A presentation by generators and relations of Nichols algebras of diagonal type

Iván Angiono (National University of Córdoba, Argentina)
ivanangiono@gmail.com

The Lifting Method of Andruskiewitsch and Schneider is the leading method to classify pointed Hopf algebras [AS]. It involves as an inicial step to know for which braided vector spaces their associated Nichols algebra is finite-dimensional; such braided vector spaces were classified by Heckenberger [H].

A second step is the following one: for each of these Nichols algebras, give a nice presentation by generators a relations. In the present talk we give an answer to this question, following [A]. We characterize convex orders on root systems associated to finite Weyl groupoids and use a description of coideal subalgebras of Nichols algebras [HS]. We describe then a set of relations using the PBW bases of [Kh].

We use such presentation to prove that every finite-dimensional pointed Hopf algebra over $\mathbb{C}$, whose group of group-like elements is abelian, is generated by its group-like and skew-primitive elements, a conjecture due to Andruskiewitsch and Schneider.

Bibliography

[AS] N. Andruskiewitsch and H.-J. Schneider, *On the classification of finite-dimensional pointed Hopf algebras.* Ann. Math. **171** (2010), No. 1, 375–417.

[A] I. Angiono, *Presentation of Nichols algebras of diagonal type by generators and relations*, submitted. arXiv:1008.4144.

[HS] I. Heckenberger and H.-J. Schneider, *Right coideal subalgebras of Nichols algebras and the Duflo order on the Weyl groupoid.* `math.QA/0909.0293`.

[H] I. Heckenberger, *Classification of arithmetic root systems*, Adv. Math. **220** (2009), 59–124.

[Kh] V. Kharchenko, *A quantum analog of the Poincare-Birkhoff-Witt theorem*, Algebra and Logic **38**, (1999), 259–276.

A presentation by generators and relations of Nichols algebras of diagonal type

Iván Angiono

Universidad Nacional de Cordoba (Arg)

Hopf algebras and tensor categories - Almería, 2011

Fix the following setting

- V vector space, $\dim V = \theta < \infty$, $X = \{x_1, \dots, x_\theta\}$ a basis,

Fix the following setting

- V vector space, $\dim V = \theta < \infty$, $X = \{x_1, \dots, x_\theta\}$ a basis,
- $(q_{ij}) \in (\Bbbk^\times)^{\theta\times\theta}$,

Fix the following setting

- V vector space, $\dim V = \theta < \infty$, $X = \{x_1, \dots, x_\theta\}$ a basis,
- $(q_{ij}) \in (\mathrm{k}^\times)^{\theta \times \theta}$,
- $\alpha_1, \dots, \alpha_\theta$ the canonical basis of $\mathbb{Z}^\theta$:

$$\chi : \mathbb{Z}^\theta \times \mathbb{Z}^\theta \to \mathrm{k}^\times \ \mathbb{Z}\text{-bilinear}, \qquad \chi(\alpha_i, \alpha_j) = q_{ij}.$$

Fix the following setting

- V vector space, $\dim V = \theta < \infty$, $X = \{x_1, \dots, x_\theta\}$ a basis,
- $(q_{ij}) \in (\Bbbk^\times)^{\theta\times\theta}$,
- $\alpha_1, \dots, \alpha_\theta$ the canonical basis of $\mathbb{Z}^\theta$:

$$\chi : \mathbb{Z}^\theta \times \mathbb{Z}^\theta \to \Bbbk^\times \ \mathbb{Z}\text{-bilinear}, \qquad \chi(\alpha_i, \alpha_j) = q_{ij}.$$

- $\mathbb{X}$ = set of words with letters in X (a basis of $T(V)$).

Fix the following setting

- V vector space, $\dim V = \theta < \infty$, $X = \{x_1, \dots, x_\theta\}$ a basis,
- $(q_{ij}) \in (\Bbbk^\times)^{\theta \times \theta}$,
- $\alpha_1, \dots, \alpha_\theta$ the canonical basis of $\mathbb{Z}^\theta$:

$$\chi : \mathbb{Z}^\theta \times \mathbb{Z}^\theta \to \Bbbk^\times \ \mathbb{Z}\text{-bilinear}, \qquad \chi(\alpha_i, \alpha_j) = q_{ij}.$$

- $\mathbb{X}$ = set of words with letters in X (a basis of $T(V)$).
- $T(V) = \oplus_{\alpha \in \mathbb{Z}^\theta} T^\alpha(V)$, with $\mathbb{Z}^\theta$-graduation $\deg(x_i) = \alpha_i$.

Fix the following setting

- V vector space, $\dim V = \theta < \infty$, $X = \{x_1, \dots, x_\theta\}$ a basis,
- $(q_{ij}) \in (\mathrm{k}^\times)^{\theta \times \theta}$,
- $\alpha_1, \dots, \alpha_\theta$ the canonical basis of $\mathbb{Z}^\theta$:

$$\chi : \mathbb{Z}^\theta \times \mathbb{Z}^\theta \to \mathrm{k}^\times \ \mathbb{Z}\text{-bilinear}, \qquad \chi(\alpha_i, \alpha_j) = q_{ij}.$$

- $\mathbb{X}$ = set of words with letters in X (a basis of $T(V)$).
- $T(V) = \oplus_{\alpha \in \mathbb{Z}^\theta} T^\alpha(V)$, with $\mathbb{Z}^\theta$-graduation $\deg(x_i) = \alpha_i$.
- Product in $T(V) \otimes T(V)$: $a, b, c, d \in T(V)$, $\beta = \deg(b), \gamma = \deg(c)$,

$$(a \otimes b)(c \otimes d) = \chi(\beta, \gamma) ac \otimes bd.$$

Fix the following setting

- V vector space, $\dim V = \theta < \infty$, $X = \{x_1, \dots, x_\theta\}$ a basis,
- $(q_{ij}) \in (\Bbbk^\times)^{\theta\times\theta}$,
- $\alpha_1, \dots, \alpha_\theta$ the canonical basis of $\mathbb{Z}^\theta$:

$$\chi : \mathbb{Z}^\theta \times \mathbb{Z}^\theta \to \Bbbk^\times \ \mathbb{Z}\text{-bilinear}, \qquad \chi(\alpha_i, \alpha_j) = q_{ij}.$$

- $\mathbb{X}$ = set of words with letters in X (a basis of $T(V)$).
- $T(V) = \oplus_{\alpha\in\mathbb{Z}^\theta} T^\alpha(V)$, with $\mathbb{Z}^\theta$-graduation $\deg(x_i) = \alpha_i$.
- Product in $T(V) \otimes T(V)$: $a, b, c, d \in T(V)$, $\beta = \deg(b), \gamma = \deg(c)$,

$$(a \otimes b)(c \otimes d) = \chi(\beta, \gamma) ac \otimes bd.$$

- $\Delta : T(V) \to T(V) \otimes T(V)$ morfism of algebras defined by $\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i$.

Proposition (Lusztig, Andruskiewitsch-Schneider)

$\exists$ a unique bilinear form in $T(V)$ such that

$\forall x, x', y, y' \in T(V)$, $1 \leq i, j \leq \theta$, $\alpha \neq \beta \in \mathbb{Z}^\theta$.

Proposition (Lusztig, Andruskiewitsch-Schneider)

$\exists$ a unique bilinear form in $T(V)$ such that

- $(x|yy') = (\Delta(x)|y \otimes y')$,

$\forall x, x', y, y' \in T(V)$, $1 \leq i, j \leq \theta$, $\alpha \neq \beta \in \mathbb{Z}^\theta$.

Proposition (Lusztig, Andruskiewitsch-Schneider)

$\exists$ a unique bilinear form in $T(V)$ such that

- $(x|yy') = (\Delta(x)|y \otimes y')$,
- $(xx'|y) = (x' \otimes x|\Delta(y))$,

$\forall x, x', y, y' \in T(V)$, $1 \leq i, j \leq \theta$, $\alpha \neq \beta \in \mathbb{Z}^\theta$.

Proposition (Lusztig, Andruskiewitsch-Schneider)

$\exists$ a unique bilinear form in $T(V)$ such that

- $(x|yy') = (\Delta(x)|y \otimes y')$,
- $(xx'|y) = (x' \otimes x|\Delta(y))$,
- $(x_i|x_j) = \delta_{ij}$,

$\forall x, x', y, y' \in T(V)$, $1 \leq i, j \leq \theta$, $\alpha \neq \beta \in \mathbb{Z}^\theta$.

Proposition (Lusztig, Andruskiewitsch-Schneider)

$\exists$ a unique bilinear form in $T(V)$ such that

- $(x|yy') = (\Delta(x)|y \otimes y')$,
- $(xx'|y) = (x' \otimes x|\Delta(y))$,
- $(x_i|x_j) = \delta_{ij}$,
- $(T^\alpha(V)|T^\beta(V)) = 0$,

$\forall x, x', y, y' \in T(V)$, $1 \leq i, j \leq \theta$, $\alpha \neq \beta \in \mathbb{Z}^\theta$.

