Generalized Hopf algebras by deforming identities

Abdenacer Makhlouf (Mulhouse University, France)
abdenacer.makhlouf@uha.fr

The purpose of my talk is to summarize present recent developments and provide some key constructions of Hom-associative and Hom-Hopf algebraic structures. The main feature of Hom-algebras is that the classical identities are twisted by a homomorphism.

The Hom-Lie algebras arise naturally in discretizations and deformations of vector fields and differential calculus, to describe the structures on some q-deformations of the Witt and the Virasoro algebras. They were developed in a general framework by Larsson and Silvestrov. The Hom-associative algebras, Hom-coassociative coalgebras and Hom-Hopf algebra were introduced by Silvestrov and myself. Recently, the Hom-type algebras were intensively investigated. A categorical point of view were discussed by Caenepeel and Goyvaerts. Also Yau showed that the enveloping algebra of a Hom-Lie algebra may be endowed by a structure of Hom-bialgebra.

Generalized Bialgebras and Hopf algebras by deforming identities

Abdenacer MAKHLOUF

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Plan

Paradigmatic example : quasi-deformation of $\mathfrak{sl}_2(\mathbb{K})$

$$[H, E] = 2E, \qquad [H, F] = -2F, \qquad [E, F] = H.$$

In terms of first order differential operators acting on some vector space of functions in the variable t:

$$E \mapsto \partial, \qquad H \mapsto -2t\partial, \qquad F \mapsto -t^2\partial.$$

To **quasi-deform** $\mathfrak{sl}_2(\mathbb{K})$ means that we replace ∂ by ∂_σ which is a σ-derivation.

Let $\mathcal{A}$ be a commutative, associative $\mathbb{K}$-algebra with unity 1.

A *σ-derivation* on $\mathcal{A}$ is a $\mathbb{K}$-linear map $\partial_\sigma : \mathcal{A} \to \mathcal{A}$ such that a σ-Leibniz rule holds:

$$\partial_\sigma(ab) = \partial_\sigma(a)b + \sigma(a)\partial_\sigma(b).$$

Example : Jackson q-derivation operator on $\mathcal{A} = \mathbb{K}[t]$

$$\partial_\sigma \quad : \; P(t) \to (D_q P)(t) = \frac{P(qt) - P(t)}{qt - t},$$

here $\sigma P(t) := P(qt)$. The operator satisfies

$$(D_q(PQ))(t) = (D_q P)(t)Q(t) + P(qt)(D_q Q)(t), \quad \sigma\text{-Leibniz rule}$$

Assume $\sigma(1) = 1$, $\sigma(t) = qt$, $\partial_\sigma(1) = 0$ and $\partial_\sigma(t) = t$
Then : $\partial_\sigma(t^2) = \partial_\sigma(t \cdot t) = \sigma(t)\partial_\sigma(t) + \partial_\sigma(t)t = (\sigma(t) + t)\partial_\sigma(t)$.
The brackets become

$$[H, F]_\sigma = 2\sigma(t)t\partial_\sigma(t)\partial_\sigma = 2qt^2\partial_\sigma = -2qF$$

$$[H, E]_\sigma = 2\partial_\sigma(t)\partial_\sigma = 2E$$

$$[E, F]_\sigma = -(\sigma(t) + t)\partial_\sigma(t)\partial_\sigma = -(q + 1)t\partial_\sigma = \frac{1}{2}(1 + q)H.$$

The new bracket satisfies

$$\circlearrowleft_{a,b,c}\ [\sigma(a)\cdot\partial_\sigma,[b\cdot\partial_\sigma,c\cdot\partial_\sigma]_\sigma]_\sigma=0. \qquad (1)$$

Definition

A **Hom-Lie algebra** is a triple $(V,[\cdot,\cdot],\alpha)$ satisfying

$$[x,y]=-[y,x] \quad \text{(skewsymmetry)}$$
$$\circlearrowleft_{x,y,z}[\alpha(x),[y,z]]=0 \quad \text{(Hom-Jacobi condition)}$$

for all $x,y,z\in V$, where $\circlearrowleft_{x,y,z}$ cyclic summation.

