra. 

Moreover, the set of possible augmented PBW-deformations of $U$ is in one-to-one correspondence with the set of all the possible differential structures on $U^!$. 

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(III) Calabi-Yau algebras
For the background of Calabi-Yau algebra, see Xiaolan Yu’s talk yesterday.

Definition. [Ginzburg] An algebra \( A \) is said to be a Calabi-Yau algebra of dimension \( d \) (CY-\( d \), for short) if:

1. \( A \) is homologically smooth, that is; \( A \) has a bounded resolution of finitely generated projective \( A \)-\( A \)-bimodules,
2. \( \text{Ext}^i_A(A, A_{\text{en}}) = 0 \) if \( i \neq d \) and \( \text{Ext}^d_A(A, A_{\text{en}}) \sim A \) as \( A \)-\( A \)-bimodules, where \( A_{\text{en}} = A \otimes A^{\text{op}} \) is the enveloping algebra of \( A \).

We call \( d \) the Calabi-Yau dimension of \( A \).
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**Definition.** [Ginzburg] An algebra $A$ is said to be a **Calabi-Yau algebra of dimension $d$ (CY-$d$, for short)** if

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(ii) $\text{Ext}^i_{A^e}(A, A^e) = 0$ if $i \neq d$ and $\text{Ext}^d_{A^e}(A, A^e) \cong A$ as $A$-$A$-bimodules, where $A^e = A \otimes A^{op}$ is the enveloping algebra of $A$.

We call $d$ the **Calabi-Yau dimension** of $A$. 

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Deformations of quadratic Hopf algebras
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- An interesting question is to find out the relation between the global dimension and the CY dimension of a CY algebra.
(IV) Main results
Theorem. [Yekutieli] If $A$ is a (positively) filtered algebra such that $gr(A)$ is a Calabi-Yau algebra, then $A$ differs from being Calabi-Yau by a filtration-preserving automorphism $\sigma$: that is, $\text{RHom}_{A^e}(A, A^e) \cong {}^1A^\sigma[d]$. 
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Denote by $\text{Aut}_{\text{filt}}(A)$ the group of automorphisms of $A$ which preserve the filtration of $A$. 
Main results

Theorem (H-Zhang)

Let $U = T(V)/(R)$ be a weakly symmetric algebra, and let $A = T(V)/(r - \varphi(r) : r \in R)$ be an augmented PBW-deformation of $U$. Then $\text{Aut}_{\text{filt}}(A) \cong Z^1(U^!, d)$, where $Z^1(U^!, d)$ is the group of 1-cocycles of the differential graded algebra $(U^!, d)$.

Moreover, if the quadratic algebra $U$ is Koszul then $\text{Aut}_{\text{filt}}(A) \cong \text{Ext}^1_A(A \mathbb{k}, A \mathbb{k})$.
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Moreover, if the quadratic algebra $U$ is Koszul then $\text{Aut}_{\text{filt}}(A) \cong \text{Ext}^1_A(A\mathbb{1}, A\mathbb{1})$.

**Corollary.** [Well known] Any universal enveloping algebra of a finite dimensional semisimple Lie algebra is Calabi-Yau.
Let $U = T(V)/(R)$ be a weakly symmetric algebra, and $\varphi : R \to V$ and $\theta : R \to \mathbb{k}$ be linear maps.

Set

$$I_2 = \{ r - \varphi(r) | r \in R \},$$

$$I'_2 = \{ r - \varphi(r) - \theta(r) | r \in R \}.$$
Let $U = T(V)/(R)$ be a weakly symmetric algebra, and $\varphi : R \to V$ and $\theta : R \to \mathbb{I}_k$ be linear maps. Set

$$l_2 = \{ r - \varphi(r) | r \in R \},$$

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Assume that both $A = T(V)/(l_2)$ and $A' = T(V)/(l'_2)$ are PBW-deformations of $U$.

Define

$$D : T(V) \to A' \otimes A'^{op},$$

$$D(x) = x \otimes 1 - 1 \otimes x, \quad \text{for all } x \in V.$$
A lemma

Let \( U = T(V)/(R) \) be a weakly symmetric algebra, and \( \varphi : R \to V \) and \( \theta : R \to \mathbb{K} \) be linear maps.

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D(x) = x \otimes 1 - 1 \otimes x,
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\( D \) induces an algebra morphism (also denoted by \( D \))

\[
D : A \to A' \otimes A'^\text{op}.
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Lemma. $A' \otimes A'^{\text{op}}$ is projective either as a left $A$-module or as a right $A$-module.
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The key point to prove the lemma is that \( U \) is a graded Hopf algebra. Then \( U \otimes U^{\text{op}} \) is a free module either as a left \( U \)-module or as a right \( U \)-module.
Main results

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Let $U = T(V)/(R)$ be a weakly symmetric algebra. Assume that both $A = T(V)/(r - \varphi(r) : r \in R)$ and $A' = T(V)/(r - \varphi(r) - \theta(r) : r \in R)$ are PBW-deformations of $U$. If $A$ is CY-d, then so is $A'$. 
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Deformations of quadratic Hopf algebras
Theorem (H-Van Oystaeyen-Zhang)

Let \( \mathfrak{g} \) be a finite dimensional Lie algebra. Then for any 2-cocycle \( f \in Z^2(\mathfrak{g}, \mathbb{k}) \), the following statements are equivalent.

(i) The Sridharan enveloping algebra \( U_f(\mathfrak{g}) \) is CY-d.

(ii) The universal enveloping algebra \( U(\mathfrak{g}) \) is CY-d.

(iii) \( \dim \mathfrak{g} = d \) and \( \mathfrak{g} \) is unimodular, that is, for any \( x \in \mathfrak{g} \), \( \text{tr}(\text{ad}_x) = 0 \).
Main results

Theorem (H-Van Oystaeyen-Zhang)

Let $A$ be a noetherian CY filtered algebra of dimension 3 such that $\text{gr}(A)$ is commutative and generated in degree 1, then $A$ is isomorphic to $\mathbb{k}\langle x, y, z \rangle/(R)$ with the commuting relations $R$ listed in the following table:

<table>
<thead>
<tr>
<th>Case</th>
<th>${x, y}$</th>
<th>${x, z}$</th>
<th>${y, z}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$z$</td>
<td>$-2x$</td>
<td>$2y$</td>
</tr>
<tr>
<td>2</td>
<td>$y$</td>
<td>$-z$</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>$z$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>$y$</td>
<td>$-z$</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>$z$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
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where $\{x, y\} = xy - yx$. 
Remarks.

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- The results can be generalized without too much difficulty to the nonquadratic algebras. That is, if the graded Hopf algebra $U$ is $N$-homogeneous with some “anti-symmetric” relations, then the same results still hold.

For example, $U = T(V)/(r)$, where

$$r = \sum_{\sigma \in S_n} \text{sgn}(\sigma)x_{\sigma(1)}x_{\sigma(2)} \cdots x_{\sigma(n)}.$$
Thank you!