Proposition (Lusztig, Andruskiewitsch-Schneider)

$\exists$ a unique bilinear form in $T(V)$ such that

- $(x|yy') = (\Delta(x)|y \otimes y')$,
- $(xx'|y) = (x' \otimes x|\Delta(y))$,
- $(x_i|x_j) = \delta_{ij}$,
- $(T^\alpha(V)|T^\beta(V)) = 0$,

$\forall x, x', y, y' \in T(V)$, $1 \leq i, j \leq \theta$, $\alpha \neq \beta \in \mathbb{Z}^\theta$.

Definition

$\mathcal{I}(V)$ radical of $(\,|\,)$, an ideal of $T(V)$. $\mathfrak{B}(V) := T(V)/\mathcal{I}(V)$ is the **Nichols algebra** asociated to the matrix (q_{ij}).

Problem

Classify all the matrices $(q_{ij})_{1\leq i,j\leq\theta}$ such that $\dim \mathfrak{B}(V) < \infty$.

For each one of these Nichols algebras, give a *minimal* presentation by generators and relations, and its dimension. [a]

[a] N. Andruskiewitsch, Contemp. Math. **294**, 1–57 (2002).

Problem

Classify all the matrices $(q_{ij})_{1\leq i,j\leq\theta}$ such that $\dim\mathfrak{B}(V)<\infty$.
For each one of these Nichols algebras, give a *minimal* presentation by generators and relations, and its dimension. [a]

[a] N. Andruskiewitsch, Contemp. Math. **294**, 1–57 (2002).

Answer to the first question: I. Heckenberger, *Classification of arithmetic root systems*, Adv. Math. **220** (2009) 59–124.

- $\mathfrak{B}(V)$ $\mathbb{Z}^\theta$-graded: *Hilbert series*

$$\mathcal{H}_{\mathfrak{B}(V)} := \sum_{\alpha \in \mathbb{N}_0^\theta} (\dim \mathfrak{B}(V)_\alpha) x^\alpha \in \mathbb{Z}[[x_1, \dots, x_\theta]], \qquad x^\alpha = x_1^{a_1} \cdots x_\theta^{a_\theta}.$$

- $\mathfrak{B}(V)$ $\mathbb{Z}^\theta$-graded: *Hilbert series*

$$\mathcal{H}_{\mathfrak{B}(V)} := \sum_{\alpha\in\mathbb{N}_0^\theta} (\dim \mathfrak{B}(V)_\alpha) x^\alpha \in \mathbb{Z}[[x_1, \dots, x_\theta]], \qquad x^\alpha = x_1^{a_1} \cdots x_\theta^{a_\theta}.$$

- Kharchenko: $\exists$ a basis PBW of $\mathfrak{B}(V)$, whose generators are $\mathbb{Z}^\theta$-homogeneous, $h : T \to \mathbb{N} \cup \{\infty\}$:
$B(T, <, h) := \{t_1^{e_1} \dots t_r^{e_r} : t_1 > \dots > t_r, t_i \in T, 0 < e_i < h(t_i)\}$.

- $\mathfrak{B}(V)$ $\mathbb{Z}^\theta$-graded: *Hilbert series*

$$\mathcal{H}_{\mathfrak{B}(V)} := \sum_{\alpha\in\mathbb{N}_0^\theta} (\dim \mathfrak{B}(V)_\alpha) x^\alpha \in \mathbb{Z}[[x_1, \dots, x_\theta]], \qquad x^\alpha = x_1^{a_1} \cdots x_\theta^{a_\theta}.$$

- Kharchenko: $\exists$ a basis PBW of $\mathfrak{B}(V)$, whose generators are $\mathbb{Z}^\theta$-homogeneous, $h : T \to \mathbb{N} \cup \{\infty\}$:
$B(T, <, h) := \{t_1^{e_1} ... t_r^{e_r} : t_1 > ... > t_r, t_i \in T, 0 < e_i < h(t_i)\}$.
- $\deg t_i = \alpha \in \mathbb{Z}^\theta$, $h(t_i) < \infty$, $\Rightarrow h(t_i) = \operatorname{ord}(\chi(\alpha, \alpha)) =: N_\alpha$.

- $\mathfrak{B}(V)$ $\mathbb{Z}^\theta$-graded: *Hilbert series*

$$\mathcal{H}_{\mathfrak{B}(V)} := \sum_{\alpha\in\mathbb{N}_0^\theta} (\dim \mathfrak{B}(V)_\alpha) x^\alpha \in \mathbb{Z}[[x_1, \dots, x_\theta]], \qquad x^\alpha = x_1^{a_1} \cdots x_\theta^{a_\theta}.$$

- Kharchenko: $\exists$ a basis PBW of $\mathfrak{B}(V)$, whose generators are $\mathbb{Z}^\theta$-homogeneous, $h : T \to \mathbb{N} \cup \{\infty\}$: $B(T, <, h) := \{t_1^{e_1} \dots t_r^{e_r} : t_1 > \dots > t_r, t_i \in T, 0 < e_i < h(t_i)\}$.
- $\deg t_i = \alpha \in \mathbb{Z}^\theta$, $h(t_i) < \infty$, $\Rightarrow h(t_i) = \operatorname{ord}(\chi(\alpha, \alpha)) =: N_\alpha$.
- $\Delta_+^V :=$ {degrees of generators of a PBW basis of $\mathfrak{B}(V)$}: it does not depend on the PBW basis.

- $\mathfrak{B}(V)$ $\mathbb{Z}^\theta$-graded: *Hilbert series*

$$\mathcal{H}_{\mathfrak{B}(V)} := \sum_{\alpha\in\mathbb{N}_0^\theta} (\dim \mathfrak{B}(V)_\alpha) x^\alpha \in \mathbb{Z}[[x_1, \dots, x_\theta]], \qquad x^\alpha = x_1^{a_1} \cdots x_\theta^{a_\theta}.$$

- Kharchenko: $\exists$ a basis PBW of $\mathfrak{B}(V)$, whose generators are $\mathbb{Z}^\theta$-homogeneous, $h : T \to \mathbb{N} \cup \{\infty\}$:
  $B(T, <, h) := \{t_1^{e_1} \dots t_r^{e_r} : t_1 > \dots > t_r, t_i \in T, 0 < e_i < h(t_i)\}$.
- $\deg t_i = \alpha \in \mathbb{Z}^\theta$, $h(t_i) < \infty$, $\Rightarrow h(t_i) = \operatorname{ord}(\chi(\alpha, \alpha)) =: N_\alpha$.
- $\Delta_+^V :=$ {degrees of generators of a PBW basis of $\mathfrak{B}(V)$}: it does not depend on the PBW basis.
- Δ_+^V **root system**:

$$\mathcal{H}_{\mathfrak{B}(V)} = \prod_{\alpha\in\Delta_+^V} (1 + x^\alpha + x^{2\alpha} + \dots + x^{\alpha(N_\alpha - 1)}).$$

$$-a_{ij} := \max\left\{n : (\mathrm{ad}_c x_i)^n x_j \neq 0\right\} = \max\left\{n : \alpha_j + n\alpha_i \in \Delta_+^V\right\},$$

$$-a_{ij} := \max\{n : (\mathrm{ad}_c x_i)^n x_j \neq 0\} = \max\left\{n : \alpha_j + n\alpha_i \in \Delta_+^V\right\},$$

$$s_i \in \mathrm{Aut}(\mathbb{Z}^\theta), \quad s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i \qquad (a_{ii} := 2).$$

$$-a_{ij} := \max\left\{n : (\mathrm{ad}_c x_i)^n x_j \neq 0\right\} = \max\left\{n : \alpha_j + n\alpha_i \in \Delta_+^V\right\},$$

$$s_i \in \mathrm{Aut}(\mathbb{Z}^\theta), \quad s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i \qquad (a_{ii} := 2).$$

Proposition (Heckenberger)

$\dim V_i = \theta$, $\widetilde{q}_{kj} = \chi(s_i(\alpha_k), s_i(\alpha_j))$,

$$\Delta_+^{V_i} = s_i\left(\Delta_+^V \setminus \{\alpha_i\}\right) \cup \{\alpha_i\}.$$

$$-a_{ij} := \max\left\{n : (\mathrm{ad}_c x_i)^n x_j \neq 0\right\} = \max\left\{n : \alpha_j + n\alpha_i \in \Delta_+^V\right\},$$

$s_i \in \mathrm{Aut}(\mathbb{Z}^\theta), \quad s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i \qquad (a_{ii} := 2).$

Proposition (Heckenberger)

$\dim V_i = \theta$, $\widetilde{q}_{kj} = \chi(s_i(\alpha_k), s_i(\alpha_j))$,

$$\Delta_+^{V_i} = s_i\left(\Delta_+^V \setminus \{\alpha_i\}\right) \cup \{\alpha_i\}.$$

$\rightsquigarrow$ Weyl groupoid: in some cases, $s_i(\Delta^V) = \Delta^{V_i} \neq \Delta^V$.