• *Makhlouf and Silvestrov, Hom-algebra structures, Journal of Generalized Lie Theory and Applications, vol* **2** *(2)* (2008)

Definition

A **Hom-associative algebra** is a triple (A, μ, α) consisting of a vector space A, a bilinear map $\mu : A \times A \to A$ and a homomorphism $\alpha : A \to A$ satisfying

$$\mu(\alpha(x), \mu(y, z)) = \mu(\mu(x, y), \alpha(z))$$

A linear map $\phi \ : A \to A'$ is a morphism of Hom-associative algebras if

$$\mu' \circ (\phi \otimes \phi) = \phi \circ \mu \quad \text{and} \qquad \phi \circ \alpha = \alpha' \circ \phi.$$

A Hom-associative algebra is said to be weakly unital if there exists a unit $\mathbf{1}$ such that

$$\mu(x, \mathbf{1}) = \mu(\mathbf{1}, x) = \alpha(x).$$

A **Hom-module** is a pair (M, α_M) consisting of a $\mathbb{K}$-module and a linear self-map $\alpha_M : M \to M$. A morphism of Hom-modules $f : (M, \alpha_M) \to (N, \alpha_N)$ is a morphism of the underlying $\mathbb{K}$-modules that is compatible with the twisting maps, in the sense that

$$\alpha_N \circ f = f \circ \alpha_M.$$

The tensor product of two Hom-modules M and N is the pair $(M \otimes N, \alpha_M \otimes \alpha_N)$.

A **Hom-Nonassociative algebra** or a **Hom-algebra** is a triple (A, μ_A, α_A) in which (A, α_A) is a Hom-module and $\mu_A : A \otimes A \to A$ is a multiplication.
A morphism of Hom-algebras and The tensor product of two Hom-algebras are as we expect.

Set

1. $HomMod$: category of Hom-modules,
2. HNA : category of Hom-Nonassociative algebras,
3. HA : category of associative algebras (multiplicative),
4. HL : category of Hom-Lie algebras (multiplicative)

There is the following adjoint pairs of functors in which F_0, F_1, F_2 are $F_2 \circ F_1 \circ F_0$ are the left adjoint

$$Mod \underset{U}{\overset{F_0}{\rightleftarrows}} HomMod \underset{U}{\overset{F_1}{\rightleftarrows}} HNA \underset{U}{\overset{F_2}{\rightleftarrows}} HA \underset{U_{HLie}}{\overset{HLie}{\rightleftarrows}} HL$$

Free Hom-associative algebra and enveloping algebra

• D. Yau, Enveloping algebras of Lie algebras, J. Gen. Lie Theory Appl 2 (2008)

Let (A, μ, α) be a Hom-Nonassociative algebra (Hom-algebra).
The products are defined using the set of weighted trees (T_n^{wt}).
Consider the map

$$\mathbb{K}[T_n^{wt}] \otimes A^{\otimes n} \longrightarrow A$$
$$(\tau; x_1, \cdots, x_n) \longrightarrow (x_1, \cdots, x_n)_\tau$$

inductively via the rules

1. $(x)_i = x$ for $x \in A$, where i denote the 1-tree,
2. If $\tau = (\tau_1 \vee \tau_2)[r]$ then $(x_1, \cdots, x_n)_\tau = \alpha^r((x_1, \cdots, x_p)_{\tau_1}(x_{p+1}, \cdots, x_{p+q})_{\tau_2})$.

The free Hom-Nonassociative algebra is

$$F_{HNAs}(A) = \oplus_{n\geq 1} \oplus_{\tau \in T_n^{wt}} A_\tau^{\otimes n}$$

where $A_\tau^{\otimes n}$ is a copy of $A^{\otimes n}$.
The multiplication μ_F is defined by

$$\mu_F((x_1, \cdots, x_n)_\tau, (x_{n+1}, \cdots, x_{n+m})_\sigma) = (x_1, \cdots, x_{n+m})_{\tau \vee \sigma}$$

and the linear map is defined by the rule

1. $\alpha_{F|A} = \alpha_V$
2. $\alpha_F((x_1, \cdots, x_n)_\tau) = (x_1, \cdots, x_n)_{\tau[1]}$.

Consider two-sided ideals $I^1 \subset I^2 \subset \cdots I^\infty \subset F_{HNAs(A)}$ where

$$I^1 =< Im(\mu_F \circ (\mu_F \circ \alpha_F - \alpha_F \circ \mu_F) >$$

and $I^{n+1} =< I^n \bigcup \alpha(I^n) >$, $I^\infty = \bigcup_{n\geq 1} I^n$.
The quotient module

$$F_{HAs}(A) = F_{HNAs(A)}/I^\infty.$$

equipped with μ_F and α_F is the free Hom-associative algebra.

The **enveloping Lie algebra** is obtained by considering the two-sided ideals J^k where

$$J^1 =< Im(\mu_F \circ (\mu_F \circ \alpha - \alpha \circ \mu); [x,y] - (xy - yx) \text{ for } x, y \in A > .$$

Hom-coalgebras, Hom-bialgebras and Hom-Hopf algebras

(1) Makhlouf A. and Silvestrov S., Hom-Lie admissible Hom-Coalgebras and Hom-Hopf Algebras, In "Generalized Lie Theory in Math., Physics and Beyond", Springer (2008),
(2) Makhlouf A. and Silvestrov S., Hom-algebras and Hom-coalgebras , Journal of Algebra and Its Applications Vol. **9** *(2010)*
(3) Yau, Hom-bialgebras and comodule algebras, e-print arXiv 0810.4866 (2008)
(4) Caenepeel S. and Goyvaerts I., Monoidal Hopf algebras, e-print arXiv:0907.0187 (2010).

