Calabi-Yau pointed Hopf algebras of finite Cartan type

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Outline

- Background
- The Calabi-Yau property of $U(\mathcal{D}, \lambda)$
- The Calabi-Yau property of Nichols algebras of finite Cartan type
- Rigid dualizing complexes of braided Hopf algebras over finite group algebras

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Calabi-Yau categories

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- The bounded derived category of coherent sheaves on a Calabi-Yau manifold has a Serre functor which is isomorphic to a power of the shift functor.
- A triangulated category satisfying this condition was defined to be a Calabi-Yau category by Kontsevich.
- Now Calabi-Yau categories appear in
  - mathematical physics;
  - representation theory of finite dimensional algebras;
  - $\vdots$

Calabi-Yau algebras

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Calabi-Yau algebras

- Throughout, we fix an algebraically closed field $\mathbb{k}$ with characteristic 0. All Hopf algebras mentioned are assumed to be Hopf algebras with bijective antipodes.
- (Ginzburg) An algebra A is called a Calabi-Yau algebra of dimension d if

  (i) A is homologically smooth. That is, A has a bounded resolution of finitely generated projective A-A-bimodules.

  (ii) There are A-A-bimodule isomorphisms

$$\mathrm{Ext}^i_{A^e}(A, A^e) \cong \begin{cases} 0, & i \neq d; \\ A, & i = d. \end{cases}$$

In the following, Calabi-Yau will be abbreviated to CY for short.

Lemma 1 (Keller)

If A is a CY algebra of dimension d, then the category $D^b_{fd}(A)$ is a CY category, where $D^b_{fd}(A)$ is the full triangulated subcategory of the derived category $D(A)$ of A consisting of complexes whose homology is of finite total dimension.

Examples of CY algebras

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- The polynomial algebra $\Bbbk[x_1, \cdots, x_n]$ is CY of dimension n.
- (Berger) The Weyl algebra
  $A_n = \Bbbk[x_1, \cdots, x_n, y_1, \cdots, y_n]/ < x_i y_j - y_j x_i - \delta_{ij} >$
  is CY of dimension $2n$.

Dualizing complexes

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- (Yekutieli) Let A be a Noetherian algebra. Roughly speaking, a complex $\mathscr{R} \in D^b(A^e)$ is called dualizing if the functor

$$\mathrm{RHom}_A(-, \mathscr{R}) : D^b_{fg}(A) \to D^b_{fg}(A^{op})$$

is a duality, with adjoint $\mathrm{RHom}_{A^{op}}(-, \mathscr{R})$.

Here $D^b_{fg}(A)$ is the full triangulated subcategory of the derive category $D(A)$ of A consisting of bounded complexes with finitely generated cohomology modules.

Rigid Dualizing complexes

- (Van den Bergh) Let A be a Noetherian algebra. A dualizing complex $\mathscr{R}$ over A is called rigid if

$$\mathrm{RHom}_{A^e}(A, {}_A\mathscr{R} \otimes \mathscr{R}_A) \cong \mathscr{R}$$

in $D(A^e)$.

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in $D(A^e)$.

- An algebra A is CY of dimension d if and only if A is homologically smooth and has a rigid dualizing complex $A[d]$.

Hopf CY algebras

- Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra. Chemla computed the rigid dualizing complex of the quantized enveloping algebra $U_q(\mathfrak{g})$ is $U_q(\mathfrak{g})[d]$, where $d = \dim \mathfrak{g}$.

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- Brown and Zhang used homological integral to give the rigid dualizing complex of an AS-Gorenstein Hopf algebra.

- He, Van Oystaeyen and Zhang used homological integral to give a necessary and sufficient condition for a Noetherian Hopf algebra to be a CY algebra.