Definition

Weyl Groupoid and **generalized root system** (Heckenberger-Yamane): set of objects $\mathcal{X}$ (for us, a certain family of matrices (q_{ij})),

Definition

Weyl Groupoid and **generalized root system** (Heckenberger-Yamane): set of objects $\mathcal{X}$ (for us, a certain family of matrices (q_{ij})),

- $\Delta^X = \Delta^X_+ \cup -\Delta^X_+$, $\Delta_+ \subset \mathbb{N}_0^\theta$ $(X \in \mathcal{X})$,

Definition

Weyl Groupoid and **generalized root system** (Heckenberger-Yamane): set of objects $\mathcal{X}$ (for us, a certain family of matrices (q_{ij})),

- $\Delta^X = \Delta^X_+ \cup -\Delta^X_+$, $\Delta_+ \subset \mathbb{N}_0^\theta$ $(X \in \mathcal{X})$,
- symmetries s_i^X, $1 \leq i \leq \theta$,

Definition

Weyl Groupoid and **generalized root system** (Heckenberger-Yamane): set of objects $\mathcal{X}$ (for us, a certain family of matrices (q_{ij})),

- $\Delta^X = \Delta_+^X \cup -\Delta_+^X$, $\Delta_+ \subset \mathbb{N}_0^\theta$ $(X \in \mathcal{X})$,
- symmetries s_i^X, $1 \le i \le \theta$,
- $s_i^X(\alpha_j) = \alpha_j - a_{ij}^X \alpha_i$, $a_{ii}^X = 2$, $a_{ij}^X \in -\mathbb{N}_0$ if $i \neq j$;

Definition

Weyl Groupoid and **generalized root system** (Heckenberger-Yamane): set of objects $\mathcal{X}$ (for us, a certain family of matrices (q_{ij})),

- $\Delta^X = \Delta^X_+ \cup -\Delta^X_+$, $\Delta_+ \subset \mathbb{N}_0^\theta$ $(X \in \mathcal{X})$,
- symmetries s_i^X, $1 \leq i \leq \theta$,
- $s_i^X(\alpha_j) = \alpha_j - a_{ij}^X \alpha_i$, $a_{ii}^X = 2$, $a_{ij}^X \in -\mathbb{N}_0$ if $i \neq j$;
- $s_i^X(\Delta^X_+ - \{\alpha_i\}) = \Delta^Y_+ - \{\alpha_i\}$, if s_i^X goes to Y.

Definition

Weyl Groupoid and **generalized root system** (Heckenberger-Yamane): set of objects $\mathcal{X}$ (for us, a certain family of matrices (q_{ij})),

- $\Delta^X = \Delta^X_+ \cup -\Delta^X_+$, $\Delta_+ \subset \mathbb{N}_0^\theta$ $(X \in \mathcal{X})$,
- symmetries s_i^X, $1 \leq i \leq \theta$,
- $s_i^X(\alpha_j) = \alpha_j - a_{ij}^X \alpha_i$, $a_{ii}^X = 2$, $a_{ij}^X \in -\mathbb{N}_0$ if $i \neq j$;
- $s_i^X(\Delta^X_+ - \{\alpha_i\}) = \Delta^Y_+ - \{\alpha_i\}$, if s_i^X goes to Y.

If $\mathcal{X} = \{X\}$ $\rightsquigarrow$ classic root system + Weyl group.

Finite root system: Δ^X finite for some (all) $X \in \mathcal{X}$, i.e. the groupoid is finite ([HY]).

Finite root system: Δ^X finite for some (all) $X \in \mathcal{X}$, i.e. the groupoid is finite ([HY]).

$$\ell(w) = \min\{k \in \mathbb{N} : \ \exists i_1, ..., i_k \in I \text{ such that } w = s_{i_1} \cdots s_{i_k}\}.$$

Finite root system: Δ^X finite for some (all) $X \in \mathcal{X}$, i.e. the groupoid is finite ([HY]).

$$\ell(w) = \min\{k \in \mathbb{N} : \exists i_1, ..., i_k \in I \text{ such that } w = s_{i_1} \cdots s_{i_k}\}.$$

Proposition (Cuntz and Heckenberger)

If $w = \mathrm{id}_X s_{i_1} \cdots s_{i_m}$ is such that $\ell(w) = m$ (**reduced expression**), then $\beta_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j}) \in \Delta^X$ are positive and all different.

Finite root system: Δ^X finite for some (all) $X \in \mathcal{X}$, i.e. the groupoid is finite ([HY]).

$$\ell(w) = \min\{k \in \mathbb{N} : \exists i_1, \dots, i_k \in I \text{ such that } w = s_{i_1} \cdots s_{i_k}\}.$$

Proposition (Cuntz and Heckenberger)

If $w = \mathrm{id}_X s_{i_1} \cdots s_{i_m}$ is such that $\ell(w) = m$ (**reduced expression**), then $\beta_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j}) \in \Delta^X$ are positive and all different.

Finite root system: Δ^X finite for some (all) $X \in \mathcal{X}$, i.e. the groupoid is finite ([HY]).

$$\ell(w) = \min\{k \in \mathbb{N} : \exists i_1, ..., i_k \in I \text{ such that } w = s_{i_1} \cdots s_{i_k}\}.$$

Proposition (Cuntz and Heckenberger)

If $w = \mathrm{id}_X s_{i_1} \cdots s_{i_m}$ is such that $\ell(w) = m$ (**reduced expression**), then $\beta_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j}) \in \Delta^X$ are positive and all different.

There exists a unique w_0^X of maximal length por any $X \in \mathcal{X}$, and so $\{\beta_j\} = \Delta^X_+$: all the roots are real and of multiplicity one.

k an algebraically closed field, $\operatorname{char} k = 0$. $\mathbb{G}_N$ group of roots of unity such that $q^N = 1$.

Theorem (General presentation)

$\dim V = \theta$, $(q_{ij}) \in (k^\times)^{\theta\times\theta}$ such that $|\Delta_+^V| < \infty$. $\mathfrak{B}(V)$ is presented by generators $x_1, \dots, x_\theta$ and relations:

k an algebraically closed field, $\operatorname{char} \Bbbk = 0$. $\mathbb{G}_N$ group of roots of unity such that $q^N = 1$.

Theorem (General presentation)

$\dim V = \theta$, $(q_{ij}) \in (\Bbbk^\times)^{\theta \times \theta}$ such that $|\Delta_+^V| < \infty$. $\mathfrak{B}(V)$ is presented by generators $x_1, \dots, x_\theta$ and relations:

1. $x_\beta^{N_\beta} = 0, \qquad \beta \in \Delta_+^V,\ N_\beta < \infty,$

k an algebraically closed field, $\operatorname{char} \Bbbk = 0$. $\mathbb{G}_N$ group of roots of unity such that $q^N = 1$.

Theorem (General presentation)

$\dim V = \theta$, $(q_{ij}) \in (\Bbbk^\times)^{\theta\times\theta}$ such that $|\Delta_+^V| < \infty$. $\mathfrak{B}(V)$ is presented by generators $x_1, \dots, x_\theta$ and relations:

1. $x_\beta^{N_\beta} = 0, \qquad \beta \in \Delta_+^V,\ N_\beta < \infty,$
2. $[x_\alpha, x_\beta]_c = \sum_{\deg u = \alpha+\beta} c_{\alpha,\beta}^u\, u, \qquad \alpha < \beta,$
   u: elements of the PBW basis written in letters $x_\gamma, \alpha \le \gamma \le \beta$.

k an algebraically closed field, char k $= 0$. $\mathbb{G}_N$ group of roots of unity such that $q^N = 1$.

Theorem (General presentation)

$\dim V = \theta$, $(q_{ij}) \in (\Bbbk^\times)^{\theta\times\theta}$ such that $|\Delta_+^V| < \infty$. $\mathfrak{B}(V)$ is presented by generators $x_1, \dots, x_\theta$ and relations:

1. $x_\beta^{N_\beta} = 0, \qquad \beta \in \Delta_+^V,\ N_\beta < \infty,$
2. $[x_\alpha, x_\beta]_c = \sum_{\deg u = \alpha+\beta} c_{\alpha,\beta}^u\, u, \qquad \alpha < \beta,$
   u: elements of the PBW basis written in letters $x_\gamma, \alpha \leq \gamma \leq \beta$.

Explicit formula for the coefficients $c_{\alpha,\beta}^u$.

k an algebraically closed field, $\operatorname{char} \Bbbk = 0$. $\mathbb{G}_N$ group of roots of unity such that $q^N = 1$.