Definition

A *Hom-coalgebra* is a triple (C, Δ, β) where C is a $\mathbb{K}$-vector space and $\Delta : C \to C \otimes C$, $\beta : C \to C$ are linear maps.
A *Hom-coassociative coalgebra* is a Hom-coalgebra satisfying

$$(\beta \otimes \Delta) \circ \Delta = (\Delta \otimes \beta) \circ \Delta. \tag{2}$$

A Hom-coassociative coalgebra is said to be *counital* if there exists a map $\varepsilon : C \to \mathbb{K}$ satisfying

$$(id \otimes \varepsilon) \circ \Delta = id \quad \text{and} \quad (\varepsilon \otimes id) \circ \Delta = id \tag{3}$$

Hom-bialgebras

Definition

A *Hom-bialgebra* is a 7-uple $(B, \mu, \alpha, \eta, \Delta, \beta, \varepsilon)$ where

(B1) (B, μ, α, η) is a Hom-associative algebra with unit η.

(B2) $(B, \Delta, \beta, \varepsilon)$ is a Hom-coassociative coalgebra with counit ε.

(B3) The linear maps Δ and ε are compatible with the multiplication μ, that is

$$\begin{cases} \Delta(e_1) = e_1 \otimes e_1 \qquad \text{where } e_1 = \eta(1) \\ \Delta(\mu(x \otimes y)) = \Delta(x) \bullet \Delta(y) = \sum_{(x)(y)} \mu(x^{(1)} \otimes y^{(1)}) \otimes \mu(x^{(2)} \otimes y^{(2)}) \\ \varepsilon(e_1) = 1 \\ \varepsilon(\mu(x \otimes y)) = \varepsilon(x)\,\varepsilon(y) \end{cases}$$

Definition (2)

One can consider a more restrictive definition where linear maps Δ and ε are morphisms of Hom-associative algebras that is the condition (B3) becomes equivalent to

$$\begin{cases} \Delta(e_1) = e_1 \otimes e_1 \qquad \text{where } e_1 = \eta(1) \\ \Delta(\mu(x \otimes y)) = \Delta(x) \bullet \Delta(y) = \sum_{(x)(y)} \mu(x^{(1)} \otimes y^{(1)}) \otimes \mu(x^{(2)} \otimes y^{(2)}) \\ \varepsilon(e_1) = 1 \\ \varepsilon(\mu(x \otimes y)) = \varepsilon(x)\, \varepsilon(y) \\ \Delta(\alpha(x)) = \sum_{(x)} \alpha(x^{(1)}) \otimes \alpha(x^{(2)}) \\ \varepsilon \circ \alpha(x) = \varepsilon(x) \end{cases}$$

Given a Hom-bialgebra $(B, \mu, \alpha, \eta, \Delta, \beta, \varepsilon)$, we show that the vector space $Hom(B, B)$ with the multiplication given by the convolution product carries a structure of Hom-associative algebra.

Proposition

Let $(B, \mu, \alpha, \eta, \Delta, \beta, \varepsilon)$ be a Hom-bialgebra. Then the algebra $Hom(B, B)$ with the multiplication given by the convolution product defined by

$$f \star g = \mu \circ (f \otimes g) \circ \Delta$$

and the unit being $\eta \circ \epsilon$, is a unital Hom-associative algebra with the homomorphism map defined by $\gamma(f) = \alpha \circ f \circ \beta$.

Hom-Hopf algebras

A Hom-Hopf algebra over a $\mathbb{K}$-vector space H is given by $(H, \mu, \alpha, \eta, \Delta, \beta, \varepsilon, S)$, where $(H, \mu, \alpha, \eta, \Delta, \beta, \varepsilon)$ is a bialgebra and S is the antipode that is the inverse of the identity over H for the convolution product.

We have the following properties :

- The antipode S is unique,
- $S(\eta(1)) = \eta(1)$,
- $\varepsilon \circ S = \varepsilon$.
- Let x be a primitive element $(\Delta(x) = \eta(1) \otimes x + x \otimes \eta(1))$, then $\varepsilon(x) = 0$.
- If x and y are two primitive elements in $\mathcal{H}$. Then we have $\varepsilon(x) = 0$ and the commutator $[x, y] = \mu(x \otimes y) - \mu(y \otimes x)$ is also a primitive element.
- The set of all primitive elements of $\mathcal{H}$, denoted by $Prim(\mathcal{H})$, has a structure of Hom-Lie algebra.