AS-Gorenstein algebras

- Let A be a Noetherian augmented algebra with a fixed augmentation map $\varepsilon : A \to \Bbbk$. A is said to be AS-Gorenstein if
  (i) $\operatorname{injdim}{}_A A = d < \infty$,
  (ii) $\dim \operatorname{Ext}^i_A({}_A\Bbbk, {}_AA) = \begin{cases} 0, & i \neq d; \\ 1, & i = d, \end{cases}$
  (iii) The right A-module versions of conditions (i) and (ii) hold,

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  where injdim stands for injective dimension.
- An AS-Gorenstein algebra A is said to be regular if in addition, the global dimension of A is finite.
- Remark: Let A be a Noetherian algebra. If the injective dimension of ${}_AA$ and A_A are both finite, then these two integers are equal. We call this common value the injective dimension of A.

Homological integrals

- (Lu, Wu and Zhang) Let A be an AS-Gorenstein algebra with injective dimension d. Then $\mathrm{Ext}_A^d({}_A\Bbbk, {}_AA)$ is a 1-dimensional right A-module. It is called the left homological integral module of A. Any non-zero element in $\mathrm{Ext}_A^d({}_A\Bbbk, {}_AA)$ is called a left homological integral of A. We write $\int_A^l$ for $\mathrm{Ext}_A^d({}_A\Bbbk, {}_AA)$.

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- Similarly, the 1-dimensional left A-module $\mathrm{Ext}^d_A(\mathbb{k}_A, A_A)$ is called the right homological integral module of A. Any non-zero element in $\mathrm{Ext}^d_A(\mathbb{k}_A, A_A)$ is called a right homological integral of A. Write $\int^r_A$ for $\mathrm{Ext}^d_A(\mathbb{k}_A, A_A)$.

Pointed Hopf algebras

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- A Hopf algebra A is called pointed, if all its simple left or right comodules are 1-dimensional. This is equivalent to saying that the coradical of A is a group algebra.
- For a pointed Hopf algebra A, its coradical filtration is a Hopf algebra filtration.
- Let $\mathrm{Gr}\, A$ be its associated graded Hopf algebra.

$$\mathrm{Gr}\, A \cong R\#\Bbbk\Gamma,$$

where $\Bbbk\Gamma$ is the coradical of A and R is a braided Hopf algebra in the category of Yetter-Drinfeld modules over $\Bbbk\Gamma$.

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Nichols algebras

- Let V be the vector space consisting of primitive elements of R. It is a Yetter-Drinfeld module over $\Bbbk\Gamma$.
- The algebra $\mathcal{B}(V)$ generated by V is a braided Hopf subalgebra of R. It is called the Nichols algebra of V.
- The algebra structure and coalgebra structure of $\mathcal{B}(V)$ depend only on the braiding of V.

Pointed Hopf algebras $U(\mathcal{D}, \lambda)$

- The pointed Hopf algebras $U(\mathcal{D}, \lambda)$ constructed by Andruskiewitsch and Schneider constitute a large class of pointed Hopf algebras with finite Gelfand-Kirillov dimension, whose group-like elements form an abelian group.

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- The pointed Hopf algebras $U(\mathcal{D}, \lambda)$ constructed by Andruskiewitsch and Schneider constitute a large class of pointed Hopf algebras with finite Gelfand-Kirillov dimension, whose group-like elements form an abelian group.
- Such an algebra $U(\mathcal{D}, \lambda)$ is viewed as a generalization of the quantized enveloping algebra $U_q(\mathfrak{g})$, $\mathfrak{g}$ a finite dimensional semisimple Lie algebra.

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- $\mathcal{D}(\Gamma,(g_i)_{1\leqslant i\leqslant\theta},(\chi_i)_{1\leqslant i\leqslant\theta},(a_{ij})_{1\leqslant i,j\leqslant\theta})$: a datum of finite Cartan type for Γ.
  - $(a_{ij})\in\mathbb{Z}^{\theta\times\theta}$ is a Cartan matrix of finite type, where $\theta\in\mathbb{N}$; Let $\mathcal{X}$ be the set of connected components of the Dynkin diagram corresponding to the Cartan matrix (a_{ij}). If $1\leqslant i,j\leqslant\theta$, then $i\sim j$ means that they belong to the same connected component;
  - $g_1,\cdots,g_\theta$ are elements in Γ and $\chi_1,\cdots,\chi_\theta$ are characters in $\widehat{\Gamma}$ such that

$$\begin{array}{rcl}\chi_j(g_i)\chi_i(g_j) & = & \chi_i(g_i)^{a_{ij}},\\ \chi_i(g_i) & \neq & 1,\ \text{for all } 1\leqslant i,j\leqslant\theta.\end{array}$$