Theorem (General presentation)

$\dim V = \theta$, $(q_{ij}) \in (\Bbbk^\times)^{\theta \times \theta}$ such that $|\Delta_+^V| < \infty$. $\mathfrak{B}(V)$ is presented by generators $x_1, \dots, x_\theta$ and relations:

1. $x_\beta^{N_\beta} = 0, \qquad \beta \in \Delta_+^V,\ N_\beta < \infty,$
2. $[x_\alpha, x_\beta]_c = \sum_{\deg u = \alpha+\beta} c^u_{\alpha,\beta}\ u, \qquad \alpha < \beta,$ $\longleftarrow\!\!\longleftarrow$
   u: elements of the PBW basis written in letters $x_\gamma, \alpha \leq \gamma \leq \beta$.

Generalization of quantum Serre relations:

$$0 = (\operatorname{ad}_c x_i)^{1-a_{ij}} x_j = [x_i, (\operatorname{ad}_c x_i)^{-a_{ij}} x_j]_c.$$

$w = \mathrm{id}_V s_{i_1} \cdots s_{i_k} \in \mathrm{Hom}(W, V)$ reduced expression:

- $L_w = \{\alpha \in \Delta_+^V : w^{-1}(\alpha) \in \Delta_-^W\}$.

$w = \mathrm{id}_V s_{i_1} \cdots s_{i_k} \in \mathrm{Hom}(W, V)$ reduced expression:

- $L_w = \{\alpha \in \Delta_+^V : w^{-1}(\alpha) \in \Delta_-^W\}$.
- order associated to $s_{i_1} \cdots s_{i_k}$:

$$\alpha_{i_1} < s_{i_1}(\alpha_{i_2}) < \ldots < s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}).$$

$w = \mathrm{id}_V s_{i_1} \cdots s_{i_k} \in \mathrm{Hom}(W, V)$ reduced expression:

- $L_w = \{\alpha \in \Delta_+^V : w^{-1}(\alpha) \in \Delta_-^W\}$.
- order associated to $s_{i_1} \cdots s_{i_k}$:

$$\alpha_{i_1} < s_{i_1}(\alpha_{i_2}) < \ldots < s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}).$$

Definition

A total order $<$ en Δ_+^V is **convex** if for each $\alpha, \beta \in \Delta^+$, $\alpha < \beta$, $\alpha + \beta \in \Delta^+$, it holds $\alpha < \alpha + \beta < \beta$.
It is **strongly convex** if for each $\beta = \sum \beta_j \in \Delta^+$, $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_n$, it holds $\beta_1 < \beta < \beta_n$.

Theorem

Given $<$ on Δ_+^V, the following statements are equivalent:

Theorem

Given $<$ on Δ_+^V, the following statements are equivalent:

1. the order is associated to an expression of the element of maximal length of $\mathrm{Hom}(\mathcal{W}, V)$,

Theorem

Given $<$ on Δ_+^V, the following statements are equivalent:

1. the order is associated to an expression of the element of maximal length of $\mathrm{Hom}(\mathcal{W}, V)$,
2. the order is strongly convex,

Theorem

Given $<$ on Δ_+^V, the following statements are equivalent:

1. the order is associated to an expression of the element of maximal length of $\mathrm{Hom}(\mathcal{W}, V)$,
2. the order is strongly convex,
3. the order is convex.

Theorem

Given $<$ on Δ_+^V, the following statements are equivalent:

1. the order is associated to an expression of the element of maximal length of $\mathrm{Hom}(\mathcal{W}, V)$,
2. the order is strongly convex,
3. the order is convex.

Theorem

The order on Kharchenko's PBW generators is convex.

Theorem

Given $<$ on Δ_+^V, the following statements are equivalent:

1. the order is associated to an expression of the element of maximal length of $\mathrm{Hom}(\mathcal{W}, V)$,
2. the order is strongly convex,
3. the order is convex.

Theorem

The order on Kharchenko's PBW generators is convex.

Proposition

The Kharchenko's PBW basis of $\mathfrak{B}(V)$ is orthogonal for $(\cdot|\cdot)$.

Remark

Remark

- Fundamental step: clssification of coideal subalgebras of $\mathfrak{B}(V)$, with a bijection with the Weyl groupoid presenving orders (Heckenberger-Schneider).

Remark

- Fundamental step: clssification of coideal subalgebras of $\mathfrak{B}(V)$, with a bijection with the Weyl groupoid presenving orders (Heckenberger-Schneider).
- Finitely generated ideal.

Remark

- Fundamental step: clssification of coideal subalgebras of $\mathfrak{B}(V)$, with a bijection with the Weyl groupoid presenving orders (Heckenberger-Schneider).
- Finitely generated ideal.
- Proof does not involve Heckenberger's classification.

Remark

- Fundamental step: clssification of coideal subalgebras of $\mathfrak{B}(V)$, with a bijection with the Weyl groupoid presenving orders (Heckenberger-Schneider).
- Finitely generated ideal.
- Proof does not involve Heckenberger's classification.
- Key step to obtain a minimal presentation.

Theorem (Minimal presentation)

$(q_{ij})_{1\le i,j\le\theta}$, $\theta = \dim V$, $\Delta_+^V = \{\beta_1, \dots, \beta_M\}$ finite. $\mathfrak{B}(V)$ presented by generators $x_1, \dots, x_\theta$ and relations:

$$x_\alpha^{N_\alpha}, \qquad \alpha \in \mathcal{O}(\chi);$$

$$(\mathrm{ad}_c x_i)^{m_{ij}+1} x_j, \qquad q_{ii}^{m_{ij}+1} \neq 1;$$

$$x_i^{N_i}, \qquad i \text{ a non Cartan vertex};$$

if $q_{ii} = q_{ij}q_{ji} = q_{jj} = -1$, $\quad ((\mathrm{ad}_c x_i)x_j)^2$;

if $q_{jj} = -1$, $q_{ik}q_{ki} = q_{ij}q_{ji}q_{jk}q_{kj} = 1$, $\quad [(\mathrm{ad}_c x_i)(\mathrm{ad}_c x_j)x_k, x_j]_c$;

if $q_{jj} = -1$, $q_{ii}q_{ij}q_{ji} \in \mathbb{G}_6$, and also $q_{ii} \in \mathbb{G}_3$ or $m_{ij} \geq 3$,

$$\left[(\mathrm{ad}_c x_i)^2 x_j, (\mathrm{ad}_c x_i)x_j\right]_c;$$

Theorem (Minimal presentation)

if $q_{ii} = \pm q_{ij} q_{ji} \in \mathbb{G}_3$, $q_{ik} q_{ki} = 1$, and also $-q_{jj} = q_{ji} q_{ij} q_{jk} q_{kj} = 1$ or $q_{jj}^{-1} = q_{ji} q_{ij} = q_{jk} q_{kj} \neq -1$,

$$\left[(\mathrm{ad}_c x_i)^2 (\mathrm{ad}_c x_j) x_k, (\mathrm{ad}_c x_i) x_j \right]_c ;$$

Theorem (Minimal presentation)

if $q_{ii} = \pm q_{ij} q_{ji} \in \mathbb{G}_3$, $q_{ik} q_{ki} = 1$, and also $-q_{jj} = q_{ji} q_{ij} q_{jk} q_{kj} = 1$ or $q_{jj}^{-1} = q_{ji} q_{ij} = q_{jk} q_{kj} \neq -1$,

$$\left[(\mathrm{ad}_c x_i)^2 (\mathrm{ad}_c x_j) x_k, (\mathrm{ad}_c x_i) x_j \right]_c ;$$

Theorem (Minimal presentation)

if $q_{ii} = \pm q_{ij} q_{ji} \in \mathbb{G}_3$, $q_{ik} q_{ki} = 1$, and also $-q_{jj} = q_{ji} q_{ij} q_{jk} q_{kj} = 1$ or $q_{jj}^{-1} = q_{ji} q_{ij} = q_{jk} q_{kj} \neq -1$,

$$\left[(\mathrm{ad}_c x_i)^2 (\mathrm{ad}_c x_j) x_k, (\mathrm{ad}_c x_i) x_j\right]_c;$$

if $q_{ik} q_{ki}, q_{ij} q_{ji}, q_{jk} q_{kj} \neq 1$,

$$[x_i, (\mathrm{ad}_c x_j) x_k]_c - \frac{1 - q_{jk} q_{kj}}{q_{kj}(1 - q_{ik} q_{ki})} [(\mathrm{ad}_c x_i) x_k, x_j]_c - q_{ij}(1 - q_{kj} q_{jk}) x_j (\mathrm{ad}_c x_i) x_k;$$

