Computing of the combinatorial rank of quantum groups

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In general an intersection of two biideals is not a biideal. By this reason one may not define a biideal generated by a set of elements, and the bialgebras do not admit a usual combinatorial representation by generators and relations. Heyneman-Radford theorem implies that each nonzero biideal of a pointed bialgebra has nonzero skew primitive element. Each ideal generated by skew primitive elements is a biideal, but certainly a biideal in general is not generared as an ideal by its skew primitive elemens. The Heyneman-Radford theorem allows one to define a combinatorial representation over the coradical in the following form

$$\mathfrak{A} = C\langle X \,||F_1 = 0|F_2 = 0|\dots|F_\kappa = 0|\rangle,$$

where X is a set of generators, F_1 is a set of skew primitive relations, F_i, $1 < i \leq \kappa$ is a set of relations that are skew primitive in $C\langle X \,||F_1 = 0|F_2 = 0|\dots|F_{i-1} = 0|\rangle$. The minimal number κ is called a *combinatorial rank* of $\mathfrak{A}$. We prove that the combinatorial rank of the multiparameter version of the Lusztig small quantum group $u_q(\mathfrak{so}_{2n+1})$, or equivalently of the Frobenius-Lusztig kernel of type B_n, equals $\lfloor \log_2(n-1) \rfloor + 2$ provided that q has a finite multiplicative order $t > 4$. In the case A_n the combinatorial rank equals $\lfloor \log_2 n \rfloor + 1$, see [1].

Bibliography

[1] V.K. Kharchenko, A. Andrade Alvarez, *On the combinatorial rank of Hopf algebras.* Contemporary Mathematics 376 (2005), 299-308.

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The extension theorem of MacWilliams

- **The Extension Problem** (MacWilliams, 62). *For $n \in \mathbf{Z}^+$, for a right linear code over a field R where $C \subseteq R^n$, and for a linear isometry $f : C \to R^n$, can f be extended to a linear isometry $T : R^n \to R^n$?*

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- **Extension Theorem** (Wood, 99). *Let* R *be a finite Frobenius ring. Suppose* $C \subset R^n$ *is a right linear code, and suppose* $f : C \to R^n$ *is a right linear homomorphism which preserves Hamming weight. Then* f *extends to a right isometry of* R^n.

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- The converse of the extension theorem holds! (Wood, 06).

The Larson-Sweedler Theorem

- **Theorem** (Larson-Sweedler, 69). *Every finite dimensional Hopf algebra is Frobenius.*

  In this way, finite quantum groups provide a material to work within the coding theory.

Algebras and bialgebras

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- Example: $f = x_1 x_2$; $\Delta(f) = f \otimes 1 + x_1 \otimes x_2 + x_2 \otimes x_1 + 1 \otimes f$;
  Biid$\langle x_1 x_2 \rangle$ is either Id$\langle x_1 \rangle$ or Id$\langle x_2 \rangle$.

Primitive elements and coradical filtration

- If $\Delta(f) = a \otimes f + f \otimes b$, then $\mathrm{Id}\langle f \rangle$ is a biideal!
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- **Theorem** (Heyneman–Radford, 74). *Let C and D be coalgebras y $\phi : C \to D$ be a morphism of coalgebras such that the restriction $\phi|_{C_1}$ is injective. Then ϕ is injective.*
- Here $C_0 \subset C_1 \subset C_2 \subset \ldots = C$ is the coradical filtration:

$$\Delta(C_n) \subseteq \sum_{i=1}^{n} C_i \otimes C_{n-i}.$$

Combinatorial rank

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- **Theorem** (Taft-Wilson, 74). *If C is pointed, then C_1 is spanned by 1 and by skew-primitive elements.*

  Corollary. *Every nonzero biideal I of a pointed bialgebra A has a nonzero skew-primitive element.*

  $$A = \langle X || \, f_1^{(1)}, \ldots, f_m^{(1)} \mid f_1^{(2)}, \ldots, f_m^{(2)} \mid \ldots | f_1^{(\kappa)}, \ldots, f_m^{(\kappa)} \rangle.$$

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- The number κ is the *combinatorial rank* of A.

$$I_1 \subset I_2 \subset I_3 \subset \dots \subset I_\kappa = I, \quad I_t/I_{t-1} = I/I_{t-1} \cap C_1(F/I_{t-1}).$$

- **Theorem** (V.K. Kharchenko, A. Álvarez, 05). *The combinatorial rank of the quantum group* $u_q(\mathfrak{sl}_{n+1})$ *equals* $\lfloor \log_2 n \rfloor + 1$ *provided that* q *has a finite multiplicative order* $t > 2$.