Generalized Hom-bialgebras

Definition

A *generalized Hom-bialgebra* is a 5-uple $(B, \mu, \alpha, \Delta, \beta)$ where
(GB1) (B, μ, α) is a Hom-associative algebra.
(GB2) (B, Δ, β) is a Hom-coassociative coalgebra.
(GB3) The linear maps Δ is compatible with the multiplication μ, that is

$$\Delta\left(\mu(x \otimes y)\right) = \Delta\left(x\right) \bullet \Delta\left(y\right) = \sum_{(x)(y)} \mu(x^{(1)} \otimes y^{(1)}) \otimes \mu(x^{(2)} \otimes y^{(2)})$$

We recover generalized bialgebra introduced by Loday when α and β are the identity map.

Theorem (Twisting Principle 1)

Let (A, μ) be an associative algebra and let $\alpha : A \to A$ be an algebra endomorphism. Then (A, μ_α, α), where $\mu_\alpha = \alpha \circ \mu$, is a Hom-associative algebra.

Let (C, Δ) be a coalgebra and let $\beta : C \to C$ be a coalgebra endomorphism. Then (C, Δ_β, β), where $\Delta_\beta = \Delta \circ \beta$, is a Hom-coassociative coalgebra.

Let (B, μ, Δ) be a generalized bialgebra and let $\alpha : B \to B$ be a generalized bialgebra endomorphism. Then $(B, \mu_\alpha, \Delta_\alpha, \alpha)$, where $\mu_\alpha = \alpha \circ \mu$ and $\Delta_\alpha = \Delta \circ \alpha$, is a generalized Hom-bialgebra.

Theorem (Twisting Principle 2 (Yau))

Let (A, μ, α) be a multiplicative Hom-associative algebra, (C, Δ, α) be a Hom-coalgebra and $(B, \mu, \Delta, , \alpha)$ be a generalized Hom-bialgebra. Then for each integer $n \geq 0$

1. $A^n = (A, \mu^{(n)} = \alpha^{2^n-1} \circ \mu, \alpha^{2^n})$, *is a Hom-associative algebra.*
2. $(C, \Delta^{(n)} = \Delta \circ \alpha^{2^n-1}, \alpha^{2^n})$, *is a Hom-coassociative coalgebra.*
3. $(B, \mu^{(n)} = \alpha^{2^n-1} \circ \mu, \Delta^{(n)} = \Delta \circ \alpha^{2^n-1}, \alpha^{2^n})$, *is a generalized Hom-bialgebra.*

Example

Let $\mathbb{K}G$ be the group-algebra over the group G. As a vector space, $\mathbb{K}G$ is generated by $\{e_g : g \in G\}$. If $\alpha : G \to G$ is a group homomorphism, then it can be extended to an algebra endomorphism of $\mathbb{K}G$ by setting

$$\alpha(\sum_{g\in G} a_g e_g) = \sum_{g\in G} a_g \alpha(e_g) = \sum_{g\in G} a_g e_{\alpha(g)}.$$

Consider the usual bialgebra structure on $\mathbb{K}G$ and α a generalized bialgebra morphism. Then, we define over $\mathbb{K}G$ a generalized Hom-bialgebra $(\mathbb{K}G, \mu, \alpha, \Delta, \alpha)$ by setting:

$$\mu(e_g \otimes e_{g'}) = \alpha(e_{g\cdot g'}),$$

$$\Delta(e_g) = \alpha(e_g) \otimes \alpha(e_g).$$

Example

Consider the polynomial algebra $\mathcal{A} = \mathbb{K}[(X_{ij})]$ in variables $(X_{ij})_{i,j=1,\cdots,n}$. It carries a structure of generalized bialgebra with the comultiplication defined by $\delta(X_{ij}) = \sum_{k=1}^{n} X_{ik} \otimes X_{kj}$ and $\delta(1) = 1 \otimes 1$. Let α be a generalized bialgebra morphism, it is defined by n^2 polynomials $\alpha(X_{ij})$. We define a generalized Hom-bialgebra $(\mathcal{A}, \mu, \alpha, \Delta, \alpha)$ by

$$\mu(f \otimes g) = f(\alpha(X_{11}), \cdots, \alpha(X_{nn}))g(\alpha(X_{11}), \cdots, \alpha(X_{nn})),$$

$$\Delta(X_{ij}) = \sum_{k=1}^{n} \alpha(X_{ik}) \otimes \alpha(X_{kj}),$$

$$\Delta(1) = \alpha(1) \otimes \alpha(1).$$

Example

Let X be a set and consider the set of non-commutative polynomials $\mathcal{A} = \mathbb{K}\langle X \rangle$. It carries a generalized bialgebra structure with a comultiplication defined for $x \in X$ by $\delta(x) = 1 \otimes x + x \otimes 1$ and $\delta(1) = 1 \otimes 1$. Let α be a generalized bialgebra morphism. We define a generalized Hom-bialgebra $(\mathcal{A}, \mu, \alpha, \Delta, \alpha)$ by

$$\begin{aligned} \mu(f \otimes g) &= f(\alpha(X))g(\alpha(X)), \\ \Delta(x) &= \alpha(1) \otimes \alpha(x) + \alpha(x) \otimes \alpha(1), \\ \Delta(1) &= \alpha(1) \otimes \alpha(1). \end{aligned}$$