Theorem (Minimal presentation)

if $q_{ii} = \pm q_{ij}q_{ji} \in \mathbb{G}_3$, $q_{ik}q_{ki} = 1$, and also $-q_{jj} = q_{ji}q_{ij}q_{jk}q_{kj} = 1$ or $q_{jj}^{-1} = q_{ji}q_{ij} = q_{jk}q_{kj} \neq -1$,

$$\left[(\mathrm{ad}_c x_i)^2(\mathrm{ad}_c x_j)x_k, (\mathrm{ad}_c x_i)x_j\right]_c;$$

if $q_{ik}q_{ki}, q_{ij}q_{ji}, q_{jk}q_{kj} \neq 1$,

$$[x_i, (\mathrm{ad}_c x_j)x_k]_c - \frac{1-q_{jk}q_{kj}}{q_{kj}(1-q_{ik}q_{ki})}[(\mathrm{ad}_c x_i)x_k, x_j]_c - q_{ij}(1-q_{kj}q_{jk})x_j(\mathrm{ad}_c x_i)x_k;$$

if $i, j, k \in \{1, \dots, \theta\}$ are such that

- $q_{ii} = q_{jj} = -1$, $(q_{ij}q_{ji})^2 = (q_{jk}q_{kj})^{-1}$, $q_{ik}q_{ki} = 1$, or

$$\left[[(\mathrm{ad}_c x_i)x_j, (\mathrm{ad}_c x_i)(\mathrm{ad}_c x_j)x_k]_c, x_j\right]_c;$$

Theorem (Minimal presentation)

if $q_{ii} = \pm q_{ij}q_{ji} \in \mathbb{G}_3$, $q_{ik}q_{ki} = 1$, and also $-q_{jj} = q_{ji}q_{ij}q_{jk}q_{kj} = 1$ or $q_{jj}^{-1} = q_{ji}q_{ij} = q_{jk}q_{kj} \neq -1$,

$$\left[(\mathrm{ad}_c x_i)^2(\mathrm{ad}_c x_j)x_k, (\mathrm{ad}_c x_i)x_j\right]_c;$$

if $q_{ik}q_{ki}, q_{ij}q_{ji}, q_{jk}q_{kj} \neq 1$,

$$[x_i, (\mathrm{ad}_c x_j)x_k]_c - \frac{1-q_{jk}q_{kj}}{q_{kj}(1-q_{ik}q_{ki})}[(\mathrm{ad}_c x_i)x_k, x_j]_c - q_{ij}(1-q_{kj}q_{jk})x_j(\mathrm{ad}_c x_i)x_k;$$

if $i, j, k \in \{1, \dots, \theta\}$ are such that

- $q_{ii} = q_{jj} = -1$, $(q_{ij}q_{ji})^2 = (q_{jk}q_{kj})^{-1}$, $q_{ik}q_{ki} = 1$, or
- $q_{jj} = q_{kk} = q_{jk}q_{kj} = -1$, $q_{ii} = -q_{ij}q_{ji} \in \mathbb{G}_3$, $q_{ik}q_{ki} = 1$, or

$$\left[[(\mathrm{ad}_c x_i)x_j, (\mathrm{ad}_c x_i)(\mathrm{ad}_c x_j)x_k]_c, x_j\right]_c;$$

Theorem (Minimal presentation)

if $q_{ii} = \pm q_{ij}q_{ji} \in \mathbb{G}_3$, $q_{ik}q_{ki} = 1$, and also $-q_{jj} = q_{ji}q_{ij}q_{jk}q_{kj} = 1$ or $q_{jj}^{-1} = q_{ji}q_{ij} = q_{jk}q_{kj} \neq -1$,

$$\left[(\mathrm{ad}_c x_i)^2(\mathrm{ad}_c x_j)x_k, (\mathrm{ad}_c x_i)x_j\right]_c;$$

if $q_{ik}q_{ki}, q_{ij}q_{ji}, q_{jk}q_{kj} \neq 1$,

$$[x_i, (\mathrm{ad}_c x_j)x_k]_c - \frac{1 - q_{jk}q_{kj}}{q_{kj}(1 - q_{ik}q_{ki})}[(\mathrm{ad}_c x_i)x_k, x_j]_c - q_{ij}(1 - q_{kj}q_{jk})x_j(\mathrm{ad}_c x_i)x_k;$$

if $i, j, k \in \{1, \dots, \theta\}$ are such that

- $q_{ii} = q_{jj} = -1$, $(q_{ij}q_{ji})^2 = (q_{jk}q_{kj})^{-1}$, $q_{ik}q_{ki} = 1$, or
- $q_{jj} = q_{kk} = q_{jk}q_{kj} = -1$, $q_{ii} = -q_{ij}q_{ji} \in \mathbb{G}_3$, $q_{ik}q_{ki} = 1$, or
- $q_{ii} = q_{jj} = q_{kk} = -1$, $q_{ij}q_{ji} = q_{kj}q_{jk} \in \mathbb{G}_3$, $q_{ik}q_{ki} = 1$, or

$$\left[[(\mathrm{ad}_c x_i)x_j, (\mathrm{ad}_c x_i)(\mathrm{ad}_c x_j)x_k]_c, x_j\right]_c;$$

Theorem (Minimal presentation)

if $q_{ii} = \pm q_{ij} q_{ji} \in \mathbb{G}_3$, $q_{ik} q_{ki} = 1$, and also $-q_{jj} = q_{ji} q_{ij} q_{jk} q_{kj} = 1$ or $q_{jj}^{-1} = q_{ji} q_{ij} = q_{jk} q_{kj} \neq -1$,

$$\left[(\mathrm{ad}_c x_i)^2 (\mathrm{ad}_c x_j) x_k, (\mathrm{ad}_c x_i) x_j\right]_c;$$

if $q_{ik} q_{ki}, q_{ij} q_{ji}, q_{jk} q_{kj} \neq 1$,

$$[x_i, (\mathrm{ad}_c x_j) x_k]_c - \frac{1 - q_{jk} q_{kj}}{q_{kj}(1 - q_{ik} q_{ki})} [(\mathrm{ad}_c x_i) x_k, x_j]_c - q_{ij}(1 - q_{kj} q_{jk}) x_j (\mathrm{ad}_c x_i) x_k;$$

if $i, j, k \in \{1, \dots, \theta\}$ are such that

- $q_{ii} = q_{jj} = -1$, $(q_{ij} q_{ji})^2 = (q_{jk} q_{kj})^{-1}$, $q_{ik} q_{ki} = 1$, or
- $q_{jj} = q_{kk} = q_{jk} q_{kj} = -1$, $q_{ii} = -q_{ij} q_{ji} \in \mathbb{G}_3$, $q_{ik} q_{ki} = 1$, or
- $q_{ii} = q_{jj} = q_{kk} = -1$, $q_{ij} q_{ji} = q_{kj} q_{jk} \in \mathbb{G}_3$, $q_{ik} q_{ki} = 1$, or
- $q_{ii} = q_{kk} = -1$, $q_{jj} = -q_{kj} q_{jk} = (q_{ij} q_{ji})^{\pm 1} \in \mathbb{G}_3$, $q_{ik} q_{ki} = 1$,

$$\left[[(\mathrm{ad}_c x_i) x_j, (\mathrm{ad}_c x_i)(\mathrm{ad}_c x_j) x_k]_c, x_j\right]_c;$$

Theorem (Minimal presentation)

if $q_{ii} = q_{jj} = -1$, $(q_{ij}q_{ji})^3 = (q_{jk}q_{kj})^{-1}$, $q_{ik}q_{ki} = 1$,

$$\left[\left[(\mathrm{ad}_c x_i)x_j, [(\mathrm{ad}_c x_i)x_j, (\mathrm{ad}_c x_i)(\mathrm{ad}_c x_j)x_k]_c\right]_c, x_j\right]_c;$$

if $q_{jj}q_{ij}q_{ji} = q_{jj}q_{kj}q_{jk} = 1$, $(q_{kj}q_{jk})^2 = (q_{lk}q_{kl})^{-1} = q_{ll}$, $q_{kk} = -1$, $q_{ik}q_{ki} = q_{il}q_{li} = q_{jl}q_{lj} = 1$,

$$\left[\left[[(\mathrm{ad}_c x_i)(\mathrm{ad}_c x_j)(\mathrm{ad}_c x_k)x_l, x_k]_c, x_j\right]_c, x_k\right]_c;$$

if $q_{jj} = q_{ij}^{-1}q_{ji}^{-1} = q_{jk}q_{kj} \in \mathbb{G}_3$,

$$\left[[(\mathrm{ad}_c x_i)(\mathrm{ad}_c x_j)x_k, x_j]_c\, x_j\right]_c;$$

if $q_{jj} = q_{ij}^{-1}q_{ji}^{-1} = q_{jk}q_{kj} \in \mathbb{G}_4$,

$$\left[\left[[(\mathrm{ad}_c x_i)(\mathrm{ad}_c x_j)x_k, x_j]_c, x_j\right]_c, x_j\right]_c;$$

