Let $k$ be an algebraically closed field of characteristic 0 and let $D_m$ be the dihedral group of order $2m$ with $m = 4t; t \geq 3$. This talk will be based on a joint work with Fernando Fantino [2], where we classify all finite-dimensional Nichols algebras over $D_m$ and all finite-dimensional pointed Hopf algebras whose group of group-likes is $D_m$, by means of the lifting method. As a byproduct we obtain new examples of finite-dimensional pointed Hopf algebras. In particular, we give an infinite family of non-abelian groups with non-trivial examples of pointed Hopf algebras over them and where the classification is completed. The difference with the case of the symmetric groups $S_3$ y $S_4$, see [1] and [3], respectively, is that each dihedral group provide an infinite family of new examples.

Bibliography


On pointed Hopf algebras over dihedral groups

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Hopf algebras and Tensor Categories

July 4-8, 2011, Almería, Spain.
Joint work with F. Fantino.

Let $m = 4t = 2n \geq 12$ and recall that

$$\mathbb{D}_m := \langle g, h \mid g^2 = 1 = h^m, gh = h^{-1}g \rangle.$$ 

**Theorem [FG]**

Let $H$ be a finite-dimensional pointed Hopf algebra with $G(H) = \mathbb{D}_m$. Then $H$ is isomorphic to

(a) $\mathcal{B}(M_I) \# k\mathbb{D}_m$ with $I = \{(i, k)\} \in \mathcal{I}$, $k \neq n$.

(b) $\mathcal{B}(M_L) \# k\mathbb{D}_m$ with $L \in \mathcal{L}$.

(c) $A_I(\lambda, \gamma)$ with $I \in \mathcal{I}$, $|I| > 1$ or $I = \{(i, n)\}$ and $\gamma \equiv 0$.

(d) $B_{I, L}(\lambda, \gamma, \theta, \mu)$ with $(I, L) \in \mathcal{K}$, $|I| > 0$ and $|L| > 0$.

Conversely, any pointed Hopf algebra of the list above is a lifting of a finite-dimensional braided Hopf algebra in $k\mathbb{D}_m \mathcal{YD}$. 

where
• $\omega \in \mathbb{G}_m$ is an $m$-th primitive root of unity.

• $I = \{ l = \prod_{s=1}^{r} (i_s, k_s) : \omega^{i_s k_s} = -1 \text{ and } \omega^{i_s k_t + i_t k_s} = 1, 1 \leq i_s < n, 1 \leq k_s < m \}.$

• $L = \{ L = \prod_{s=1}^{r} \ell_s : 1 \leq \ell_1, \ldots, \ell_r < n, \text{ odd} \}.$

• $K = \{ (I, L) : I \in I, L \in L \text{ and } \omega^{i \ell} = -1, k \text{ odd } \forall (i, k) \in I, (\ell, L) \}$.

• $\lambda = \{ \lambda_{p,q,i,k} (p,q), (i,k) \in I, \gamma = \{ \gamma_{p,q,i,k} (p,q), (i,k) \in I, \theta = \{ \theta_{p,q,l} (p,q) \in I, \ell \in L, \text{ and } \mu = \{ \mu_{p,q,l} (p,q) \in I, \ell \in L \text{ family of } k \text{ that satisfy:} \}$

\[
\lambda_{p,m-k,i,k} = \lambda_{i,k,p,m-k} \quad \text{and} \quad \gamma_{p,k,i,k} = \gamma_{i,k,p,k}.
\]
If $I = \{(i, k)\}$, $k \neq n$, then $\mathcal{B}(M_I)\#kD_m$ is generated by $g, h, x, y$ which satisfy

\begin{align*}
g^2 &= 1 = h^m, & ghg &= h^{m-1}, \\
gx &= yg, & hx &= \omega^k x h, & hy &= \omega^{-k} y h, \\
x^2 &= 0, & y^2 &= 0, & xy + yx &= 0
\end{align*}

