Clifford theory for graded fusion categories

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I will report on progress towards the classification of module categories over graded fusion categories. We develop a categorical analogue over graded fusion categories of Clifford theory for strongly graded rings. We describe module categories over a fusion category graded by a group G as induced from module categories over fusion subcategories associated with the subgroups of G. We define invariant C_e-module categories and extensions of C_e-module categories. The construction of module categories over C is reduced to determine invariant module categories for subgroups of G and the indecomposable extensions of this modules categories. We associate a G-crossed product fusion category to each G-invariant C_e-module category and give a criterion for a graded fusion category to be a group-theoretical fusion category. We give necessary and sufficient conditions for an indecomposable module category to be extended.

Clifford theory for fusion categories

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Hopf algebras and tensor categories
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Talk based on

- C. G., *Clifford theory for fusion categories*, accepted by Israel Journal of Mathematics. ArXiv:1010.5283
- C. G., *Clifford theory for tensor categories*, J. Lond. Math. Soc., (2) 83 (2011) 57–78. ArXiv:0902.1088
- C. G., Seung-Moon Hong, Eric Rowell, *Generalized and quasi-localizations of braid group representations*. ArXiv:1105.5048

Other approaches

- E. Meir and E. Musicantov, *Module categories over graded fusion categories*, preprint arxiv:1010.4333.
- Camille Mével, *Exemples et applications des groupoïdes quantiques finis*, Ph.D. Thesis. Université de Caen Basse-Normandie.

Graded fusion categories

Let $\mathcal{C}$ be a fusion category and let G be a finite group.
$\mathcal{C}$ is G-graded if

$$\mathcal{C} = \oplus_{\sigma \in G} \mathcal{C}_\sigma, \quad (\mathcal{C}_g \neq 0),$$

for any $\sigma, \tau \in G$, one has $\otimes : \mathcal{C}_\sigma \times \mathcal{C}_\tau \to \mathcal{C}_{\sigma\tau}$.

Module categories over tensor categories

A (left) $\mathcal{C}$*-module category* is a (semisimple) category $\mathcal{M}$ together with a bifunctor $\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ and natural isomorphisms

$$m_{X,\,Y\,M} : (X \otimes Y) \otimes M \to X \otimes (Y \otimes M),$$

for all $M \in \mathcal{M}$, $X, Y \in \mathcal{C}$, satisfying the pentagon and triangular axioms.

Module functors and $\mathcal{C}^*_{\mathcal{M}}$

A $\mathcal{C}$-module functor $(F, \phi) : \mathcal{M} \to \mathcal{N}$ consists of a functor $F : \mathcal{M} \to \mathcal{N}$ and natural isomorphisms

$$\phi_{X,M} : F(X \otimes M) \to X \otimes F(M),$$

such that

$$(X \otimes \phi_{Y,M})\phi_{X,Y \otimes M} F(m_{X,Y,M}) = m_{X,Y,F(M)} \phi_{X \otimes Y,M} \tag{1}$$

for all $X, Y \in \mathcal{C}$, $M \in M$.

We shall denote by $\mathcal{C}^*_{\mathcal{M}}$ the tensor category of $\mathcal{C}$-module functor from $\mathcal{M}$ to $\mathcal{M}$.

Tensor product of module categories [ENO3]

For $\mathcal{C}$-module categories $\mathcal{M}$ and $\mathcal{N}$ their tensor product

$$\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$$

using a universal property.

If $\mathcal{M}$ is a $\mathcal{C}$-bimodule category, then

$$\mathcal{M} \boxtimes_{\mathcal{C}} \mathcal{N}$$

is again a left $\mathcal{C}$-module categories.