Representations of Hom-algebras

Let (A, μ_A, α_A) be a Hom-associative algebra.
An A-**module** is a Hom-module (M, α_M) together with a linear map $\rho : A \otimes M \to M$, such that

$$\begin{aligned} \alpha_M \circ \rho &= \rho \circ (\alpha_A \otimes \alpha_M) \quad \text{(multiplicativity)} \\ \rho \circ (\alpha_A \otimes \rho) &= \rho \circ (\mu_A \otimes \alpha_M) \quad \text{(Hom-associativity)} \end{aligned}$$

A morphism $f : (M, \alpha_M) \to (N, \alpha_N)$ of A-modules is a morphism of the underlying Hom-modules that is compatible with the structure maps, in the sense that

$$f \circ \rho_M = \rho_N \circ (id_A \circ f).$$

Multiplicativity is equivalent to ρ being a morphism of Hom-modules.

Let (C, Δ_C, α_C) be a Hom-coassociative coalgebra.
A C-**comodule** is a Hom-module (M, α_M) together with a linear map $\rho : M \to C \otimes M$, such that

$$\begin{aligned} \rho \circ \alpha_M &= (\alpha_C \otimes \alpha_M) \circ \rho \quad \text{(comultiplicativity)} \\ \rho \circ (\alpha_A \otimes \rho) &= \rho \circ (\Delta_C \otimes \alpha_M) \quad \text{(Hom-coassociativity)} \end{aligned}$$

A morphism $f : (M, \alpha_M) \to (N, \alpha_N)$ of C-comodules is a morphism of the underlying Hom-modules that is compatible with the structure maps, in the sense that

$$(id_C \circ f) \circ \rho_M = \rho_N \circ f.$$

Twisting principle (Yau)

Let (A, μ_A, α_A) be a Hom-associative algebra (multiplicative) and (M, α_M) be an A-module with structure map $\rho : A \otimes M \to M$. For each integer $n, k \geq 0$ define the map

$$\rho^{n,k} = \alpha_M^{2^k - 1} \circ \rho \circ (\alpha_A^n \otimes Id_M) : A \otimes M \to M.$$

Then each $\rho^{n,k}$ gives the Hom-module $M^k = (M, \alpha_M^{2^k})$ the structure of an A^k-module, where A^k is the k-derived Hom-associative algebra $(A, \mu_A^{(k)} = \alpha_A^{2^k - 1} \circ \mu_A, \alpha_A^{2^k})$.
Note that $\rho^{0,0} = \rho$, $\rho^{1,0} = \rho \circ (\alpha_A \otimes Id_M)$, $\rho^{0,1} = \alpha_M \circ \rho$, $\rho^{n+1,0} = \rho^{n,0} \circ (\alpha_A \otimes Id_M)$ and $\rho^{0,k+1} = \alpha_M^{2^k - 1} \circ \rho$.
The A^k-module M^k with the structure $\rho^{n,k}$ is called the (n, k)-derived module of M.

Twisting principle (2)

Let (A, μ_A) be an associative algebra and M be an A-module (classical sense) with structure map $\rho : A \otimes M \to M$. Suppose $\alpha_A : A \to A$ is an algebra morphism and $\alpha_M : M \to M$ is a linear self-map such that

$$\alpha_M \circ \rho = \rho \circ (\alpha_A \otimes \alpha_M)$$

For any integer $n, k \geq 0$ define the map

$$\rho_\alpha^{n,k} = \alpha_M^{2^k} \circ \rho \circ (\alpha_A^n \otimes Id_M) : A \otimes M \to M.$$

Then each $\rho^{n,k}$ gives the Hom-module $M_\alpha^k = (M, \alpha_M^{2^k})$ the structure of an A_β-module, where $\beta = \alpha_A^{2^k}$ and A_β is the Hom-associative algebra $(A, \mu_\beta = \beta \circ \mu_A, \beta)$.