A datum $\mathcal{D}$ is called generic if each $\chi_i(g_i)$ is not a root of unity. For simplicity, we define $q_{ij}=\chi_j(g_i)$, $1\leqslant i,j\leqslant\theta$.

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- λ: a family of linking parameters for $\mathcal{D}$. That is, $\lambda=(\lambda_{ij})_{1\leqslant i<j\leqslant\theta}$ is a family of elements in $\Bbbk$ such that $\lambda_{ij}=0$ if $g_ig_j=1$ or $\chi_i\chi_j\neq\varepsilon$.

- Given a datum $\mathcal{D}$, we define a braided vector space V of diagonal type with basis $x_1, \cdots, x_\theta$ whose braiding is given by

$$c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i, \ \ 1 \leqslant i, j \leqslant \theta.$$

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- The algebra $U(\mathcal{D}, \lambda)$ is defined to be the quotient Hopf algebra of the smash product $\Bbbk\langle x_1, \cdots, x_\theta\rangle \# \Bbbk\Gamma$ modulo the ideal generated by the following relations

$$\begin{array}{ll} (\mathrm{ad}_c\, x_i)^{1-a_{ij}}(x_j) = 0, & 1 \leqslant i, j \leqslant \theta, \ \ i \neq j, \ \ i \sim j, \\ x_i x_j - \chi_j(g_i) x_j x_i = \lambda_{ij}(1 - g_i g_j), & 1 \leqslant i < j \leqslant \theta, \ \ i \nsim j, \end{array}$$

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- $\mathrm{Gr}\, U(\mathcal{D}, \lambda) \cong U(\mathcal{D}, 0) \cong \mathcal{B}(V) \# \Bbbk\Gamma.$

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- Let $\mathcal{D}(\Gamma, (g_i)_{1\leqslant i\leqslant\theta}, (\chi_i)_{1\leqslant i\leqslant\theta}, (a_{ij})_{1\leqslant i,j\leqslant\theta})$ be a generic datum of finite Cartan type, Φ the root system of the Cartan matrix (a_{ij}) and $\{\alpha_1, \cdots, \alpha_\theta\}$ a set of simple roots.

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- Assume that $w_0 = s_{i_1} \cdots s_{i_p}$ is a reduced decomposition of the longest element in the Weyl group $\mathcal{W}$ as a product of simple reflections.
- Then

$$\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1}(\alpha_{i_2}), \cdots, \beta_p = s_{i_1} \cdots s_{i_{p-1}}(\alpha_{i_p})$$

are the positive roots.

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  are the positive roots.
- If $\beta_i=\sum_{j=1}^{\theta}m_j\alpha_j$, $1\leqslant i\leqslant p$, then we define

$$g_{\beta_i}=g_1^{m_1}\cdots g_\theta^{m_\theta} \text{ and } \chi_{\beta_i}=\chi_1^{m_1}\cdots\chi_\theta^{m_\theta}.$$

The homological integral of $U(\mathcal{D}, \lambda)$

Theorem 2

Let $\mathcal{D}$ be a generic datum of finite Cartan type for a free abelian group Γ of rank s, λ a family of linking parameters for $\mathcal{D}$, and A the Hopf algebra $U(\mathcal{D}, \lambda)$. Then A is Noetherian AS-regular of global dimension $p + s$, where p is the number of the positive roots of the Cartan matrix in $\mathcal{D}$.
The left homological integral module $\int_A^l$ of A is isomorphic to $\Bbbk_\xi$, where $\xi : A \to \Bbbk$ is an algebra homomorphism defined by $\xi(g) = (\prod_{i=1}^{p} \chi_{\beta_i})(g)$ for all $g \in \Gamma$ and $\xi(x_i) = 0$ for all $1 \leqslant i \leqslant \theta$.