From now on $\mathcal{C}$ shall denote a G-graded fusion category

Let $\mathcal{M}$ be a $\mathcal{C}$-module category, and let $\mathcal{N} \subset \mathcal{M}$ be a full abelian subcategory. We shall denote by $\mathcal{C}_\sigma \overline{\otimes} \mathcal{N}$ the full abelian subcategory given by $\mathrm{Ob}(\mathcal{C} \overline{\otimes} \mathcal{N}) = \{$subquotients of $V \otimes N : V \in \mathcal{C}_\sigma, N \in N\}$.

Remark: If $\mathcal{N} \subset \mathcal{M}$ is a $\mathcal{C}_e$-module, the bifunctor $\otimes$ induces a canonical $\mathcal{C}_e$-module equivalence $\mu_\sigma : \mathcal{C}_\sigma \boxtimes_{\mathcal{C}_e} \mathcal{N} \to \mathcal{C}_\sigma \overline{\otimes} \mathcal{N}$.

Clifford theory for fusion categories

Given a $\mathcal{C}$-module category $\mathcal{M}$, we shall denote by $\Omega_{\mathcal{C}_e}(\mathcal{M})$ the set of equivalence classes of indecomposable $\mathcal{C}_e$-submodule categories of $\mathcal{M}$.

The group G acts on $\Omega_{\mathcal{C}_e}(\mathcal{M})$ by

$$\begin{aligned} G \times \Omega_{\mathcal{C}_e}(\mathcal{M}) &\to \Omega_{\mathcal{C}_e}(\mathcal{M}) \\ (g, [\mathcal{N}]) &\mapsto [\mathcal{C}_\sigma \boxtimes_{\mathcal{C}_e} \mathcal{N}] \end{aligned}$$

Clifford theory for fusion categories

Theorem

Let $\mathcal{C}$ be a G-graded fusion category and let $\mathcal{M}$ be an indecomposable $\mathcal{C}$-module category. Then:

1. *The action of G on $\Omega_{\mathcal{C}_e}(\mathcal{M})$ is transitive,*
2. *Let $\mathcal{N}$ be an indecomposable $\mathcal{C}_e$-submodule subcategory of $\mathcal{M}$. Let $H = st([\mathcal{N}])$ be the stabilizer subgroup of $[\mathcal{N}] \in \Omega_{\mathcal{C}_e}(\mathcal{M})$, and let also*

$$\mathcal{M}_{\mathcal{N}} = \sum_{h \in H} \mathcal{C}_H \overline{\otimes} \mathcal{N}.$$

Then $\mathcal{M}_{\mathcal{N}}$ is an indecomposable $\mathcal{C}_H$-module category and $\mathcal{M}$ is equivalent to $Ind_{\mathcal{C}_H}^{\mathcal{C}}(\mathcal{M}_{\mathcal{N}}) = \mathcal{C} \boxtimes_{\mathcal{C}_H} \mathcal{M}_{\mathcal{N}}$ as $\mathcal{C}$-module categories.

Definition

Let $\mathcal{C}$ be a G-graded fusion category. If $(\mathcal{M}, \otimes)$ is a $\mathcal{C}_e$-module category, then a $\mathcal{C}$-extension of $\mathcal{M}$ is a $\mathcal{C}$-module category $(\mathcal{M}, \odot)$ such that $(\mathcal{M}, \otimes)$ is obtained by restriction to $\mathcal{C}_e$.

Corollary

Let $\mathcal{M}$ be an indecomposable $\mathcal{C}$-category, and $\mathcal{N}$ an indecomposable $\mathcal{C}_e$-submodule category. Then there exists a subgroup $S \subset G$, and a $\mathcal{C}_S$-extension $(\mathcal{N}, \odot)$ of $\mathcal{N}$, such that $\mathcal{M} \cong \mathcal{C} \boxtimes_{\mathcal{C}_S} \mathcal{N}$ as $\mathcal{C}$-module categories.

Remark: The subgroup $S = \{\sigma \in G | \mathcal{C}_\sigma \overline{\otimes} \mathcal{N} = \mathcal{N}\}$.