Hom-quantum group: $\mathcal{U}_q(\mathfrak{sl}_2)_\alpha$

Consider the quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$ generated as usual by $E, F, K^{\pm 1}$ satisfying the relations

$$KK^{-1} = \mathbf{1} = K^{-1}K,$$
$$KE = q^2 EK,\ \ KF = q^{-2} FK,$$
$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

and the bialgebra morphism $\alpha_\lambda : \mathcal{U}_q(\mathfrak{sl}_2) \to \mathcal{U}_q(\mathfrak{sl}_2)$ defined by

$$\alpha_\lambda(E) = \lambda E,\ \ \alpha_\lambda(F) = \lambda^{-1} F,\ \ \alpha_\lambda(K^{\pm 1}) = K^{\pm 1}.$$

Then $\mathcal{U}_q(\mathfrak{sl}_2)_\alpha = (\mathcal{U}_q(\mathfrak{sl}_2), \mu_{\alpha_\lambda}, \Delta_{\alpha_\lambda}, \alpha_\lambda)$ is a Hom-bialgebra.

Finite-dimensional modules over $\mathcal{U}_q(\mathfrak{sl}_2)_\alpha$

Assume $q \in \mathbb{C} - \{0\}$ is not a root of unity.
For each integer $n \geq 0$ and $\epsilon \in \{\pm 1\}$,
there is an $(n+1)$-dimensional simple $\mathcal{U}_q(\mathfrak{sl}_2)$-module $V(\epsilon, n)$.
Let $\{v_i\}_{1 \leq i \leq n}$ be a basis of $V(\epsilon, n)$. The action is defined by

$$Kv_i = \epsilon q^{n-2i} v_i, \ \ Ev_i = \epsilon [n-i+1]_q v_{i-1}, \ \ Fv_i = [i+1]_q v_{i+1}.$$

where $v_{-1} = 0 = v_{n+1}$.
Pick any scalar $\xi \in \mathbb{C}$ and define $\alpha_\xi : V(\epsilon, n) \to V(\epsilon, n)$ by setting

$$\alpha_\xi(v_i) = \xi \lambda^{-i} v_i, \quad \forall i$$

Then

$$\alpha_\xi(Uv) = \alpha_\lambda(U)\alpha_\xi(v) \quad \forall U \in \mathcal{U}_q(\mathfrak{sl}_2), v \in V(\epsilon, n).$$

Finite-dimensional modules over $\mathcal{U}_q(\mathfrak{sl}_2)_\alpha$ (2)

By twisting principle, one constructs an uncountable, four parameter family $V(\epsilon, n)_\alpha^{r,k}$ of $(n+1)$-dimensional derived $\mathcal{U}_q(\mathfrak{sl}_2)_\beta$-module, where $\beta = \alpha_\lambda^{2^k} = \alpha_{\lambda^{2^k}}$ and the structure map is given for any $r, k \geq 0$ by

$$\rho_\alpha^{n,k}(K^{\pm 1} \otimes v_i) = (\epsilon q^{n-2i})^{\pm 1} (\xi \lambda^{-i})^{2^k} v_i,$$

$$\rho_\alpha^{n,k}(E \otimes v_i) = \epsilon [n-i+1]_q \xi^{2^k} \lambda^{r-2^k(i-1)} v_{i-1},$$

$$\rho_\alpha^{n,k}(F \otimes v_i) = \epsilon [i+1]_q \xi^{2^k} \lambda^{-r-2^k(i+1)} v_{i+1}.$$

Classical case corresponds to $\lambda = \xi = 1$.

Module Hom-algebras

Definition

Let $(H, \mu, \Delta_H, \alpha_H)$ be a Hom-bialgebra and (A, μ_A, α_A) be a Hom-associative algebra.
*An H-**module Hom-algebra** structure on A consists of an H-module structure $\rho : H \otimes A \to A$ such that the module Hom-algebra axiom*

$$\alpha_H^2(x)(ab) = \sum_{(x)} (x_{(1)} a)(x_{(2)} b)$$

is satisfied for all $x \in H$ and $a, b \in A$, where $\rho(x \otimes a) = xa$.

In element-free form the Hom-algebra axiom is

$$\rho \circ (\alpha_H^2 \otimes \mu_A) = \mu_A \circ \rho^{\otimes 2} \circ (23) \circ (\Delta_H \otimes id_A \otimes id_A).$$

Remark

Let $(H, \mu, \Delta_H, \alpha_H)$ be a Hom-bialgebra and let (M, α_M) and (N, α_N) be H-modules with structure maps ρ_M and ρ_N. Then $M \otimes N$ is an H-module with structure map

$$\rho_{MN} = (\rho_M \otimes \rho_N) \circ (23) \circ (\Delta \otimes Id_M \otimes Id_N) : H \otimes M \otimes N \to M \otimes N$$

Theorem

Let $(H, \mu_H, \Delta_H, \alpha_H)$ be a Hom-bialgebra, (A, μ_A, α_A) be a Hom-associative algebra and $\rho : H \otimes A \to A$ be an H-module structure on A

Then the module Hom-algebra axiom is satisfied if and only if $\mu_A : A \otimes A \to A$ is a morphism of H-modules, in which $A \otimes A$ and A are given the H-module structure maps ρ_{AA} and $\rho^{2,0}$, respectively.