The CY property of $U(\mathcal{D}, \lambda)$

Theorem 3

Let $\mathcal{D}$ be a generic datum of finite Cartan type for a free abelian group Γ of rank s, and λ a family of linking parameters for $\mathcal{D}$.

(1) *The rigid dualizing complex of the Hopf algebra $A = U(\mathcal{D}, \lambda)$ is ${}_{\psi}A[p+s]$, where p is the number of the positive roots and s is the rank of Γ. The algebra automorphism ψ is defined by $\psi(x_k) = \prod_{i=1, i\neq j_k}^{p} \chi_{\beta_i}(g_k)x_k$, for all $1 \leqslant k \leqslant \theta$, and $\psi(g) = (\prod_{i=1}^{p} \chi_{\beta_i})(g)$ for any $g \in \Gamma$, where each j_k is the integer such that $\beta_{j_k} = \alpha_k$.*

(2) *The algebra A is CY if and only if $\prod_{i=1}^{p} \chi_{\beta_i} = \varepsilon$ and $\mathcal{S}_A^2$ is an inner automorphism.*

Remark: For a pointed Hopf algebra $U(\mathcal{D}, \lambda)$, it is CY if and only if its associated graded algebra $U(\mathcal{D}, 0)$ is CY.

Classification

In this classification, we assume that $\mathbb{k} = \mathbb{C}$.

CY pointed Hopf algebras $U(\mathcal{D}, \lambda)$ of dimension 3

Case	Cartan matrix	Generators	Relations
Case 1	trivial	y_h, y_h^{-1} $1 \leqslant h \leqslant 3$	$y_h^{\pm 1} y_m^{\pm 1} = y_m^{\pm 1} y_h^{\pm 1}$ $y_h^{\pm 1} y_h^{\mp 1} = 1$ $1 \leqslant h, m \leqslant 3$
Case 2 (I)	$A_1 \times A_1$	$y_1^{\pm 1}, x_1, x_2$	$y_1 y_1^{-1} = y_1^{-1} y_1 = 1$ $y_1 x_1 = q x_1 y_1$ $y_1 x_2 = q^{-1} x_2 y_1$, $0 < \lvert q \rvert < 1$ $x_1 x_2 - q^{-k} x_2 x_1 = 0$, $k \in \mathbb{Z}^+$
Case 2 (II)	$A_1 \times A_1$	$y_1^{\pm 1}, x_1, x_2$	$y_1 y_1^{-1} = y_1^{-1} y_1 = 1$ $y_1 x_1 = q x_1 y_1$ $y_1 x_2 = q^{-1} x_2 y_1$, $0 < \lvert q \rvert < 1$ $x_1 x_2 - q^{-k} x_2 x_1 = (1 - y_1^{2k})$, $k \in \mathbb{Z}^+$

Remark: $U_q(\mathfrak{sl}_2)$ belongs to Case 2 (II).