It is a Hopf algebra with

\begin{align*}
\Delta(g) &= g \otimes g, & \Delta(h) &= h \otimes h, \\
\Delta(x) &= x \otimes 1 + h^i \otimes x, & \Delta(y) &= y \otimes 1 + h^{-i} \otimes y.
\end{align*}
Let $L \in \mathcal{L}$, $\mathcal{B}(M_L) \# kD_m$ is generated by $z_\ell, w_\ell, \ell \in L$ which satisfy:

\[
\begin{align*}
    g^2 &= 1 = h^m, & ghg &= h^{m-1}, \\
    gz_\ell &= w_\ell g, & hz_\ell &= \omega^\ell z_\ell h, & hw_\ell &= \omega^{-\ell} w_\ell h, \\
    z_\ell^2 &= 0, & w_\ell^2 &= 0, & z_\ell w_\ell + w_\ell z_\ell &= 0, & z_\ell z_\ell + z_\ell w_\ell z_\ell &= 0.
\end{align*}
\]

It is a Hopf algebra with

\[
\begin{align*}
    \Delta(g) &= g \otimes g, & \Delta(h) &= h \otimes h, \\
    \Delta(z_\ell) &= z_\ell \otimes 1 + h^n \otimes z_\ell, & \Delta(w_\ell) &= w_\ell \otimes 1 + h^n \otimes w_\ell.
\end{align*}
\]
For any $I \in \mathcal{I}$, $A_I(\lambda, \gamma)$ is the algebra generated by $g, h, x_{p,q}, y_{p,q}$ with $(p, q) \in I$ satisfying:

\[
\begin{align*}
g^2 &= 1 = h^m, \\
ghg &= h^{m-1}, \\
gx_{p,q} &= y_{p,q}g, \\
hx_{p,q} &= \omega^q x_{p,q}h, \\
y_{p,q} &= \omega^{-q} y_{p,q}h, \\
x_{p,q}x_{i,k} + x_{i,k}x_{p,q} &= \delta_{q,m-k} \lambda_{p,q,i,k}(1 - h^{p+i}), \\
x_{p,q}y_{i,k} + y_{i,k}x_{p,q} &= \delta_{q,k} \gamma_{p,q,i,k}(1 - h^{p-i}).
\end{align*}
\]

It is a Hopf algebra with

\[
\begin{align*}
\Delta(g) &= g \otimes g, \\
\Delta(h) &= h \otimes h, \\
\Delta(x_{p,q}) &= x_{p,q} \otimes 1 + h^p \otimes x_{p,q}, \\
\Delta(y_{p,q}) &= y_{p,q} \otimes 1 + h^{-p} \otimes y_{p,q}.
\end{align*}
\]
**Introduction**

Let \((I, L) \in \mathcal{K}, B_{I, L}(\lambda, \gamma, \theta, \mu)\) is the algebra generated by \(g, h, x_{p,q}, y_{p,q}, z_{\ell}, w_{\ell}, (p, q) \in I, \ell \in L\), satisfying:

\begin{align*}
gx_{p,q} &= y_{p,q}g, & hx_{p,q} &= \omega^q x_{p,q}h, \\
gz_{\ell} &= w_{\ell}g, & hz_{\ell} &= \omega^\ell z_{\ell}h, \\
x_{p,q}^2 &= 0 = y_{p,q}^2, & z_{\ell}w_{\ell'} + w_{\ell'}z_{\ell} &= 0 \\
x_{p,q}x_{i,k} + x_{i,k}x_{p,q} &= \delta_{q,m-k} \lambda_{p,q,i,k}(1 - h^{p+i}), \\
x_{p,q}y_{i,k} + y_{i,k}x_{p,q} &= \delta_{q,k} \gamma_{p,q,i,k}(1 - h^{p-i}), \\
x_{p,q}z_{\ell} + z_{\ell}x_{p,q} &= \delta_{q,m-\ell} \theta_{p,q,\ell}(1 - h^{n+p}), \\
x_{p,q}w_{\ell} + w_{\ell}x_{p,q} &= \delta_{q,\ell} \mu_{p,q,\ell}(1 - h^{n+p}).
\end{align*}

It is a Hopf algebra with \(g, h\) group-likes and

\begin{align*}
\Delta(x_{p,q}) &= x_{p,q} \otimes 1 + h^p \otimes x_{p,q}, & \Delta(y_{p,q}) &= y_{p,q} \otimes 1 + h^{-p} \otimes y_{p,q}, \\
\Delta(z_{\ell}) &= z_{\ell} \otimes 1 + h^n \otimes z_{\ell}, & \Delta(w_{\ell}) &= w_{\ell} \otimes 1 + h^n \otimes w_{\ell}.
\end{align*}
Let $H$ be a pointed Hopf algebra, $H_0 = \mathbb{k} G(H)$.

