Hopf algebras with triality

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In this joint work with G. Benkart and J.M. Pérez-Izquierdo, we revisit and extend the constructions of Glauberman and Doro on groups with triality and Moufang loops to Hopf algebras. We prove that the universal enveloping algebra of any Lie algebra with triality is a Hopf algebra with triality. This allows a new construction of the universal enveloping algebras of Malcev algebras. Our work relays on the approach of Grishkov and Zavarnitsine to groups with triality.

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joint work with G. Benkart & J.M.Pérez-Izquierdo

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Definitions I

Definition

If a group G admits two automorphisms ρ, σ such that $\forall x \in G$

$$\sigma^2 = \mathrm{Id} = \rho^3, \quad \sigma\rho = \rho^2\sigma$$

and

$$(x^{-1}x^{\sigma})(x^{-1}x^{\sigma})^{\rho}(x^{-1}x^{\sigma})^{\rho^2} = 1$$

then it is called **group with triality**.

This notion of group with triality firstly appears, in relation with Moufang loops, in the work by Glauberman in 1968 and it was studied also by Doro in 1978.

Definitions II

Definition

A **loop** $(Q, \cdot, e)$ is a set with a binary operation $\cdot \colon Q \times Q \to Q$ $(a, b) \mapsto ab$ with unit element $e \in Q$ such that the multiplication operators $L_a \colon b \mapsto ab$ and $R_b \colon a \mapsto ab$ are bijective for any $a, b \in Q$.

Definition

If a loop satisfies the Moufang identity

$$a(x(ay)) = ((ax)a)y$$

then it is called **Moufang loop**.

Infinitesimal analogous I

Definition

If a Lie algebra L admits two automorphisms ρ, σ such that $\forall x \in L$

$$\sigma^2 = \mathrm{Id} = \rho^3, \quad \sigma\rho = \rho^2\sigma$$

and

$$x - \sigma(x) + \rho(x) - \rho\sigma(x) + \rho^2(x) - \rho^2\sigma(x) = 0$$

then it is called **Lie algebra with triality**.

The concept of Lie algebra with triality appeared in a work of Mikheev and was studied by Grishkov in connection with Malcev algebras. It is the infinitesimal version of a group with triality.

Infinitesimal analogous II

Definition

A **Malcev algebra** is a vector space $\mathfrak{m}$ endowed with a bilinear binary operation $[\cdot,\cdot]:\mathfrak{m}\times\mathfrak{m}\to\mathfrak{m}$ satisfying

$$[x,y]=-[y,x] \quad \text{and} \quad [J(x,y,z),x]=J(x,y,[x,z])$$

$\forall x,y,z\in\mathfrak{m}$ where

$$J(x,y,z)=[[x,y],z]+[[y,z],x]+[[z,x],y]$$

is the Jacobian.

Malcev algebras are the infinitesimal version of Moufang loops (Malcev, 1955) and generalize the concept of Lie algebra.

Objects appearing

Lie algebras with triality

Malcev algebras

Groups with triality

Moufang loops

Doro, Grishkov and Zavarnistine

If G is a group with triality then

$$\mathcal{M}(G) = \{g^{-1}g^{\sigma} \mid g \in G\}$$

is a Moufang loop with respect to the product

$$m \cdot n = m^{-\rho} n m^{-\rho^2} = n^{-\rho^2} m n^{-\rho}$$

$\forall m, n \in \mathcal{M}(G)$.

Objects appearing

Lie algebras with triality

Malcev algebras

Groups with triality $\xrightarrow{G \mapsto \mathcal{M}(G)}$ Moufang loops

Autotopies of a Moufang loop

Definition

Given a Moufang loop Q, an **autotopy** of Q is a triple (A_1, A_2, A_3) with $A_i \in \mathrm{Bij}(Q)$ such that $(xy)A_1 = (xA_2)(yA_3)\ \forall x, y \in Q$.

The set $\mathrm{Atp}(Q)$ of all autotopies of Q is a group with triality with the componentwise composition and automorphisms

$$(A_1, A_2, A_3)^\rho = (JA_2J, A_3, JA_1J)$$

$$(A_1, A_2, A_3)^\sigma = (A_3, JA_2J, A_1)$$

where $J\colon x \mapsto x^{-1}$ for any $x \in Q$.

Grishkov and Zavarnistine groups revisited

For every Moufang loop Q

$$\mathcal{M}(\mathrm{Atp}(Q)) \cong Q$$

and $Z_S(\mathrm{Atp}(Q)) = 1$.

Universal property: If G is a group with triality with $\mathcal{M}(G) \cong Q$ and $Z_S(G) = 1$, then there exists a monomorphism of groups with triality

$$\tau : G \hookrightarrow \mathrm{Atp}(Q).$$

Objects appearing

Lie algebras with triality

Malcev algebras

Groups with triality

Moufang loops

$G \mapsto \mathcal{M}(G)$

$\mathrm{Atp}(Q), \mathcal{W}(Q), \mathcal{D}(Q) \leftarrow\!\shortmid Q$

From Malcev algebras to ...