CY pointed Hopf algebras $U(\mathcal{D}, \lambda)$ of dimension 4

Case	Cartan matrix	Generators	Relations
Case 1	trivial	y_h, y_h^{-1} $1 \leqslant h \leqslant 4$	$y_h^{\pm 1} y_m^{\pm 1} = y_m^{\pm 1} y_h^{\pm 1}$ $y_h^{\pm 1} y_h^{\mp 1} = 1$ $1 \leqslant h, m \leqslant 4$
Case 2 (I)	$A_1 \times A_1$	$y_1^{\pm 1}, y_2^{\pm 1}, x_1, x_2$	$y_h^{\pm 1} y_m^{\pm 1} = y_m^{\pm 1} y_h^{\pm 1}$ $y_h^{\pm 1} y_h^{\mp 1} = 1$ $1 \leqslant h, m \leqslant 2$ $y_1 x_1 = q_1 x_1 y_1,\ y_1 x_2 = q_1^{-1} x_2 y_1$ $y_2 x_1 = q_2 x_1 y_2,\ y_2 x_2 = q_2^{-1} x_2 y_2$ $0 < \lvert q_1 \rvert < 1$ $x_1 x_2 - q_1^{-k} x_2 x_1 = 0,\ k \in \mathbb{Z}^+$
Case 2 (II)	$A_1 \times A_1$	$y_1^{\pm 1}, y_2^{\pm 1}, x_1, x_2$	$y_h^{\pm 1} y_m^{\pm 1} = y_m^{\pm 1} y_h^{\pm 1}$ $y_h^{\pm 1} y_h^{\mp 1} = 1$ $1 \leqslant h, m \leqslant 2$ $y_1 x_1 = q_1 x_1 y_1,\ y_1 x_2 = q_1^{-1} x_2 y_1$ $y_2 x_1 = q_2 x_1 y_2,\ y_2 x_2 = q_2^{-1} x_2 y_2$ $0 < \lvert q_1 \rvert < 1$ $x_1 x_2 - q_1^{-k} x_2 x_1 = 1 - y_1^{2k},\ k \in \mathbb{Z}^+$

Let A and B be two algebras in Case (I) (or (II)) defined by triples (k, q_1, q_2) and (k', q_1', q_2') respectively. They are isomorphic if and only if $k = k'$, $q_1 = q_1'$ and there is some integer b, such that $q_2' = q_1^b q_2$ or $q_2' = q_1^b q_2^{-1}$.

Case 2 (III)	$A_1 \times A_1$	$y_1^{\pm 1}, y_2^{\pm 1}, x_1, x_2$	$y_h^{\pm 1} y_m^{\pm 1} = y_m^{\pm 1} y_h^{\pm 1}$ $y_h^{\pm 1} y_h^{\mp 1} = 1$ $1 \leqslant h, m \leqslant 2$ $y_1 x_1 = q x_1 y_1,\ y_1 x_2 = q^{-1} x_2 y_1$ $y_2 x_1 = q^{\frac{k}{l}} x_1 y_2,\ y_2 x_2 = q^{-\frac{k}{l}} x_2 y_2$ $x_1 x_2 - q^{-k} x_2 x_1 = 0$ $k, l \in \mathbb{Z}^+,\ 0 < \lvert q \rvert < 1$
Case 2 (IV)	$A_1 \times A_1$	$y_1^{\pm 1}, y_2^{\pm 1}, x_1, x_2$	$y_h^{\pm 1} y_m^{\pm 1} = y_m^{\pm 1} y_h^{\pm 1}$ $y_h^{\pm 1} y_h^{\mp 1} = 1$ $1 \leqslant h, m \leqslant 2$ $y_1 x_1 = q x_1 y_1,\ y_1 x_2 = q^{-1} x_2 y_1$ $y_2 x_1 = q^{\frac{k}{l}} x_1 y_2,\ y_2 x_2 = q^{-\frac{k}{l}} x_2 y_2$ $x_1 x_2 - q^{-k} x_2 x_1 = 1 - y_1^k y_2^l$ $k, l \in \mathbb{Z}^+,\ 0 < \lvert q \rvert < 1$