Theorem (Minimal presentation)

if $q_{ii} = -1$, $q_{jj}^{-1} = -q_{ij}q_{ji}q_{jk}q_{kj} \notin \{-1, q_{ij}q_{ji}\}$, $q_{ik}q_{ki} = 1$,

$$[(\mathrm{ad}_c x_i)x_j, (\mathrm{ad}_c x_i)(\mathrm{ad}_c x_j)x_k]_c;$$

if $q_{jk}q_{kj} = 1$, $q_{ii} \in \mathbb{G}_3$, $q_{ij}q_{ji}, q_{ki}q_{ik} \neq q_{ii}^{-1}$,

$$\left[(\mathrm{ad}_c x_i)^2 x_j, (\mathrm{ad}_c x_i)^2 x_k\right]_c;$$

if $-q_{ii}, -q_{jj}, q_{ii}q_{ij}q_{ji}, q_{jj}q_{ji}q_{ij} \neq 1$,

$$(1 - q_{ij}q_{ji})q_{jj}q_{ji}\left[x_i, [(\mathrm{ad}_c x_i)x_j, x_j]_c\right]_c - (1 + q_{jj})(1 - q_{jj}q_{ji}q_{ij})\left((\mathrm{ad}_c x_i)x_j\right)^2;$$

if $q_{jj} = -1$, $q_{ii}q_{ij}q_{ji} \notin \mathbb{G}_6$, and also $m_{ij} \in \{4, 5\}$, or $m_{ij} = 3$, $q_{ii} \in \mathbb{G}_4$,

$$\left[x_i, \left[(\mathrm{ad}_c x_i)^2 x_j, (\mathrm{ad}_c x_i)x_j\right]_c\right]_c - \frac{1 - q_{ii}q_{ji}q_{ij} - q_{ii}^2 q_{ji}^2 q_{ij}^2 q_{jj}}{(1 - q_{ii}q_{ij}q_{ji})q_{ji}}\left((\mathrm{ad}_c x_i)^2 x_j\right)^2;$$

Theorem (Minimal presentation)

if $4\alpha_i + 3\alpha_j \notin \Delta_+^\chi$, $q_{jj} = -1$ or $m_{ji} \geq 2$, and $m_{ij} \geq 3$, or $m_{ij} = 2$, $q_{ii} \in \mathbb{G}_3$,

$$[x_{3\alpha_i+2\alpha_j}, (\mathrm{ad}_c x_i) x_j]_c;$$

if $3\alpha_i + 2\alpha_j \in \Delta_+^\chi$, $5\alpha_i + 3\alpha_j \notin \Delta_+^\chi$, and $q_{ii}^3 q_{ij} q_{ji}, q_{ii}^4 q_{ij} q_{ji} \neq 1$,

$$[(\mathrm{ad}_c x_i)^2 x_j, x_{3\alpha_i+2\alpha_j}]_c;$$

if $4\alpha_i + 3\alpha_j \in \Delta_+^\chi$, $5\alpha_i + 4\alpha_j \notin \Delta_+^\chi$,

$$[x_{4\alpha_i+3\alpha_j}, (\mathrm{ad}_c x_i) x_j]_c;$$

if $q_{jj} = -1$, $5\alpha_i + 4\alpha_j \in \Delta_+^\chi$,

$$[x_{2\alpha_i+\alpha_j}, x_{4\alpha_i+3\alpha_j}]_c - \frac{b - (1+q_{ii})(1-q_{ii}\zeta)(1+\zeta+q_{ii}\zeta^2) q_{ii}^6 \zeta^4}{a\ q_{ii}^3 q_{ij}^2 q_{ji}^3} x_{3\alpha_i+2\alpha_j}^2.$$

Andruskiewitsch-Schneider Conjecture: Any finite-dimensional pointed Hopf algebra is generated by group-like and skew-primitive elements.

Andruskiewitsch-Schneider Conjecture: Any finite-dimensional pointed Hopf algebra is generated by group-like and skew-primitive elements.

Theorem

True when $G(H)$ is abelian.

Andruskiewitsch-Schneider Conjecture: Any finite-dimensional pointed Hopf algebra is generated by group-like and skew-primitive elements.

Theorem

True when $G(H)$ is abelian.

That is, every f.d. pointed Hopf algebra over an abelian group is a deformation of some $\mathfrak{B}(V)\#\Bbbk\Gamma$.

Andruskiewitsch-Schneider Conjecture: Any finite-dimensional pointed Hopf algebra is generated by group-like and skew-primitive elements.

Theorem

True when $G(H)$ is abelian.

Problem: Obtain all the deformations (*liftings*) of $H = \mathfrak{B}(V)\#\Bbbk\Gamma$, Γ abelian, which are pointed Hopf algebras.
Work in progress: Andruskiewitsch - A. - García Iglesias

About the proof: use Lusztig's isomorphisms T_i moving through the Weyl groupoid (Heckenberger).

About the proof: use Lusztig's isomorphisms T_i moving through the Weyl groupoid (Heckenberger).

$U(\chi) = D(T(V,\chi)\#\mathbb{Z}^\theta)$, $\mathcal{U}(\chi) = D(\mathfrak{B}(V,\chi)\#\mathbb{Z}^\theta)$,

About the proof: use Lusztig's isomorphisms T_i moving through the Weyl groupoid (Heckenberger).

$U(\chi) = D(T(V,\chi)\#\mathbb{Z}^\theta)$, $\mathcal{U}(\chi) = D(\mathfrak{B}(V,\chi)\#\mathbb{Z}^\theta)$,
$I_i(\chi)$ ideal generated by $(\mathrm{ad}_c E_i)^{1-a_{ij}} E_j$, $(\mathrm{ad}_c F_i)^{1-a_{ij}} F_j$ and/or $E_i^{N_i}, F_i^{N_i}$, depending on i,

About the proof: use Lusztig's isomorphisms T_i moving through the Weyl groupoid (Heckenberger).

$U(\chi) = D(T(V,\chi)\#\mathbb{Z}^\theta)$, $\mathcal{U}(\chi) = D(\mathfrak{B}(V,\chi)\#\mathbb{Z}^\theta)$,
$I_i(\chi)$ ideal generated by $(\mathrm{ad}_c E_i)^{1-a_{ij}} E_j$, $(\mathrm{ad}_c F_i)^{1-a_{ij}} F_j$ and/or $E_i^{N_i}, F_i^{N_i}$, depending on i,

$$\begin{array}{ccc} U(\chi)/I_i(\chi) & \twoheadrightarrow & \mathcal{U}(\chi) \\ \downarrow T_i & & \downarrow T_i \\ U(s_i^*\chi)/I_i(s_i^*\chi) & \twoheadrightarrow & \mathcal{U}(s_i^*\chi) \end{array} .$$

About the proof: use Lusztig's isomorphisms T_i moving through the Weyl groupoid (Heckenberger).

$\widetilde{\mathcal{U}}(\chi) = D(\widetilde{\mathfrak{B}}(V,\chi)\#\mathbb{Z}^\theta)$, $\widetilde{\mathfrak{B}}(V,\chi) = T(V,\chi)/I(\chi)$,
$I(\chi)$: enough relations to ensure the existence of all the isomorphisms. Just does not contain the power root vectors.

$$
\begin{array}{ccccc}
U(\chi)/I_i(\chi) & \twoheadrightarrow & \widetilde{\mathcal{U}}(\chi) & \twoheadrightarrow & \mathcal{U}(\chi) \\
\downarrow{\scriptstyle T_i} & & \downarrow{\scriptstyle T_i} & & \downarrow{\scriptstyle T_i} \\
U(s_i^*\chi)/I_i(s_i^*\chi) & \twoheadrightarrow & \widetilde{\mathcal{U}}(s_i^*\chi) & \twoheadrightarrow & \mathcal{U}(s_i^*\chi)
\end{array}
$$

Generalized Dynkin diagrams (Heckenberger)

$$\begin{pmatrix} q_{ii} & q_{ij} \\ q_{ji} & q_{jj} \end{pmatrix}: \quad \circ^{q_{ii}} \qquad \circ^{q_{jj}} \qquad q_{ij}q_{ji} = 1$$

Generalized Dynkin diagrams (Heckenberger)

$$\begin{pmatrix} q_{ii} & q_{ij} \\ q_{ji} & q_{jj} \end{pmatrix}: \quad \circ^{q_{ii}} \qquad \circ^{q_{jj}} \qquad q_{ij}q_{ji} = 1$$

$$\circ^{q_{ii}} \overset{q_{ij}q_{ji}}{\text{———}} \circ^{q_{jj}} \qquad q_{ij}q_{ji} \neq 1.$$

Example (Matrices 'super')

Type $G(3)$: $q \in \mathrm{k}^{\times}$, $q^3, q^2 \neq 1$,

Generalized Dynkin diagrams (Heckenberger)

$$\begin{pmatrix} q_{ii} & q_{ij} \\ q_{ji} & q_{jj} \end{pmatrix}: \quad \circ^{q_{ii}} \qquad \circ^{q_{jj}} \qquad q_{ij}q_{ji} = 1$$

$$\circ^{q_{ii}} \overset{q_{ij}q_{ji}}{\text{———}} \circ^{q_{jj}} \qquad q_{ij}q_{ji} \neq 1.$$

