# The Kronecker-Bohl lemma for counting zeros of Dirichlet polynomials


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The functions $L_n(z) := 1 - \sum_{k=2}^{n} k^z$, $n \geq 2$, are the prototype of *non-lattice* Dirichlet polynomials [2], that is, functions of the form $f(z) = 1 - \sum_{k=2}^{n} m_k a_k^z$ with $m_j \in \mathbb{C}$ and $a_j > 0$ such that the additive subgroup $G = \mathbb{Z}\log a_2 + ... + \mathbb{Z}\log a_n$ is dense in $\mathbb{R}$.

In this paper we prove, by means of Kronecker-Bohl lemma and by using some techniques of [3], the existence of an infinite amount of $r$-rectangles (rectangular sets with two little semicircle of radius $r$) in the critical strip of $L_n(z)$ for which the exact number of zeros inside them is given by

$$\left[\frac{T \log n}{2\pi}\right] + 1, \tag{1}$$

where $T$ is the height of the $r$-rectangle and [ ] represents the integer part.

In the extensive literature on the related question with the topic of the zeros, the formulae to determine the number of them in certain regions, mainly rectangles, contain a term which expresses the maximum error, namely $n-1$ or $\frac{n}{2}$ in the best of the cases (see for example [4] and [1] respectively), where $n$ is the "degree" of the exponential polynomial; however, in our $r$-rectangles, the error in the formula (1) is reduced to 0.


Keywords. Zeros of entire functions, exponential polynomials, Dirichlet polynomials.


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