Nichols algebras with many cubic relations

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The talk is based on a joint work with A. Lochmann and L. Vendramin. We classify Nichols algebras of irreducible Yetter-Drinfeld modules over groups under the assumption that the underlying rack is braided and the homogeneous component of degree three of the Nichols algebra satisfies a given inequality. This assumption turns out to be equivalent to a factorization assumption on the Hilbert series. Besides the known Nichols algebras, a new example is obtained. The proof is based on a combinatorial invariant of the Hurwitz orbits with respect to the action of the braid group on three strands.

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Philipps-Universität Marburg

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$V=\mathrm{span}_{\Bbbk}\{x_1,x_2\}$, $c(x_i\otimes x_j)=q_{ij}x_j\otimes x_i$, $p,r,\zeta\in\Bbbk^\times$, $(q_{ij})=\begin{pmatrix} p & r \\ p^{-1}r^{-1} & \zeta\end{pmatrix}$, $\zeta^2+\zeta+1=0$, assume $N:=\min\{m\in\mathbb{N}\,|\,(m)_p:=1+p+p^2+\cdots+p^{m-1}=0\}<\infty$.
$\mathcal{B}(V)=TV/(x_1x_{12}-prx_{12}x_1,x_1^N,x_2^3)$,
$x_{12}=x_1x_2-rx_2x_1$.

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known (for diagonal type):

1. PBW type theorem due to Kharchenko
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(Milinski-Schneider, Fomin-Kirillov) $3 \leq n \leq 5$, $G = \mathfrak{S}_n$, $g = (12)$, $G^g = \mathfrak{S}_2 \times \mathfrak{S}_{n-2} \subseteq \mathfrak{S}_n$, $V_g = \Bbbk v$, $(i\,j)v = -v$ for all $(i\,j) \in \mathfrak{S}_2 \times \mathfrak{S}_{n-2}$, $V = M(g, V_g)$.

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$\mathcal{B}(V) = TV/I(V)$, $I(V)$ is the ideal generated by

$$\begin{aligned} v_{(i\,j)}^2, &\qquad |\{i,j\}| = 2, \\ v_{(i\,j)}v_{(k\,l)} + v_{(k\,l)}v_{(i\,j)}, &\qquad |\{i,j,k,l\}| = 4, \\ v_{(i\,j)}v_{(j\,k)} + v_{(j\,k)}v_{(k\,i)} + v_{(k\,i)}v_{(i\,j)}, &\qquad |\{i,j,k\}| = 3. \end{aligned}$$

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(H., Lochmann, Vendramin) $X = (\mathrm{Ad} A_4)(2\,3\,4) \subseteq A_4$ $= \{g_1 = (2\,3\,4), g_2 = (1\,4\,3), g_3 = (1\,2\,4), g_4 = (1\,3\,2)\} \subseteq A_4$, $G = G_X$, $G^{g_1} = \langle g_1, g_2 g_4 \rangle$,
$V = \mathrm{span}_{\mathbb{k}}\{a, b, c, d\}$,
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$\dim \mathcal{B}(V) = 5184$.

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Theorem. (Andruskiewitsch, Graña, '03) Let $V \in {}^G_G\mathcal{YD}$ and let $I \subseteq TV$ be an $\mathbb{N}_0$-graded Hopf ideal of TV in ${}^G_G\mathcal{YD}$ such that $I \cap \Bbbk = I \cap V = 0$. Let $m \in \mathbb{N}_0$. Assume that

$$\dim T^m V/(I \cap T^m V) = 1,$$

$$\dim T^n V/(I \cap T^n V) = 0 \quad \text{for all } n > m.$$

If $\mathcal{B}(V)(m) \neq 0$ then $I = I(V)$.

Racks

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Then $\triangleright : X \times X \to X$ and for all $x, y, z \in X$ we have

1. $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$, and
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for all $x_1, \ldots, x_n \in X$. It is called the **Hurwitz action** of $\mathcal{B}_n$, the orbits are called the **Hurwitz orbits**.

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$T^n V$ with $n \geq 2$ decomposes into subspaces graded by Hurwitz orbits. Estimates of the rank of $\mathcal{S}_n$ on such subspaces give estimates of the Hilbert series of $\mathcal{B}(V)$.

Known examples

Table: Known examples of f.d. Nichols algebras (V simple)

dim V	dim $\mathcal{B}(V)$	Hilbert series	origin
1	N	$(N)_t$	
3	12	$(2)_t^2(3)_t$	MS, FK
3	432	$(3)_t(4)_t(6)_t(6)_{t^2}$	HS ($\mathrm{char}\,\Bbbk = 2$)
4	36	$(2)_t^2(3)_t^2$	GHV ($\mathrm{char}\,\Bbbk = 2$)
4	72	$(2)_t^2(3)_t(6)_t$	AG ($\mathrm{char}\,\Bbbk \neq 2$)
4	5184	$(6)_t^4(2)_{t^2}^2$	HLV
5	1280	$(4)_t^4(5)_t$	AG (twice)
6	576	$(2)_t^2(3)_t^2(4)_t^2$	MS, FK (twice)
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7	326592	$(6)_t^6(7)_t$	G (twice)
10	8294400	$(4)_t^4(5)_t^2(6)_t^4$	FK, GG
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Observation: All Hilbert series factorize into products of polynomials of the form
$(m)_{t^r} = 1 + t^r + t^{2r} + \cdots + t^{(m-1)r}$, $m, r \in \mathbb{N}$.

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Open problems:

- Does the Hilbert series of $\mathcal{B}(V)$ always factorize in this way? (True for all known examples.)
- If so, is it possible to use this information to calculate the Hilbert series without determining an explicit basis of $\mathcal{B}(V)$?

First classification

Theorem. (Graña, H., Vendramin) G group, $V \in {}^G_G\mathcal{YD}$ f.d. absolutely irreducible, $G = \langle \mathrm{supp} V \rangle$, $d = \dim V$. The following assertions are equivalent.

1. $\dim \mathcal{B}(V)(2) \le d(d+1)/2$.
2. $\dim \ker(1_{V\otimes V} + c) \ge d(d-1)/2$, where $c \in \mathrm{Aut}(V \otimes V)$.
3. There are $n_1, n_2, \dots, n_d \in \mathbb{Z}_{\ge 2}$ such that
$$H_{\mathcal{B}(V)}(t) = (n_1)_t (n_2)_t \cdots (n_d)_t.$$
4. V is contained in a given list.

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(4)⇒(3): computer algebra. (3)⇒(2)⇒(1) trivial.
Difficult part: (1)⇒(4).

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Theorem. (H., Lochmann, Vendramin) G group, $V \in {}^G_G\mathcal{YD}$ f.d. absolutely irreducible, $G = \langle \mathrm{supp}\, V \rangle$, $d = \dim V$. Suppose that for all $x, y \in X$ we have $x \triangleright y = y$ or $x \triangleright (y \triangleright x) = y$. The following assertions are equivalent.

1. $\dim \ker(1 + c_{12} + c_{12}c_{23}) \geq d(d^2 - 1)/3$.
2. There exist $r \in \mathbb{N}_0$, $n_1, \ldots, n_d$, $m_1, \ldots, m_r \in \mathbb{Z}_{\geq 2}$ with
$$H_{\mathcal{B}(V)}(t) = \prod_{i=1}^{d} (n_i)_t \prod_{j=1}^{r} (m_j)_{t^2}.$$
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4. (1) and the estimates in 3. give restrictions on X. Such X can be classified. The rest is similar to the proof of the previous theorem.

Thank you for you attention!