Example (Matrices 'super')

Type $G(3)$: $q \in \Bbbk^{\times}$, $q^3, q^2 \neq 1$,

Generalized Dynkin diagrams (Heckenberger)

$\begin{pmatrix} q_{ii} & q_{ij} \\ q_{ji} & q_{jj} \end{pmatrix}$: $\quad \overset{q_{ii}}{\circ} \qquad \overset{q_{jj}}{\circ} \qquad q_{ij}q_{ji} = 1$

$\overset{q_{ii}}{\circ} \xrightarrow{\ q_{ij}q_{ji}\ } \overset{q_{jj}}{\circ} \qquad q_{ij}q_{ji} \neq 1.$

Example (Matrices 'super')

Type $G(3)$: $q \in \Bbbk^{\times}$, $q^3, q^2 \neq 1$,

(a) $\overset{-1}{\circ} \overset{q^{-1}}{\text{———}} \overset{q}{\circ} \overset{q^{-3}}{\text{———}} \overset{q^3}{\circ}$,

Generalized Dynkin diagrams (Heckenberger)

$\begin{pmatrix} q_{ii} & q_{ij} \\ q_{ji} & q_{jj} \end{pmatrix}$: $\circ^{q_{ii}} \qquad \circ^{q_{jj}} \qquad q_{ij}q_{ji} = 1$

$\circ^{q_{ii}} \overset{q_{ij}q_{ji}}{\text{———}} \circ^{q_{jj}} \qquad q_{ij}q_{ji} \neq 1.$

Example (Matrices 'super')

Type $G(3)$: $q \in \Bbbk^{\times}$, $q^3, q^2 \neq 1$,

(a) $\circ^{-1} \overset{q^{-1}}{\text{———}} \circ^{q} \overset{q^{-3}}{\text{———}} \circ^{q^3}$, (b) $\circ^{-1} \overset{q}{\text{———}} \circ^{-1} \overset{q^{-3}}{\text{———}} \circ^{q^3}$,

Generalized Dynkin diagrams (Heckenberger)

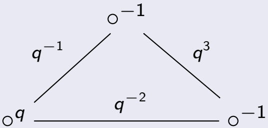
$$\begin{pmatrix} q_{ii} & q_{ij} \\ q_{ji} & q_{jj} \end{pmatrix}: \qquad \circ^{q_{ii}} \qquad\qquad \circ^{q_{jj}} \qquad q_{ij}q_{ji} = 1$$

$$\circ^{q_{ii}} \ \overset{q_{ij}q_{ji}}{\text{―――――}} \ \circ^{q_{jj}} \qquad q_{ij}q_{ji} \neq 1.$$

Example (Matrices 'super')

Type $G(3)$: $q \in \Bbbk^{\times}$, $q^3, q^2 \neq 1$,

(a) $\circ^{-1} \overset{q^{-1}}{\text{―――――}} \circ^{q} \overset{q^{-3}}{\text{―――――}} \circ^{q^3}$, (b) $\circ^{-1} \overset{q}{\text{―――――}} \circ^{-1} \overset{q^{-3}}{\text{―――――}} \circ^{q^3}$,

(c) 

,

Generalized Dynkin diagrams (Heckenberger)

$$\begin{pmatrix} q_{ii} & q_{ij} \\ q_{ji} & q_{jj} \end{pmatrix}: \qquad \circ^{q_{ii}} \qquad\quad \circ^{q_{jj}} \qquad q_{ij}q_{ji} = 1$$

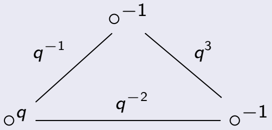
$$\circ^{q_{ii}} \overset{q_{ij}q_{ji}}{\text{———}} \circ^{q_{jj}} \qquad q_{ij}q_{ji} \neq 1.$$

Example (Matrices 'super')

Type $G(3)$: $q \in \mathrm{k}^{\times}$, $q^3, q^2 \neq 1$,

(a) $\circ^{-1} \overset{q^{-1}}{\text{———}} \circ^{q} \overset{q^{-3}}{\text{———}} \circ^{q^3}$, (b) $\circ^{-1} \overset{q}{\text{———}} \circ^{-1} \overset{q^{-3}}{\text{———}} \circ^{q^3}$,

(c)



, (d) $\circ^{-q^{-1}} \overset{q^2}{\text{———}} \circ^{-1} \overset{q^{-3}}{\text{———}} \circ^{q^3}$.

Generalized Dynkin diagrams (Heckenberger)

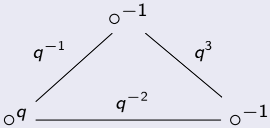
$$\begin{pmatrix} q_{ii} & q_{ij} \\ q_{ji} & q_{jj} \end{pmatrix}: \qquad \circ^{q_{ii}} \qquad\qquad \circ^{q_{jj}} \qquad q_{ij}q_{ji} = 1$$

$$\circ^{q_{ii}} \overset{q_{ij}q_{ji}}{\text{———}} \circ^{q_{jj}} \qquad q_{ij}q_{ji} \neq 1.$$

Example (Matrices 'super')

Type $G(3)$: $q \in \Bbbk^{\times}$, $q^3, q^2 \neq 1$,

(a) $\circ^{-1} \overset{q^{-1}}{\text{———}} \circ^{q} \overset{q^{-3}}{\text{———}} \circ^{q^3}$, (b) $\circ^{-1} \overset{q}{\text{———}} \circ^{-1} \overset{q^{-3}}{\text{———}} \circ^{q^3}$,

(c) 

, (d) $\circ^{-q^{-1}} \overset{q^2}{\text{———}} \circ^{-1} \overset{q^{-3}}{\text{———}} \circ^{q^3}$.

$$\Delta_+^a \neq \Delta_+^b \neq \Delta_+^c \neq \Delta_+^d$$

Example (Super $G(3)$)

(a) Admits a presentation by generators x_1, x_2, x_3 and relations

$$x_1^2 = x_\alpha^{N_\alpha} = 0, \qquad \alpha \in \Delta_+^\chi, N_\alpha \neq 2,$$
$$(\mathrm{ad}_c x_2)^2 x_1 = (\mathrm{ad}_c x_1) x_3 = (\mathrm{ad}_c x_2)^4 x_3 = (\mathrm{ad}_c x_3)^2 x_2 = 0.$$

Example (Super $G(3)$)

(a) Admits a presentation by generators x_1, x_2, x_3 and relations

$$x_1^2 = x_\alpha^{N_\alpha} = 0, \qquad \alpha \in \Delta_+^\chi, N_\alpha \neq 2,$$
$$(\mathrm{ad}_c x_2)^2 x_1 = (\mathrm{ad}_c x_1) x_3 = (\mathrm{ad}_c x_2)^4 x_3 = (\mathrm{ad}_c x_3)^2 x_2 = 0.$$

Example (Super $G(3)$)

(a) Admits a presentation by generators x_1, x_2, x_3 and relations

$$x_1^2 = x_\alpha^{N_\alpha} = 0, \qquad \alpha \in \Delta_+^\chi, N_\alpha \neq 2,$$
$$(\mathrm{ad}_c x_2)^2 x_1 = (\mathrm{ad}_c x_1) x_3 = (\mathrm{ad}_c x_2)^4 x_3 = (\mathrm{ad}_c x_3)^2 x_2 = 0.$$

Add too $\left[\left[[(\mathrm{ad}_c x_i)(\mathrm{ad}_c x_j) x_k, x_j]_c, x_j\right]_c, x_j\right]_c$, if $q \in \mathbb{G}_4$.

Example (Super $G(3)$)

(a) Admits a presentation by generators x_1, x_2, x_3 and relations

$$x_1^2 = x_\alpha^{N_\alpha} = 0, \qquad \alpha \in \Delta_+^\chi, N_\alpha \neq 2,$$
$$(\mathrm{ad}_c x_2)^2 x_1 = (\mathrm{ad}_c x_1) x_3 = (\mathrm{ad}_c x_2)^4 x_3 = (\mathrm{ad}_c x_3)^2 x_2 = 0.$$

Add too $\left[\left[[(\mathrm{ad}_c x_i)(\mathrm{ad}_c x_j) x_k, x_j]_c, x_j\right]_c, x_j\right]_c$, if $q \in \mathbb{G}_4$.

(b) Admits a presentation by generators x_1, x_2, x_3 and relations

$$x_1^2 = x_2^2 = x_\alpha^{N_\alpha} = 0, \qquad \alpha \in \Delta_+^\chi, N_\alpha \neq 2,$$
$$\left[[(\mathrm{ad}_c x_1) x_2, [(\mathrm{ad}_c x_1) x_2, (\mathrm{ad}_c x_1)(\mathrm{ad}_c x_2) x_3]_c]_c, x_2\right]_c = (\mathrm{ad}_c x_3)^2 x_2 = 0.$$

Example (Strange type)

ζ root of unity of order 12.