\{H_i\}_{i\geq 0}$ coradical filtration of $H$.

**Fact:** If $H_0$ is a Hopf subalgebra, then 
$\text{gr } H = \bigoplus_{n \geq 0} \text{gr } H(n)$ is a graded Hopf algebra, 
$\text{gr } H(n) = H_n/H_{n-1}$, $H_{-1} = 0$.

If $\pi : \text{gr } H \to H_0$ denotes the homogeneous projection, then 

$$R = (\text{gr } H)^{\text{co } \pi} = \{ h \in H : (\text{id} \otimes \pi) \Delta(h) = h \otimes 1 \}$$

is the diagram of $H$; and $\text{gr } H \simeq R \# \mathbb{k} G(H)$.
• $R$ is a (braided) Hopf algebra in the category $^{H_0}_{H_0}\mathcal{YD}$ of Yetter-Drinfel’d modules over $H_0$.

• $R$ is a graded subalgebra of $\text{gr } H$.

• The linear subspace $R(1)$, together with the braiding of $^{H_0}_{H_0}\mathcal{YD}$, is called the infinitesimal braiding of $H$ and coincides with

$$P(R) = \{ r \in R : \Delta_R(r) = r \otimes 1 + 1 \otimes r \}.$$

• The subalgebra of $R$ generated by $P(R) = V$ is (isomorphic to) the Nichols algebra $\mathcal{B}(V)$. 
Let $G$ be a finite group and $H_0 = \mathbb{k}G$. Main steps for classifying finite-dimensional pointed Hopf algebras over $G$ are

(a) determine all Yetter-Drinfel’d modules $V$ such that $\mathcal{B}(V)$ is finite-dimensional,

(b) For such $V$, determine all Hopf algebras $H$ such that $\text{gr } H \simeq \mathcal{B}(V)\#H_0$, $H$ is called a \textit{lifting} of $\mathcal{B}(V)$ over $G$.

(c) Prove that any finite-dimensional pointed Hopf algebra over $G$ is generated by group-likes and skew-primitives.
It was introduced by Andruskiewitsch and Schneider.

Complete classification of finite-dimensional pointed Hopf algebras over $G$ (with non-trivial examples) where

- $G$ finite and abelian with $(|G|, 210) = 1$ [AS].
- $G = S_3$, [AS & Heckenerger].
- $G = S_4$, [AHS] and [G. & A. García Iglesias].
Let $G$ be a finite group. Recall that a Yetter-Drinfel’d module over $\mathbb{k}G$ is a $G$-module and a $\mathbb{k}G$-comodule $M$ such that

$$\delta(g.m) = ghg^{-1} \otimes g.m, \quad \forall \ m \in M_h, g, h \in G,$$

where $M_h = \{m \in M : \delta(m) = h \otimes m\}$, $M = \bigoplus_{h \in G} M_h$.

**Proposition**

- Finite-dimensional Yetter-Drinfel’d modules over $G$ are completely reducible.
- Irreducible modules are parametrized by pairs $(O, \rho)$, where $O$ is a conjugacy class of $G$ and $(\rho, V)$ is an irreducible representation of the centralizer $C_G(\sigma)$ of some $\sigma \in O$.

