Optimality of function spaces in Sobolev embeddings

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In this minicourse we shall concentrate on the study of sharpness of function spaces appearing in various types of Sobolev embeddings.
We shall first present a general method of reducing a multi-dimensional Sobolev and Poincaré type inequalities involving gradients of scalar-valued functions of several variables to (far simpler) one-dimensional inequalities involving suitable Hardy-type integral operators acting on functions defined on an interval. Such results we call *reduction theorems*.
We shall explain the details of our reduction methods. The first step is the reduction of the *first order Sobolev embedding*. This part is based on a combination of techniques involving isoperimetric inequalities and various variants of the Pólya–Szegö principle. The second major step is the extension of the reduction to *higher order Sobolev embeddings*, which seems to constitute a considerably more difficult task. To this end, we develop a powerful method based on an iteration of sharp first-order results, which are at this stage already at our disposal.
Once the reduction theorems (of all orders) are established, we will apply them to the construction of the *optimal range* space when a domain space is given, and vice versa. We shall work out important examples illustrating the results in several concrete situations involving certain well-established scales of function spaces (Lebesgue spaces, Lorentz spaces, Lorentz–Zygmund spaces, Orlicz spaces, classical Lorentz spaces of type Lambda and Gamma and Lorentz and Marcinkiewicz endpoint spaces). A nontrivial effort has to be spent here in order to work out manageable versions of rather implicit conditions that one has to grapple with.
We will mention how the methods and results could be of use when *compactness* of the Sobolev embeddings is studied.
We shall present three major types of applications of the results obtained. These will involve Euclidean Sobolev embeddings on possibly irregular domains, trace Sobolev embeddings, and Sobolev embeddings on product probability measure spaces (of which the Gaussian measure is a distinctive example).
The techniques we use will include, for example, various inequalities connecting gradients and rearrangements, weighted inequalities for integral and supremal operators, interpolation theory, and more.

Keywords. Sobolev spaces, rearrangement-invariant spaces, optimal range, optimal domain, reduction theorems, interpolation, duality, compact embeddings, fundamental functions, boundary traces, Gaussian embeddings

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