1 Koszul duality (Ralph)

Recall the following theorems of J.D.S. Jones relating the Hochschild and cyclic homology of the cochains of a space $X$ with the (equivariant) cohomology of the free loop space $\mathcal{L}X$:

$$HH_*(C^*(X)) \cong H^*(\mathcal{L}X)$$

$$HC_*^-(C^*(X)) \cong H_{S^1}^*(\mathcal{L}X)$$

Likewise T. Goodwillie encountered a similar link between the Hochschild and cyclic homology of the chains of the loop space $\Omega X$ with the (equivariant) homology of the free loop space $LX$:

$$HH_*(C_*(\Omega X)) \cong H_*(\mathcal{L}X)$$

$$HC_*(C_*(\Omega X)) \cong H_{S^1}^*(\mathcal{L}X)$$

These two pairs of isomorphisms can be explained by Koszul duality. Let $C$ be a coalgebra over a ring $k$ (above $C$ is chain coalgebra $C_*(X)$). One can construct two algebras from it. First the dual $C^* = \text{Hom}(C, k)$ and second the cobar construction $\Omega C$. Y. Félix, L. Menichi and J.-C. Thomas proved that the Hochschild cohomology of these algebras are isomorphic.

**Question.** What is the relation between the categories of modules over $C^*$ and over $\Omega C$ (pair of adjoint functors giving the duality).
**Conjecture.** The duality induces an isomorphism as well in algebraic $K$-theory (Waldhausen’s $A$ theory $A(X) = K_{alg}(C_*(\Omega X))$):

$$\text{Hom}_{K_{alg}(\mathbb{Z})}(K_{alg}(C^*(X); K_{alg}(\mathbb{Z}))) \cong K_{alg}(C_*(\Omega X))$$

## 2 String structure for pseudo-isotopies (Ralph)

Let $X$ be a simply connected space and consider Waldhausen’s fibration

$$\tilde{A}(X) \to A(X) \to A(*)$$

Consider on the other hand the Borel construction $ES^1 \times_{S^1} \mathcal{L}X$ where $S^1$ acts on the free loop space by rotating the loops and apply Quillen’s $Q$-construction $(\Omega^\infty \Sigma^\infty)$ to it so as to define $B(X) = Q(ES^1 \times_{S^1} \mathcal{L}X)$. The fibration $\mathcal{L}X \to ES^1 \times_{S^1} \mathcal{L}X \to BS^1$ yields thus another fibration

$$\tilde{B}(X) \to B(X) \to B(*)$$

where by definition $\tilde{B}(X)$ is the homotopy fiber of the map $Q(ES^1 \times_{S^1} \mathcal{L}X) \to Q(BS^1)$. Dundas showed that these two fibrations are equivalent up to $p$-completion at any prime $p$.

Waldhausen proved that the group of pseudo-isotopies $\text{colim}_k \text{Diff}(M \times I^k)$ splits as a direct summand of $A(M)$ for any manifold $M$.

**Question.** Can one translate the string topology structure to a structure either on $A(M)$ or on the space of pseudo-isotopies?

## 3 Category of $D$-branes (Ralph)

Consider a collection of submanifolds $\{L_i\}$ of a given manifold $M$. Recall that $\mathcal{P}_M(L_i, L_j)$ denotes the space of paths in $M$ originating in $L_i$ and ending in $L_j$.

**Question.** Can one define a category whose objects are the chains $C_*(L_i)$ and morphisms $C_*(\mathcal{P}_M(L_i, L_j))$? Or maybe a 2-category? Or maybe an $A_\infty$-category? Or maybe a motivic category as suggested by Sasha?
4 Stable string structure (Ralph)

Chas and Sullivan proved that there is an intersection product turning $H_\ast(\mathcal{L}M)$ into an algebra and a Lie bracket turning $H_\ast(ES^1 \times_{S^1} \mathcal{L}M)$ into a Lie algebra. The homology groups of any space are nothing but the homotopy groups of this space smashed with the Eilenberg-Mac Lane spectrum $HZ$. So $\pi_\ast(ES^1 \times_{S^1} \mathcal{L}M \wedge HZ)$ is a Lie algebra. One could therefore also look at the $S^1$-equivariant function spectrum $Map^{S^1}(ES^1, \mathcal{L}M \wedge HZ)$.

**Question.** Which string structure does $\pi_\ast Map^{S^1}(ES^1, \mathcal{L}M \wedge HZ)$ carry? This is related to work of L. Hesselholt and I. Madsen.

**Homework.** Does $H_\ast(M^{S^n} \times_{SO(n)} ESO(n))$ carry a Lie algebra structure?

5 Higher dimensional trousers (Ralph)

The space of cacti with $k$ components $C(k)$ is homotopy equivalent to the classifying space of the homeomorphism group of $k$-legged trousers.

**Question.** Is the space of higher cacti with $k$-components homotopy equivalent to the classifying space of the homeomorphism group of higher dimensional $k$-legged trousers?

6 Homology of higher cacti (Sasha)

The rational homology of the space $C^n(1)$ of higher cacti is known.

**Question.** Compute $H_\ast(C^n(\_); \mathbb{Q})$ as an operad. Or compute maybe only the part coming from the higher cacti having multi-degree $(1, \ldots, 1)$ (the framing from $S^n$ to the $n$-dimensional cactus has degree 1 when restricted to any component of the cactus).

7 Gromov-Witten moduli space (Sasha)

Consider a manifold $M$ and a class $\beta \in H_2(M; \mathbb{Z})$. Define the Gromov-Witten moduli space $\mathcal{M}_{g,n}(M, \beta)$ (of genus $g$ with $n$ punctures) as the space of pairs $(X_{g,n}, \varphi)$ where $X_{g,n}$ is an $n$-punctured surface of genus $g$ and $\varphi$ a holomorphic map $X_{g,n} \to M$ modulo isomorphisms of pairs acting as the identity on $M$. 

A theorem of Mumford, Harer, and Penner states the ribbon graphs model the moduli space $\mathcal{M}_{g,n}$ of $n$-punctured surfaces of genus $g$ modulo isomorphisms preserving punctures.

**Question.** Find a combinatorial model for the Gromov-Witten moduli space.