$u_q(\mathfrak{sl}_{n+1})$ and $u_q(\mathfrak{so}_{2n+1})$

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- **Theorem** (V.K. Kharchenko, M.L. Díaz Sosa, 10). *The combinatorial rank of the quantum group $u_q(\mathfrak{so}_{2n+1})$ equals $\lfloor \log_2(n-1) \rfloor + 2$ provided that q has a finite multiplicative order $t > 4$.*

Main steps of the proof

- Triangular decomposition:

$$u_q(\mathfrak{so}_{2n+1}) = u_q^-(\mathfrak{so}_{2n+1}) \otimes H \otimes u_q^+(\mathfrak{so}_{2n+1}).$$

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- By definition $u_q^+(\mathfrak{so}_{2n+1}) = G\langle x_1, \dots, x_n \rangle / \mathbf{\Lambda}$, where $\mathbf{\Lambda}$ is the biggest biideal with trivial intersection with the space spanned by $x_1, \dots, x_n$.

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- $\Delta(x_i) = x_i \otimes 1 + g_i \otimes x_i$; $\Delta(g_i) = g_i \otimes g_i$; $x_i g_j = p_{ij} g_j x_i$, where p_{ij} are arbitrary parameters satisfying:

$$p_{nn} = q, \ \ p_{ii} = q^2, \ \ p_{i\,i+1} p_{i+1\,i} = q^{-2}, \ 1 \leq i < n;$$

$$p_{ij} p_{ji} = 1, \ j > i+1.$$

Main steps of the proof

- $u_q^+(\mathfrak{so}_{2n+1}) = G\langle x_1, \dots, x_n \,||\, [u_{km}]^{t_u},\ k \leq m \leq 2n-k\rangle,$

  $[u_{km}] = [\dots[[[[\dots[x_k, x_{k+1}]\cdots x_n,]x_n,]x_{n-1},]x_{n-2},]\cdots x_{2n-m+1}],$

  here $[u,v] = uv - p(u,v)vu$, while the bimultiplicative map $p(u,v)$ is so that $p(x_i, x_j) = p_{ij}$; and $t_u = t$ if $m = n$ or t is odd and $t_u = t/2$ otherwise.

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  here $[u,v] = uv - p(u,v)vu$, while the bimultiplicative map $p(u,v)$ is so that $p(x_i,x_j) = p_{ij}$; and $t_u = t$ if $m = n$ or t is odd and $t_u = t/2$ otherwise.
- The quantum Serre relations $S_{ij}(x_i,x_j)$ are skew-primitive, hence instead of the homomorphism $G\langle X\rangle \rightarrow u_q^+(\mathfrak{so}_{2n+1})$ we may consider $U_q^+(\mathfrak{so}_{2n+1}) \rightarrow u_q^+(\mathfrak{so}_{2n+1})$ and work with elements of $U_q^+(\mathfrak{so}_{2n+1})$.

Main steps of the proof

Proposition.

- The elements $T_u = [u]^{t_u}$, $u = u_{km}$ generate an algebra C of quantum polynomials, $T_u T_v = q_{uv} T_v T_u$, $q_{uv} q_{vu} = 1$.
- GC is a Hopf subalgebra.
- $U_q^+(\mathfrak{so}_{2n+1})$ is a free finitely generated module over GC of rank t^{n^2} if t is odd, and $t^n (t/2)^{n^2-n}$ if t is even.

Main steps of the proof

Lemma.

- If t is odd or $m \neq n$, then $[u_{km}]^t \in \mathbf{\Lambda}_i$ if and only if, $m - k < 2^i - 1 + \varepsilon_m^n$. Here $\varepsilon_m^n = 0$ if $m \leq n$, and $\varepsilon_m^n = 1$ otherwise.
- If t is even and $m = n$, then $m - k < 2^{i-1}$ implies $[u_{km}]^{t/2} \in \mathbf{\Lambda}_i$, while $m - k \geq 2^i - 1$ implies $[u_{km}]^{t/2} \notin \mathbf{\Lambda}_i$.

Problems

- Find the combinatorial rank of $u_q(\mathfrak{g})$, where $\mathfrak{g}$ is a simple Lie algebra of type C, D, E, F or G.

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- It is likely that the proposition is still valid.
- In order to prove the lemma we have used an explicit formula for the coproduct:

$$\Delta([u_{km}]) = [u_{km}] \otimes 1 + g_{km} \otimes [u_{km}] + \sum_{i=k}^{m-1} \alpha_i g_{ki} [u_{1+i\,m}] \otimes [u_{ki}],$$

which is not proven for the other classes yet.

Main results

$A_n: \quad \lfloor \log_2 n \rfloor + 1$

$B_n: \quad \lfloor \log_2 (n-1) \rfloor + 2$

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Thank you!