We denote by $M(O, \rho)$ the Yetter-Drinfel’d module and by $\mathcal{B}(O, \rho)$ the associated Nichols algebra.
Conjugacy classes of $\mathbb{D}_m$ are

- $O_{hn} = \{h^n\}, \ C_{hn} = \mathbb{D}_m$.
- $O_{hi} = \{h^{\pm i}\}, \ C_{hi} = \langle h \rangle \simeq \mathbb{Z}/m$, Rep: $\chi(k), \ \chi(k)(h) = \omega^k$.
- $O_g = \{gh^j : j \text{ even}\}, \ O_{gh} = \{gh^j : j \text{ odd}\}$

Recall the irreducible representations of $\mathbb{D}_m$:

- $n - 1$ irred. repr. of degree 2, $\rho_\ell : \mathbb{D}_m \to \text{GL}(2, k)$,

\[ \rho_\ell(g^ah^b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^a \begin{pmatrix} \omega^\ell & 0 \\ 0 & \omega^{-\ell} \end{pmatrix}^b, \quad 1 \leq \ell < n. \]

- 4 irred. repr. of degree 1:

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$1$</th>
<th>$h^n$</th>
<th>$h^i$, $1 \leq b \leq n - 1$</th>
<th>$g$</th>
<th>$gh$</th>
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<tr>
<td>$\chi_1$</td>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
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<td>1</td>
<td>$-1$</td>
<td>$-1$</td>
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<td>$(-1)^i$</td>
<td>1</td>
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</tr>
<tr>
<td>$\chi_4$</td>
<td>1</td>
<td>$(-1)^n$</td>
<td>$(-1)^i$</td>
<td>$-1$</td>
<td>1</td>
</tr>
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</table>
Andruskiewitsch & Fantino determined the dimension of $\mathcal{B}(O_{h^n}, \pi)$ and $\mathcal{B}(O_{h^i}, \chi(k))$.

For the others we have

**Lemma [FG]**

\[
\dim \mathcal{B}(O_g, \rho) = \dim \mathcal{B}(O_{gh}, \eta) = \infty \text{ for all } \rho \in \widehat{C_{D_m}(g)} \text{ and } \\
\eta \in \widehat{C_{D_m}(gh)}.
\]

Summarizing
### Conjecture: Conjugacy Classes, Centers, Representations, and Dimensions

<table>
<thead>
<tr>
<th>Conj. class</th>
<th>Centr.</th>
<th>Rep.</th>
<th>dim $\mathcal{B}(V)$</th>
</tr>
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<tr>
<td>$e$</td>
<td>$D_m$</td>
<td>any</td>
<td>$\infty$ [AF]</td>
</tr>
<tr>
<td>$O_{h^n} = { h^n }$,</td>
<td>$D_m$</td>
<td>$\chi_1, \chi_2, \chi_3, \chi_4$,</td>
<td>$\infty$ [AF]</td>
</tr>
<tr>
<td>$</td>
<td>O_{h^n}</td>
<td>= 1$</td>
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</tr>
<tr>
<td></td>
<td></td>
<td>$\rho_l$, $l$ odd</td>
<td>$4$ [AF] $\mathcal{B}(M_l)$</td>
</tr>
<tr>
<td>$O_{h^i} = { h^{\pm i} }$, $i \neq 0, n$,</td>
<td>$\mathbb{Z}/m \cong \langle h \rangle$</td>
<td>$\chi(k)$, $\omega^{ik} = -1$</td>
<td>$4$ [AF] $\mathcal{B}(M_{i,k})$</td>
</tr>
<tr>
<td>$</td>
<td>O_{h^i}</td>
<td>= 2$</td>
<td></td>
</tr>
<tr>
<td>$O_g = { gh^j : j$ even $}$</td>
<td>$\mathbb{Z}/2 \times \mathbb{Z}/2 \cong \langle g \rangle \oplus \langle h^n \rangle$</td>
<td>any</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$</td>
<td>O_g</td>
<td>= n$</td>
<td></td>
</tr>
<tr>
<td>$O_{gh} = { gh^j : j$ odd $}$</td>
<td>$\mathbb{Z}/2 \times \mathbb{Z}/2 \cong \langle gh \rangle \oplus \langle h^n \rangle$</td>
<td>any</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$</td>
<td>O_{gh}</td>
<td>= n$</td>
<td></td>
</tr>
</tbody>
</table>
Define $\mathcal{I} = \{ I = \coprod_{s=1}^{r} \{(i_s, k_s)\} : \omega^{i_s k_s} = -1 \text{ and } \omega^{i_s k_t + i_t k_s} = 1, 1 \leq i_s < n, 1 \leq k_s < m \}$ and

$$M_I = \bigoplus_{(i, k) \in I} M_{(i, k)}$$

Then $\mathcal{B}(M_I) = \bigwedge M_I$ and $\dim \mathcal{B}(M_I) = 4^{|I|}$.