Example (Strange type)

ζ root of unity of order 12.

Example (Strange type)

ζ root of unity of order 12.

(a) $\overset{\zeta^8}{\circ} \overset{\zeta}{\text{———}} \overset{\zeta^8}{\circ}$,

Example (Strange type)

ζ root of unity of order 12.

(a) $\overset{\zeta^8}{\circ} \xrightarrow{\ \zeta\ } \overset{\zeta^8}{\circ}$, (b) $\overset{\zeta^8}{\circ} \xrightarrow{\ \zeta^3\ } \overset{-1}{\circ}$,

Example (Strange type)

ζ root of unity of order 12.

(a) $\circ^{\zeta^8} \overset{\zeta}{\text{———}} \circ^{\zeta^8}$, (b) $\circ^{\zeta^8} \overset{\zeta^3}{\text{———}} \circ^{-1}$,

(c) $\circ^{\zeta^5} \overset{\zeta^9}{\text{———}} \circ^{-1}$.

- $\Delta_+^a = \{\alpha_1, 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_2\}$.

Example (Strange type)

ζ root of unity of order 12.

(a) $\overset{\zeta^8}{\circ} \xrightarrow{\ \zeta\ } \overset{\zeta^8}{\circ}$, (b) $\overset{\zeta^8}{\circ} \xrightarrow{\ \zeta^3\ } \overset{-1}{\circ}$,

(c) $\overset{\zeta^5}{\circ} \xrightarrow{\ \zeta^9\ } \overset{-1}{\circ}$.

- $\Delta_+^a = \{\alpha_1, 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_2\}$.
- $s_1(\alpha_1) = -\alpha_1$, $s_1(\alpha_2) = \alpha_2 + 2\alpha_1$: $s_1(\Delta^a) = \Delta^b$.

Example (Strange type)

ζ root of unity of order 12.

(a) $\overset{\zeta^8}{\circ} \overset{\zeta}{\text{———}} \overset{\zeta^8}{\circ}$, (b) $\overset{\zeta^8}{\circ} \overset{\zeta^3}{\text{———}} \overset{-1}{\circ}$,

(c) $\overset{\zeta^5}{\circ} \overset{\zeta^9}{\text{———}} \overset{-1}{\circ}$.

- $\Delta_+^a = \{\alpha_1, 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_2\}$.
- $s_1(\alpha_1) = -\alpha_1,\ s_1(\alpha_2) = \alpha_2 + 2\alpha_1$: $s_1(\Delta^a) = \Delta^b$.

- $\Delta_+^b = \{\alpha_1, 2\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2, \alpha_1 + \alpha_2, \alpha_2\}$.

Example (Strange type)

ζ root of unity of order 12.

(a) $\circ^{\zeta^8} \overset{\zeta}{\text{———}} \circ^{\zeta^8}$, (b) $\circ^{\zeta^8} \overset{\zeta^3}{\text{———}} \circ^{-1}$,

(c) $\circ^{\zeta^5} \overset{\zeta^9}{\text{———}} \circ^{-1}$.

- $\Delta_+^a = \{\alpha_1, 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_2\}$.
- $s_1(\alpha_1) = -\alpha_1$, $s_1(\alpha_2) = \alpha_2 + 2\alpha_1$: $s_1(\Delta^a) = \Delta^b$.

- $\Delta_+^b = \{\alpha_1, 2\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2, \alpha_1 + \alpha_2, \alpha_2\}$.
- $s_2(\alpha_1) = \alpha_1 + \alpha_2$, $s_2(\alpha_2) = -\alpha_2$: $s_2(\Delta^b) = \Delta^c$.

Example (Strange type)

ζ root of unity of order 12.

(a) $\circ^{\zeta^8} \overset{\zeta}{\text{———}} \circ^{\zeta^8}$, (b) $\circ^{\zeta^8} \overset{\zeta^3}{\text{———}} \circ^{-1}$,

(c) $\circ^{\zeta^5} \overset{\zeta^9}{\text{———}} \circ^{-1}$.

- $\Delta_+^a = \{\alpha_1, 2\alpha_1+\alpha_2, \alpha_1+\alpha_2, \alpha_1+2\alpha_2, \alpha_2\}$.
- $s_1(\alpha_1) = -\alpha_1$, $s_1(\alpha_2) = \alpha_2 + 2\alpha_1$: $s_1(\Delta^a) = \Delta^b$.

- $\Delta_+^b = \{\alpha_1, 2\alpha_1+\alpha_2, 3\alpha_1+2\alpha_2, \alpha_1+\alpha_2, \alpha_2\}$.
- $s_2(\alpha_1) = \alpha_1+\alpha_2$, $s_2(\alpha_2) = -\alpha_2$: $s_2(\Delta^b) = \Delta^c$.

- $\Delta_+^c = \{\alpha_1, 3\alpha_1+\alpha_2, 2\alpha_1+\alpha_2, \alpha_1+\alpha_2, \alpha_2\}$.

Example (Strange type)

ζ root of unity of order 12.

(a) $\circ^{\zeta^8} \overset{\zeta}{\text{———}} \circ^{\zeta^8}$, (b) $\circ^{\zeta^8} \overset{\zeta^3}{\text{———}} \circ^{-1}$,

(c) $\circ^{\zeta^5} \overset{\zeta^9}{\text{———}} \circ^{-1}$.

- $\Delta_+^a = \{\alpha_1, 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_2\}$.
- $s_1(\alpha_1) = -\alpha_1$, $s_1(\alpha_2) = \alpha_2 + 2\alpha_1$: $s_1(\Delta^a) = \Delta^b$.

- $\Delta_+^b = \{\alpha_1, 2\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2, \alpha_1 + \alpha_2, \alpha_2\}$.
- $s_2(\alpha_1) = \alpha_1 + \alpha_2$, $s_2(\alpha_2) = -\alpha_2$: $s_2(\Delta^b) = \Delta^c$.

- $\Delta_+^c = \{\alpha_1, 3\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, \alpha_2\}$.
- $s_1(\alpha_1) = -\alpha_1$, $s_1(\alpha_2) = \alpha_2 + 3\alpha_1$: fixes Δ^c.

Example (Strange)

$\dim \mathfrak{B}(V) = 432$.

(a) Admits a presentation by generators x_1, x_2 and relations

$$x_1^3 = x_2^3 = x_{\alpha_1+\alpha_2}^{12} = [x_1, x_{\alpha_1+2\alpha_2}]_c + \frac{(1+\zeta^8)(1-\zeta^7)q_{12}}{1-\zeta^9} x_{\alpha_1+\alpha_2}^2 = 0.$$

Example (Strange)

$\dim \mathfrak{B}(V) = 432$.

(a) Admits a presentation by generators x_1, x_2 and relations

$$x_1^3 = x_2^3 = x_{\alpha_1+\alpha_2}^{12} = [x_1, x_{\alpha_1+2\alpha_2}]_c + \frac{(1+\zeta^8)(1-\zeta^7)q_{12}}{1-\zeta^9} x_{\alpha_1+\alpha_2}^2 = 0.$$

Example (Strange)

$\dim \mathfrak{B}(V) = 432$.

(a) Admits a presentation by generators x_1, x_2 and relations

$$x_1^3 = x_2^3 = x_{\alpha_1+\alpha_2}^{12} = [x_1, x_{\alpha_1+2\alpha_2}]_c + \frac{(1+\zeta^8)(1-\zeta^7)q_{12}}{1-\zeta^9} x_{\alpha_1+\alpha_2}^2 = 0.$$

(b) Admits a presentation by generators x_1, x_2 and relations

$$x_1^3 = x_2^2 = x_{\alpha_1+\alpha_2}^{12} = [x_{3\alpha_1+2\alpha_2}, x_{\alpha_1+\alpha_2}]_c = 0.$$

Example (Strange)

$\dim \mathfrak{B}(V) = 432$.

(a) Admits a presentation by generators x_1, x_2 and relations

$$x_1^3 = x_2^3 = x_{\alpha_1+\alpha_2}^{12} = [x_1, x_{\alpha_1+2\alpha_2}]_c + \frac{(1+\zeta^8)(1-\zeta^7)q_{12}}{1-\zeta^9} x_{\alpha_1+\alpha_2}^2 = 0.$$

(b) Admits a presentation by generators x_1, x_2 and relations

$$x_1^3 = x_2^2 = x_{\alpha_1+\alpha_2}^{12} = [x_{3\alpha_1+2\alpha_2}, x_{\alpha_1+\alpha_2}]_c = 0.$$

(c) Admits a presentation by generators x_1, x_2 and relations

$$x_1^{12} = x_2^2 = (\mathrm{ad}_c x_1)^4 x_2 = [x_{2\alpha_1+\alpha_2}, x_{\alpha_1+\alpha_2}]_c = 0.$$