On the left and right Brylinski-Kostant filtrations

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Abstract

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{b}$ a Borel subalgebra, and $\mathfrak{h} \subset \mathfrak{b}$ a Cartan subalgebra. Let V be a finite dimensional simple $U(\mathfrak{g})$ module. Based on a principal s-triple (e, h, f) and following work of Kostant, Brylinski [?] defined a filtration $\mathcal{F}_e$ for all weight subspaces V_μ of V and calculated the dimensions of the graded subspaces for μ dominant. In [?] these dimensions were calculated for all μ.

Let $\delta M(0)$ be the finite dual of the Verma module of highest weight 0. It identifies with the global functions on a Weyl group translate of the open Bruhat cell and so inherits a natural degree filtration. On the other hand, up to an appropriate shift of weights, there is a unique

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$U(\mathfrak{b})$ module embedding of V into $\delta M(0)$ and so the degree filtration induces a further filtration $\mathcal{F}$ on each weight subspace V_μ.

A casual reading of [?] might lead one to believe that $\mathcal{F}$ and $\mathcal{F}_e$ coincide. However this is quite false. Rather one should view $\mathcal{F}_e$ as coming from a left action of $U(\mathfrak{n})$ and then there is a second (Brylinski-Kostant) filtration $\mathcal{F}'_e$ coming from a right action. It is $\mathcal{F}'_e$ which may coincide with $\mathcal{F}$.

In this paper the above claim is made precise. First it is noted that $\mathcal{F}$ is itself not canonical, but depends on a choice of variables. Then it is shown that a particular choice can be made to ensure that $\mathcal{F} = \mathcal{F}'_e$.

An explicit form for the unique left $U(\mathfrak{b})$ module embedding $V \hookrightarrow \delta M(0)$ is given using right action of $U(\mathfrak{n})$. This is used to give a purely algebraic proof of Brylinski's main result in [?] which is much simpler than [?].

It is noted that the dimensions of the graded subspaces of $\mathrm{gr}_{\mathcal{F}_e} V_\mu$ and $\mathrm{gr}_{\mathcal{F}'_e} V_\mu$ can be very different. Their interrelation may involve the Kashiwara involution. Indeed a combinatorial formula for multiplicities in tensor products involving crystal bases and the Kashiwara involution is recovered. Though the dimensions for the graded subspaces of $\mathrm{gr}_{\mathcal{F}'_e} V_\mu$ are determined by polynomial degree, their values remain unknown.