Case 2 (V)	$A_1 \times A_1$	$y_1^{\pm 1}, y_2^{\pm 1}, x_1, x_2$	$y_h^{\pm 1} y_m^{\pm 1} = y_m^{\pm 1} y_h^{\pm 1}$ $y_h^{\pm 1} y_h^{\mp 1} = 1$ $1 \leqslant h, m \leqslant 2$ $y_1 x_1 = q x_1 y_1,\ y_1 x_2 = q^{-1} x_2 y_1$ $y_2 x_1 = q^{\frac{k-l_1}{l_2}} x_1 y_2,\ y_2 x_2 = q^{-\frac{k-l_1}{l_2}} x_2 y_2$ $x_1 x_2 - q^{-k} x_2 x_1 = 0$ $k, l_1, l_2 \in \mathbb{Z}^+,\ 0 < l_1 < l_2,\ 0 < \lvert q \rvert < 1$
Case 2 (VI)	$A_1 \times A_1$	$y_1^{\pm 1}, y_2^{\pm 1}, x_1, x_2$	$y_h^{\pm 1} y_m^{\pm 1} = y_m^{\pm 1} y_h^{\pm 1}$ $y_h^{\pm 1} y_h^{\mp 1} = 1$ $1 \leqslant h, m \leqslant 2$ $y_1 x_1 = q x_1 y_1,\ y_1 x_2 = q^{-1} x_2 y_1$ $y_2 x_1 = q^{\frac{k-l_1}{l_2}} x_1 y_2,\ y_2 x_2 = q^{-\frac{k-l_1}{l_2}} x_2 y_2$ $x_1 x_2 - q^{-k} x_2 x_1 = 1 - y_1^{k+l_1} y_2^{l_2}$ $k, l_1, l_2 \in \mathbb{Z}^+,\ 0 < l_1 < l_2,\ 0 < \lvert q \rvert < 1$

Example

Let A be the algebra with generators x_i, $y_j^{\pm 1}$, $1 \leqslant i, j \leqslant 3$, subject to the relations

$$y_i^{\pm 1} y_j^{\pm 1} = y_j^{\pm 1} y_i^{\pm 1}, \quad y_j^{\pm 1} y_j^{\mp 1} = 1, \quad 1 \leqslant i, j \leqslant 3,$$

$$y_j(x_i) = \chi_i(y_j) x_i y_j, \quad 1 \leqslant i, j \leqslant 3,$$
$$x_1^2 x_2 - q x_1 x_2 x_1 - q^2 x_1 x_2 x_1 + q^3 x_2 x_1^2 = 0,$$
$$x_2^2 x_1 - q^{-2} x_2 x_1 x_2 - q^{-1} x_2 x_1 x_2 + q^{-3} x_1 x_2^2 = 0,$$
$$x_1 x_3 = x_3 x_1.$$

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- A is a CY pointed Hopf algebra of type $A_2 \times A_1$ of dimension 7.
- The non-trivial liftings of A are also CY.

Outline

- Background
- The Calabi-Yau property of $U(\mathcal{D}, \lambda)$
- The Calabi-Yau property of Nichols algebras of finite Cartan type
- Rigid dualizing complexes of braided Hopf algebras over finite group algebras

- Let $\mathcal{D}$ be a generic datum of finite Cartan type and λ a family of linking parameters for $\mathcal{D}$.

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- The Nichols algebra $\mathcal{B}(V)$ is generated by x_i, $1 \leqslant i \leqslant \theta$, subject to the relations

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- The Nichols algebra $\mathcal{B}(V)$ is an $\mathbb{N}^{p+1}$-filtered algebra, whose associated graded algebra $\mathbb{G}\mathrm{r}\mathcal{B}(V)$ is isomorphic to the following algebra:

$$\Bbbk\langle x_{\beta_1}, \cdots, x_{\beta_p} \mid x_{\beta_i} x_{\beta_j} = \chi_{\beta_j}(g_{\beta_i}) x_{\beta_j} x_{\beta_i}, \ \ 1 \leqslant i < j \leqslant p\rangle,$$

where $x_{\beta_1}, \cdots, x_{\beta_p}$ are the root vectors of $\mathcal{B}(V)$.

The CY property of Nichols algebras

Theorem 4

Let V be a generic braided vector space of finite Cartan type, and $R = \mathcal{B}(V)$ the Nichols algebra of V. For each $1 \leqslant k \leqslant \theta$, let j_k be the integer such that $\beta_{j_k} = \alpha_k$.