Invariant and G-graded module categories

Definition

A $\mathcal{C}_e$-module category $\mathcal{M}$ is called G-invariant if $\mathcal{C}_\sigma \boxtimes_{\mathcal{C}_e} \mathcal{M}$ is equivalent to $\mathcal{M}$ as $\mathcal{C}_e$-module categories, for all $\sigma \in G$.

Definition

A graded tensor category over a group G will be called a crossed product tensor category if every homogeneous component has at least one multiplicatively invertible object.

Invariant and G-graded module categories

Proposition

Let $\mathcal{C}$ be a G-graded fusion category. An indecomposable $\mathcal{C}_e$-module category $\mathcal{N}$ is invariant if and only if $\mathcal{C}^*_{\mathcal{C}\boxtimes_{\mathcal{C}_e}\mathcal{N}}$ is G^{op}-crossed product fusion category.

Semi-direct product fusion category

Given $* : \underline{G} \to \underline{\mathrm{Aut}_{\otimes}(\mathcal{C})}$, the semi-direct product fusion category $\mathcal{C} \rtimes G$, is defined as follows: As an abelian category $\mathcal{C} \rtimes G = \bigoplus_{\sigma \in G} \mathcal{C}_\sigma$, where $\mathcal{C}_\sigma = \mathcal{C}$ as an abelian category, the tensor product is

$$[X, \sigma] \otimes [Y, \tau] := [X \otimes \sigma_*(Y), \sigma\tau], \quad X, Y \in \mathcal{C}, \quad \sigma, \tau \in G,$$

and the unit object is $[\mathbf{1}, e]$.

$\mathcal{C} \rtimes G$ is G-graded by

$$\mathcal{C} \rtimes G = \bigoplus_{\sigma \in G} (\mathcal{C} \rtimes G)_\sigma, \quad \text{where} \quad (\mathcal{C} \rtimes G)_\sigma = \mathcal{C}_\sigma,$$

and the objects $[\mathbf{1}, \sigma] \in (\mathcal{C} \rtimes G)_\sigma$ are invertible, with inverse $[\mathbf{1}, \sigma^{-1}] \in (\mathcal{C} \rtimes G)_{\sigma^{-1}}$.

Classification of $\mathcal{C}$-extension of a $\mathcal{C}_e$-module categories

Theorem

Let $\mathcal{C}$ be a G-graded fusion category. Then an indecomposable left $\mathcal{C}_e$-module category $\mathcal{M}$ has an extension $(\mathcal{M}, \odot)$ if and only if $\mathcal{C}^*_{\mathcal{M}}$ is a semi-direct product fusion category. There is a one-to-one correspondence between equivalence classes of $\mathcal{C}$-extensions of $\mathcal{M}$ and conjugacy classes of graded tensor functors $\mathrm{Vec}_{G^{op}} \to \mathcal{C}^*_{\mathcal{M}}$.

Algorithm for constructing module categories

Theorem 3 and Corollary 1, reduce the problem of constructing module categories over a graded fusion category $\mathcal{C} = \bigoplus_{\sigma \in G}$, to the following steps:

1. Classifying the indecomposable $\mathcal{C}_e$-module categories.
2. Finding the subgroup S and the indecomposable $\mathcal{C}_e$-module categories $\mathcal{N}$, such that $\mathcal{N}$ is S-invariant.
3. Determining if $\mathcal{F}_{\mathcal{C}_S}(\mathrm{Ind}_{\mathcal{C}_e}^{\mathcal{C}_S}(\mathcal{N}), \mathrm{Ind}_{\mathcal{C}_e}^{\mathcal{C}_S}(\mathcal{N}))$ is equivalent to a semi-direct S^{op}-product fusion category.
4. Finding all graded functors from $\mathrm{Vec}_{S^{op}}$ to $\mathcal{F}_{\mathcal{C}_S}(\mathcal{N}, \mathcal{N})$, up to conjugation.