Define $\mathcal{L} = \{ L = \coprod_{s=1}^{r} \{\ell_s\} : 1 \leq \ell_1, \ldots, \ell_r < n, \text{ odd} \}$ and

$$M_L = \bigoplus_{\ell \in L} M_{\ell}.$$ 

Then $\mathcal{B}(M_L) = \bigwedge M_L$ and $\dim \mathcal{B}(M_L) = 4^{|L|}$. 
Define $\mathcal{K} = \{(I, L) : I \in \mathcal{I}, L \in \mathcal{L} \text{ and } \omega^{ik} = -1, k \text{ odd, } \forall (i, k) \in I, \ell \in L\}$ and

$$M_{I,L} = \left( \bigoplus_{(i,k) \in I} M_{(i,k)} \right) \oplus \left( \bigoplus_{\ell \in L} M_{\ell} \right).$$

Then $\mathcal{B}(M_{I,L}) \simeq \bigwedge M_{I,L}$ and $\dim \mathcal{B}(M_{I,L}) = 4|I| + |L|.$

**Theorem [FG]**

Let $\mathcal{B}(M)$ be a finite-dimensional Nichols algebra in $k\mathcal{D}_m^n \mathcal{YD}$. Then $\mathcal{B}(M) \simeq \bigwedge M$, with $M$ isomorphic to $M_I$ with $I \in \mathcal{I}$, or $M_L$ with $L \in \mathcal{L}$, or $M_{I,L}$ with $(I, L) \in \mathcal{K}.$
Using that all finite-dimensional Nichols algebras are exterior algebras one can prove the generation in degree one:

**Theorem**

Let $H$ be a finite-dimensional pointed Hopf algebra with $G(H) = \mathbb{D}_m$. Then $H$ is generated by group-likes and skew-primitives.

i. e. $\text{gr } H \cong \mathcal{B}(M) \# k\mathbb{D}_m$ for some $M$. 
Let $M \in \mathbb{k}\mathbb{D}_m \mathcal{YD}$. For all $1 \leq r, s < m$, let

$$M^s_r = \{ a \in M : \delta(a) = h^s \otimes a, h \cdot a = \omega^r a \}. \quad \text{Then } M = \bigoplus_{r,s} M^s_r.$$

Using the description obtained above we find the possible deformations of the relations of the Nichols algebras over $\mathbb{D}_m$:

**Proposition [FG]**

Let $H$ be a finite-dimensional pointed Hopf algebra with $G(H) = \mathbb{D}_m$ and infinitesimal braiding $M$. Let $a \in M^s_r, b \in M^v_u$ with $1 \leq r, s, u, v < m$ and denote $x = \sigma(a\#1), y = \sigma(b\#1)$. Then there exists $\lambda \in \mathbb{k}^\times$ such that

$$xy + yx = \delta_{u,m-r} \lambda (1 - h^{s+v}).$$
Not all Nichols algebras admit deformations:

**Lemma [FG]**

Let $H$ be a finite-dimensional such that its infinitesimal braiding $M$ is isomorphic to $M_I$ with $I = (i, k) \subseteq I$, $k \neq n$ or $M_L$ with $L \in \mathcal{L}$. Then $H \simeq \mathcal{B}(M_I) \# k\mathbb{D}_m$ or $H \simeq \mathcal{B}(M_L) \# k\mathbb{D}_m$, resp.

Using the proposition we define the quadratic algebras $A_I(\lambda, \gamma)$ and $B_{I,L}(\lambda, \gamma, \theta, \mu)$ as above and the first part of the main theorem is proved.

To prove that these algebras are liftings one first shows that

\[
\dim A_I(\lambda, \gamma) \leq |\mathbb{D}_m| \dim \mathcal{B}(M_I) \quad \text{and} \\
\dim B_{I,L}(\lambda, \gamma, \theta, \mu) \leq |\mathbb{D}_m| \dim \mathcal{B}(M_{I,L}).
\]

The equality follows by finding a representation whose restriction to $\mathbb{D}_m$ is faithful and is not trivial on the skew-primitives.
References