(1) *The rigid dualizing complex is isomorphic to ${}_\varphi R[p]$, where φ is the algebra automorphism defined by*

$$\varphi(x_k) = \Big(\prod_{i=1}^{j_k-1} \chi_k^{-1}(g_{\beta_i})\Big)\Big(\prod_{i=j_k+1}^{p} \chi_{\beta_i}(g_k)\Big)x_k = \prod_{i=1, i\neq j_k}^{p} \chi_{\beta_i}(g_k)x_k,$$

for any $1 \leqslant k \leqslant \theta$.

The CY property of Nichols algebras

(2) The algebra R is a CY algebra if and only if

$$\prod_{i=1}^{j_k-1} \chi_k(g_{\beta_i}) = \prod_{i=j_k+1}^{p} \chi_{\beta_i}(g_k),$$

for any $1 \leqslant k \leqslant \theta$.

- $\operatorname{Gr} U(\mathcal{D}, \lambda) \cong U(\mathcal{D}, 0) \cong \mathcal{B}(V) \# \Bbbk\Gamma$.

- $\operatorname{Gr} U(\mathcal{D},\lambda) \cong U(\mathcal{D},0) \cong \mathcal{B}(V)\#\Bbbk\Gamma$.

Proposition 5

If $A = U(\mathcal{D},\lambda)$ is a CY algebra, then the rigid dualizing complex of the Nichols algebra $R = \mathcal{B}(V)$ is isomorphic to ${}_{\varphi}R[p]$, where φ is defined by $\varphi(x_k) = \chi_k^{-1}(g_k)x_k$, for all $1 \leqslant k \leqslant \theta$.

- Gr $U(\mathcal{D},\lambda) \cong U(\mathcal{D},0) \cong \mathcal{B}(V)\#\Bbbk\Gamma$.

Proposition 5

If $A = U(\mathcal{D},\lambda)$ is a CY algebra, then the rigid dualizing complex of the Nichols algebra $R = \mathcal{B}(V)$ is isomorphic to ${}_{\varphi}R[p]$, where φ is defined by $\varphi(x_k) = \chi_k^{-1}(g_k)x_k$, for all $1 \leqslant k \leqslant \theta$.

Proposition 6

If the Nichols algebra $R = \mathcal{B}(V)$ is a CY algebra, then the rigid dualizing complex of $A = U(\mathcal{D},\lambda)$ is isomorphic to ${}_{\psi}A[p+s]$, where ψ is defined by $\psi(x_k) = x_k$ for all $1 \leqslant k \leqslant \theta$ and $\psi(g) = \prod_{i=1}^{p} \chi_{\beta_i}(g)$ for all $g \in \Gamma$.

- **Question**:
  Let H be a Hopf algebra, and R a braided Hopf algebra in the category of Yetter-Drinfeld modules over H. What is the relation between the CY property of R and that of $R\#H$?

Outline

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- The Calabi-Yau property of $U(\mathcal{D}, \lambda)$
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- **Question**: Let H be a Hopf algebra, and R a braided Hopf algebra in the category of Yetter-Drinfeld modules over H. What is the relation between the CY property of R and that of $R\#H$?
  - If R is CY, when is $R\#H$ CY?

- Let R be a p-Koszul CY algebra (not necessarily a braided Hopf algebra) and H an involutory CY Hopf algebra. Liu, Wu and Zhu showed that the smash product $R\#H$ is CY if and only if the homological determinant of the H-action is trivial.

- Let R be a p-Koszul CY algebra (not necessarily a braided Hopf algebra) and H an involutory CY Hopf algebra. Liu, Wu and Zhu showed that the smash product $R\#H$ is CY if and only if the homological determinant of the H-action is trivial.
- (Jørgensen-Zhang) Let R be an AS-Gorenstein algebra of injective dimension d. There is a left H-action on $\mathrm{Ext}^d_R(\Bbbk, R)$ induced by the left H-action on R. Let $\mathbf{e}$ be a non-zero element in $\mathrm{Ext}^d_R(\Bbbk, R)$. Then there is an algebra homomorphism $\eta : H \to \Bbbk$ satisfying $h \cdot \mathbf{e} = \eta(h)\mathbf{e}$ for all $h \in H$.
  - (i) The composite map $\eta\mathcal{S}_H : H \to \Bbbk$ is called the homological determinant of the H-action on R, and it is denoted by hdet (or more precisely hdet_R).
  - (ii) The homological determinant hdet_R is said to be trivial if $\mathrm{hdet}_R = \varepsilon_H$, where ε_H is the counit of the Hopf algebra H.