By twisting principle (2), every module Hom-algebra gives rise to a derived double-sequence of module Hom-algebras.
Twisting principle (1) for module Hom-algebras:

Theorem

Let (H, μ_H, Δ_H) be a bialgebra, (A, μ_A) be an associative algebra and $\rho : H \otimes A \to A$ be an H-module structure on A. Suppose $\alpha_H : H \to H$ is a bialgebra morphism and $\alpha_A : A \to A$ is an algebra morphism such that $\alpha_A \circ \rho = \rho \circ (\alpha_H \otimes \alpha_A)$.
For any integer $n, k \geq 0$ define the map

$$\rho_\alpha^{n,k} = \alpha_A^{2^k} \circ \rho \circ (\alpha_H^n \otimes Id_A) : H \otimes A \to A.$$

Then each $\rho^{n,k}$ gives A_β the structure of an H_γ-module Hom-algebra, where $\beta = \alpha_A^{2^k}$, $\gamma = \alpha_H^{2^k}$, A_β is the Hom-associative algebra $(A, \mu_\beta = \beta \circ \mu_A, \beta)$ and H_γ is the Hom-bialgebra $(H, \mu_\gamma = \gamma \circ \mu_H, \Delta_\gamma = \Delta \circ \gamma, \gamma)$.

Hom-quantum plane

Assume $q \in \mathbb{C} - \{0\}$ is not a root of unity. The $\mathcal{U}_q(\mathfrak{sl}_2)$-module algebra structure on the quantum plane

$$\mathbb{A}_q^{2|0} = \mathbb{K} < x, y > /(yx - qxy)$$

is defined using the following quantum partial derivatives

$$\partial_{q,x}(x^n y^m) = [m]_q x^{m-1} y^n \quad \text{and} \quad \partial_{q,y}(x^n y^m) = [n]_q x^m y^{n-1}$$

and for $P = P(x, y) \in \mathbb{A}_q^{2|0}$ we define

$$\sigma_x(P) = P(qx, y) \quad \text{and} \quad \sigma_y(P) = P(x, qy).$$

$$\rho : \mathcal{U}_q(\mathfrak{sl}_2) \otimes \mathbb{A}_q^{2|0} \to \mathbb{A}_q^{2|0}$$

is determined by

$$EP = x(\partial_{q,y} P), \;\; FP = (\partial_{q,x} P)y,$$
$$KP = \sigma_x \sigma_y^{-1}(P) = P(qx, q^{-1}y),$$
$$K^{-1}P = \sigma_y \sigma_x^{-1}(P) = P(q^{-1}x, qy).$$

The bialgebra morphism $\alpha_\lambda : \mathcal{U}_q(\mathfrak{sl}_2) \to \mathcal{U}_q(\mathfrak{sl}_2)$ defined by

$$\alpha_\lambda(E) = \lambda E, \;\; \alpha_\lambda(F) = \lambda^{-1} F, \;\; \alpha_\lambda(K^{\pm 1}) = K^{\pm 1},$$

and the algebra morphism $\alpha : \mathbb{A}_q^{2|0} \to \mathbb{A}_q^{2|0}$ defined

$$\alpha(x) = \xi x \quad \text{and} \quad \alpha(y) = \xi \lambda^{-1} y$$

satisfy

$$\alpha \circ \rho = \rho \circ (\alpha_\lambda \otimes \alpha).$$

By the Theorem, for any integer $l, k \geq 0$ the map

$$\rho_\alpha^{l,k} = \alpha^{2^k} \circ \rho \circ (\alpha_\lambda^l \otimes Id) : \mathcal{U}_q(\mathfrak{sl}_2) \otimes \mathbb{A}_q^{2|0} \to \mathbb{A}_q^{2|0},$$

gives the Hom-quantum-plane $(\mathbb{A}_q^{2|0})_\beta$ the structure of a $(\mathcal{U}_q(\mathfrak{sl}_2))_\gamma$-module. We have

$$\rho_\alpha^{l,k}(E \otimes x^m y^n) = [n]_q \xi^{2^k(m+n)} \lambda^{l-2^k(n-1)} x^{m+1} y^{n-1},$$
$$\rho_\alpha^{l,k}(F \otimes x^m y^n) = [m]_q \xi^{2^k(m+n)} \lambda^{-l-2^k(n+1)} x^{m-1} y^{n+1},$$
$$\rho_\alpha^{l,k}(K^{\pm 1} \otimes P) = P(q^{\pm 1} \xi^{2^k} x, q^{\mp 1} (\xi \lambda^{-1})^{2^k} y).$$

Twistings

Drinfeld's: gauge transformations of quasi-Hopf algebras
Giaquinto-Zhang: twists of algebraic structures based on action of a bialgebra.

