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1 Koszul duality (Ralph)

Recall the following theorems of J.D.S. Jones relating the Hochschild and cyclic homology of the cochains of a space X with the (equivariant) cohomology of the free loop space $\mathcal{L}X$:

$$HH_*(C^*(X)) \cong H^*(\mathcal{L}X)$$

$$HC_*^-(C^*(X)) \cong H_{S^1}^*(\mathcal{L}X)$$

Likewise T. Goodwillie encountered a similar link between the Hochschild and cyclic homology of the chains of the loop space ΩX with the (equivariant) homology of the free loop space ŁX:

$$HH_*(C_*(\Omega X)) \cong H_*(\mathcal{L}X)$$

$$HC_*(C_*(\Omega X)) \cong H_*^{S^1}(\mathcal{L}X)$$

These two pairs of isomorphisms can be explained by Koszul duality. Let C be a coalgebra over a ring k (above C is chain coalgebra $C_*(X)$). One can construct two algebras from it. First the dual $C^* = Hom(C,k)$ and second the cobar construction ΩC. Y. Félix, L. Menichi and J.-C. Thomas proved that the Hochschild cohomology of these algebras are isomorphic.

Question. What is the relation between the categories of modules over C^* and over ΩC (pair of adjoint functors giving the duality).

Conjecture. The duality induces an isomorphism as well in algebraic K-theory (Waldhausen's A theory $A(X) = K_{alg}(C_*(\Omega X))$:

$$Hom_{K_{alg}(\mathbb{Z})}(K_{alg}(C^*(X); K_{alg}(\mathbb{Z})) \cong K_{alg}(C_*(\Omega X))$$

2 String structure for pseudo-isotopies (Ralph)

Let X be a simply connected space and consider Waldhausen's fibration

$$\tilde{A}(X) \longrightarrow A(X) \longrightarrow A(*)$$

Consider on the other hand the Borel construction $ES^1 \times_{S^1} \mathcal{L}X$ where S^1 acts on the free loop space by rotating the loops and apply Quillen's Q-construction ($\Omega^\infty\Sigma^\infty$) to it so as to define $B(X) = Q(ES^1 \times_{S^1} \mathcal{L}X)$. The fibration $\mathcal{L}X \longrightarrow ES^1 \times_{S^1} \mathcal{L}X \rightarrow BS^1$ yields thus another fibration

$$\tilde{B}(X) \longrightarrow B(X) \longrightarrow B(*)$$

where by definition $\tilde{B}(X)$ is the homotopy fiber of the map $Q(ES^1 \times_{S^1} \mathcal{L}X) \rightarrow Q(BS^1)$. Dundas showed that these two fibrations are equivalent up to p-completion at any prime p.

Waldhausen proved that the group of pseudo-isotopies $colim_k Diff(M \times I^k)$ splits as a direct summand of $A(M)$ for any manifold M.

Question. Can one translate the string topology structure to a structure either on $A(M)$ or on the space of pseudo-isotopies?

3 Category of D-branes (Ralph)

Consider a collection of submanifolds $\{ L_i \}$ of a given manifold M. Recall that $\mathcal{P}_M(L_i, L_j)$ denotes the space of paths in M originating in L_i and ending in L_j.

Question. Can one define a category whose objects are the chains $C_*(L_i)$ and morphisms $C_*(\mathcal{P}_M(L_i, L_j))$? Or maybe a 2-category? Or maybe an A_∞-category? Or maybe a motivic category as suggested by Sasha?

4 Stable string structure (Ralph)

Chas and Sullivan proved that there is an intersection product turning $H_*(\mathcal{L}M)$ into an algebra and a Lie bracket turning $H_*(ES^1 \times_{S^1} \mathcal{L}M)$ into a Lie algebra. The homology groups of any space are nothing but the homotopy groups of this space smashed with the Eilenberg-Mac Lane spectrum $H\mathbb{Z}$. So $\pi_*(ES^1 \times_{S^1} \mathcal{L}M \wedge H\mathbb{Z})$ is a Lie algebra. One could therefore also look at the S^1-equivariant function spectrum $Map^{S^1}(ES^1, \mathcal{L}M \wedge H\mathbb{Z})$.

Question. Which string structure does $\pi_* Map^{S^1}(ES^1, \mathcal{L}M \wedge H\mathbb{Z})$ carry? This is related to work of L. Hesseholt and I. Madsen.

Homework. Does $H_*(M^{S^n} \times_{SO(n)} ESO(n))$ carry a Lie algebra structure?

5 Higher dimensional trousers (Ralph)

The space of cacti with k components $C(k)$ is homotopy equivalent to the classifying space of the homeomorphism group of k-legged trousers.

Question. Is the space of higher cacti with k-components homotopy equivalent to the classifying space of the homeomorphism group of higher dimensional k-legged trousers?

6 Homology of higher cacti (Sasha)

The rational homology of the space $C^n(1)$ of higher cacti is known.

Question. Compute $H_*(C^n(.);\mathbb{Q})$ as an operad. Or compute maybe only the part coming from the higher cacti having multi-degree $(1, \dots, 1)$ (the framing from S^n to the n-dimensional cactus has degree 1 when restricted to any component of the cactus).

7 Gromov-Witten moduli space (Sasha)

Consider a manifold M and a class $\beta \in H_2(M;\mathbb{Z})$. Define the Gromov-Witten moduli space $\mathcal{M}_{g,n}(M,\beta)$ (of genus g with n punctures) as the space of pairs $(X_{g,n}, \varphi)$ where $X_{g,n}$ is an n-punctured surface of genus g and φ a holomorphic map $X_{g,n} \to M$ modulo isomorphisms of pairs acting as the identity on M.

A theorem of Mumford, Harer, and Penner states the ribbon graphs model the moduli space $\mathcal{M}_{g,n}$ of n-punctured surfaces of genus g modulo isomorphisms preserving punctures.

Question. Find a combinatorial model for the Gromov-Witten moduli space.