Proposition 7

Let H be a finite dimensional semisimple Hopf algebra and R a braided Hopf algebra in the category ${}^H_H\mathcal{YD}$. If R is an AS-regular algebra of global dimension d_R, then $A = R\#H$ is also AS-regular of global dimension d_R.
In this case, if $\int_R^l = \Bbbk_{\xi_R}$ where $\xi_R : R \to \Bbbk$ is an algebra homomorphism, then $\int_A^l = \Bbbk_\xi$, where $\xi : A \to \Bbbk$ is defined by

$$\xi(r\#h) = \xi_R(r)\,\mathrm{hdet}(h),$$

for all $r\#h \in R\#H$.

$R \to R\#H$

Theorem 8

Let H be a finite dimensional semisimple Hopf algebra and R a Noetherian braided Hopf algebra in the category ${}^H_H\mathcal{YD}$ of Yetter-Drinfeld modules. Suppose that the algebra R is CY of dimension d_R. Then $R\#H$ is CY if and only if the homological determinant of the H-action is trivial and the algebra automorphism ϕ defined by

$$\phi(r\#h) = \mathcal{S}_H(r_{(-1)})(\mathcal{S}_R^2(r_{(0)}))\mathcal{S}_H^2(h)$$

for any $r\#h \in R\#H$ is an inner automorphism.

- **Question**: Let H be a Hopf algebra, and R a braided Hopf algebra in the category of Yetter-Drinfeld modules over H. What is the relation between the CY property of R and that of $R\#H$?
    - If R is CY, when is $R\#H$ CY?
    - If $R\#H$ is CY, when is R CY?

Rigid dualizing complexes

Theorem 9

Let Γ be a finite group and R a braided Hopf algebra in the category ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ of Yetter-Drinfeld modules. Assume that R is an AS-Gorenstein algebra with injective dimension d. If $\int_R^l \cong \Bbbk_{\xi_R}$, for some algebra homomorphism $\xi_R : R \to \Bbbk$, then R has a rigid dualizing complex ${}_{\varphi}R[d]$, where φ is the algebra automorphism defined by

$$\varphi(r) = \sum_{g\in\Gamma} \xi_R(r^1)\,\mathrm{hdet}(g)g^{-1}(\mathcal{S}_R^2((r^2)_g))$$

for all $r \in R$. Here hdet *denotes the homological determinant of the group action.*

We use $\Delta(r) = r^1 \otimes r^2$ to denote the comultiplication for a braided Hopf algebra. If Γ is a finite group and the algebra R is a Γ-comodule. then R is a Γ-graded module. Let δ denote the Γ-comodule structure. Then $R = \oplus_{g\in\Gamma} R_g$, where $R_g = \{r \in R \mid \delta(r) = g \otimes r\}$. If $r = \sum_{g\in\Gamma} r_g$ with $r_g \in R_g$, then $\delta(r) = \sum_{g\in\Gamma} g \otimes r_g$.

$R\#H \to R$

Theorem 10

Let Γ be a finite group and R a braided Hopf algebra in the category ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ of Yetter-Drinfeld modules. Define an algebra automorphism φ of R by

$$\varphi(r) = \sum_{g \in \Gamma} g^{-1}(\mathcal{S}_R^2(r_g)),$$

for any $r \in R$. If $R\#\Bbbk\Gamma$ is a CY algebra, then R is CY if and only if the algebra automorphism φ is an inner automorphism.

Thank you!