Complex $*$-category

Definition

A $\mathbb{C}$-linear category $\mathcal{D}$ is called a ***complex $*$-category*** *if:*

1. *There is an involutive antilinear contravariant endofunctor $*$ of $\mathcal{D}$ which is the identity on objects. The image of f under $*$ will be denoted by f^*.*
2. *For each $f \in Hom_{\mathcal{D}}(X, Y)$, $f^* f = 0$ implies $f = 0$.*

Complex $*$-category

Let X and Y be objects in a $*$-category. A morphism $u : X \to Y$ is **unitary** if $uu^* = \mathrm{id}_Y$ and $u^*u = \mathrm{id}_X$. A morphism $a : X \to X$ is **self-adjoint** if $a^* = a$.
A natural transformation $\gamma : F \to G$, between functors $F, G : \mathcal{D}_1 \to \mathcal{D}_2$ with $\mathcal{D}_2$ a $*$-category is called **unitary natural transformation** if γ_X is unitary for each $X \in \mathcal{D}_1$.

Unitary fusion categories

Definition

*A **unitary fusion category** is a fusion category $(\mathcal{C}, \otimes, \alpha)$, where $\mathcal{C}$ is a $*$-category, the constraints are unitary natural transformations, and $(f \otimes g)^* = f^* \otimes g^*$, for every pair of morphisms f, g in $\mathcal{C}$.*

Example

1. *Hilb_f, with the tensor product of Hilbert spaces is a unitary fusion category.*
2. *A finite dimensional (quasi) Kac algebra is a (quasi) Hopf algebra H, such that H is a C^*-algebra, Δ and ε are $*$-algebras morphisms, and if H is a quasi-Hopf algebra the associator must satisfy $\Phi^* = \Phi^{-1}$. In this case the category of unitary H-modules is a unitary fusion category.*

$\mathcal{C}$-module $*$-category

Definition

*Let $\mathcal{C}$ be a unitary fusion category. A $\mathcal{C}$***-module** $*$**-category** *is a left $\mathcal{C}$-module category $(\mathcal{M}, \overline{\otimes}, \mu)$ such that $\mathcal{M}$ is a $*$-category, the constraints are unitary natural transformations, and $(f \overline{\otimes} g)^* = f^* \overline{\otimes} g^*$ for all $f \in \mathcal{C}, g \in \mathcal{M}$.*

Definition

Let $\mathcal{C}$ be a fusion category. We shall say that $\mathcal{C}$ is ***completely unitary*** *if the following properties are satisfied:*

1. *$\mathcal{C}$ is monoidally equivalent to a unique (up to $*$-monoidal equivalences) unitary fusion category.*
2. *Every $\mathcal{C}$-module category is equivalent to a unique (up to $\mathcal{C}$-module $*$-functor equivalences) $\mathcal{C}$-module $*$-category.*
3. *Every $\mathcal{C}$-module functor equivalence between $\mathcal{C}$-module $*$-categories is equivalent to a unique (up to unitary $\mathcal{C}$-module natural isomorphisms) $\mathcal{C}$-module $*$-functor equivalence.*

Weakly group-theoretical fusion categories are completely unitary

Theorem

Every weakly group theoretical fusion category is a completely unitary fusion category.

Corollary

Every weakly group-theoretical (quasi)-Hopf algebra is isomorphic to a (quasi)-Kac algebra.

1. Question 7.8 in [A]: Given a semisimple Hopf algebra H, does it admit a compact involution? Corollary 2 gives an affirmative answer for weakly group theoretical Hopf algebras.
2. It is not known ([ENO2] Question 2) if there exist weakly integral fusion categories that are not weakly group-theoretical. Theorem 4 inspires the following **question**: *Is every weakly integral fusion category completely unitary or unitary?*

Bibliography

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- P. Etingof, D. Nikshych and V. Ostrik, *Weakly group-theoretical and solvable fusion categories*, Adv. Math. 226 (2011) 176–205.
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