Definition

Let $(B, \mu_B, \Delta_B, \mathbf{1}_B, \varepsilon_B)$ be a bialgebra.
An element $F \in B \otimes B$ is a twisting element (based on B) if

1. $(\varepsilon_B \otimes Id)F = \mathbf{1} \otimes \mathbf{1} = (Id \otimes \varepsilon_B)F,$
2. $[(\Delta \otimes Id)(F)](F \otimes 1) = [(Id \otimes \Delta)(F)](1 \otimes F).$

Theorem (Giaquinto-Zhang)

Let $F \in B \otimes B$ be a twisting element.

1. *If A is a left B-module algebra, then $A_F = (A, \mu_A \circ F_l, \mathbf{1}_A)$ is an associative algebra.*
2. *If C is a right B-module coalgebra, then $C_F = (C, F_l \circ \Delta_C, \varepsilon_C)$ is an coassociative algebra.*

Hom-Twistings

Definition

Let $(B, \mu_B, \Delta_B, \mathbf{1}_B, \varepsilon_B, \alpha_B)$ be a Hom-bialgebra where α_B is invertible.
An element $F \in B \otimes B$ is a Hom-twisting element (based on B) if

$$[(\alpha_B^{-1} \otimes \alpha_B^{-1} \otimes Id)(\Delta \otimes Id)(F)][(\alpha_B^2 \otimes \alpha_B^2 \otimes Id)(F \otimes \mathbf{1})] = [(Id \otimes \alpha_B^{-1} \otimes \alpha_B^{-1})(\mathbf{1} \otimes \Delta)(F)][(Id \otimes \alpha_B^2 \otimes \alpha_B^2)(\mathbf{1} \otimes F)].$$

Theorem

Let $F \in B \otimes B$ be a Hom-twisting element. Let $(A, \mu_A, \mathbf{1}_A, \alpha_A)$ be a Hom-associative algebra, where α_A is surjective.
Assume that $b\,\mathbf{1}_B = \mathbf{1}_B b = \alpha_B(b)$ and $a\,\mathbf{1}_A = \mathbf{1}_A a = \alpha_A(a)$
If A is a left B-module Hom-algebra, where $\mathbf{1}_B a = \alpha_A(a)$,
then

$$A_F = (A, \mu_A \circ [(\alpha_B^2 \otimes \alpha_B^2)F]_l, \alpha_A)$$

is a Hom-associative algebra.

Let $(B, \mu_B, \Delta_B, \mathbf{1}_B, \varepsilon_B)$ be a bialgebra. and $F \in B \otimes B$ a twisting element.
Let $(A, \mu_A, \mathbf{1}_A)$ be a left B-module algebra.
Suppose $\alpha_B : B \to B$ is a bialgebra morphism which is involutive and $\alpha_A : A \to A$ is an algebra morphism such that

$$\alpha_A \circ \rho = \rho \circ (\alpha_H \otimes \alpha_A).$$

Then F is a twisting element of the Hom-bialgebra B_{α_B} and the Hom-associative algebra A_{α_A} deforms to $(A_{\alpha_A})_F$

Hom-Yang-Baxter Equation

• D. Yau *Hom-Yang-Baxter equation, Hom-Lie algebras, and quasi-triangular bialgebras*, J. Phys. A **42** (2009).
The Hom-Yang-Baxter equation and Hom-Lie algebras, arXiv:0905.1887v2, (2009).
The classical Hom-Yang-Baxter equation and Hom-Lie bialgebras, arXiv:0905.1890v1, (2009).
Hom-quantum groups I:quasi-triangular Hom-bialgebras, arXiv:0906.4128v1, (2009).
Hom-quantum groups II:cobraided Hom-bialgebras and Hom-quantum geometry, arXiv:0907.1880v1, (2009).
Hom-quantum groups III:representations and Module Hom-algebras, arXiv:0907.1880v1, (2009).

The Hom-Yang-Baxter Equation (HYBE) is

$$(\alpha \otimes B) \circ (B \otimes \alpha) \circ (\alpha \otimes B) = (B \otimes \alpha) \circ (\alpha \otimes B) \circ (B \otimes \alpha)$$

- Each solution of the HYBE can be extended to operators that satisfy the braid relations (Yau).
- Each quasi-triangular Hom-bialgebra comes with a solution of the quantum Hom-Yang-Baxter equation (Yau).
- Hochschild type cohomology of multipl. Hom-ass. algebras (Ammar, Ejbehi, Makhlouf, Silvestrov)
- Hochschild type cohomology of multipl. Hom-coassociative coalgebras (Dekkar, Makhlouf)