Given a Malcev algebra $\mathfrak{m}$ over a field F $(\mathrm{char} \neq 2, 3)$ the Lie algebra $\mathcal{L}(\mathfrak{m})$ generated by $\{\lambda_a, \rho_a \mid a \in \mathfrak{m}\}$ with relations

$$\begin{aligned} \lambda_{\alpha a+\beta b} &= \alpha\lambda_a + \beta\lambda_b & \rho_{\alpha a+\beta b} &= \alpha\rho_a + \beta\rho_b \\ [\lambda_a, \lambda_b] &= \lambda_{[a,b]} - 2[\lambda_a, \rho_b] & [\rho_a, \rho_b] &= -\rho_{[a,b]} - 2[\lambda_a, \rho_b] \\ [\lambda_a, \rho_b] &= [\rho_a, \lambda_b] \end{aligned}$$

for any $\alpha, \beta \in F$ is a Lie algebra with triality relative to the automorphisms ζ, η determined by

$$\begin{aligned} \zeta(\lambda_a) &= \lambda_a + \rho_a & \eta(\lambda_a) &= -\lambda_a \\ \zeta(\rho_a) &= -\rho_a & \eta(\rho_a) &= \lambda_a + \rho_a \end{aligned}$$

Objects appearing

$\mathcal{L}(\mathfrak{m}) \leftarrow\!\shortmid \mathfrak{m}$

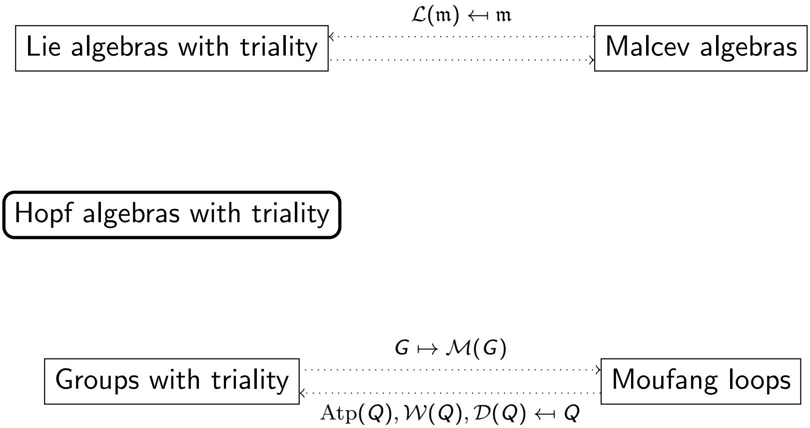
Lie algebras with triality ⇄ Malcev algebras

$G \mapsto \mathcal{M}(G)$

Groups with triality ⇄ Moufang loops

$\mathrm{Atp}(Q), \mathcal{W}(Q), \mathcal{D}(Q) \leftarrow\!\shortmid Q$

Goal I


$\mathcal{L}(\mathfrak{m}) \leftarrow\!\shortmid \mathfrak{m}$
Lie algebras with triality
Malcev algebras
Hopf algebras with triality
$G \mapsto \mathcal{M}(G)$
Groups with triality
Moufang loops
$\mathrm{Atp}(Q), \mathcal{W}(Q), \mathcal{D}(Q) \leftarrow\!\shortmid Q$

More definitions

Definition

If a Hopf algebra H admits two automorphisms ρ, σ such that $\sigma^2 = \mathrm{Id} = \rho^3$, $\sigma\rho = \rho^2\sigma$ and

$$\sum P(x_{(1)})\rho(P(x_{(2)}))\rho^2(P(x_{(3)})) = \epsilon(x)1,$$

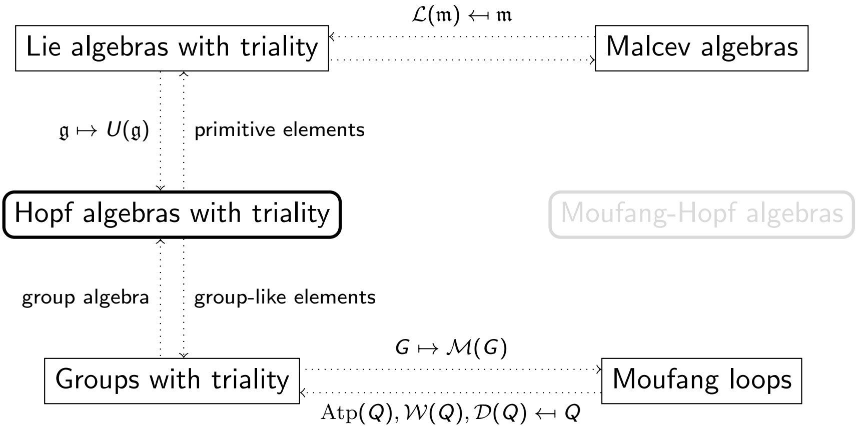
where $P(x) = \sum \sigma(x_{(1)})S(x_{(2)})$, then it is called **Hopf algebra with triality**.

An interesting result

Theorem (B,M and P-I)

Let $\mathfrak{g}$ be a Lie algebra with triality then $U(\mathfrak{g})$ is a Hopf algebra with triality.

Objects appearing



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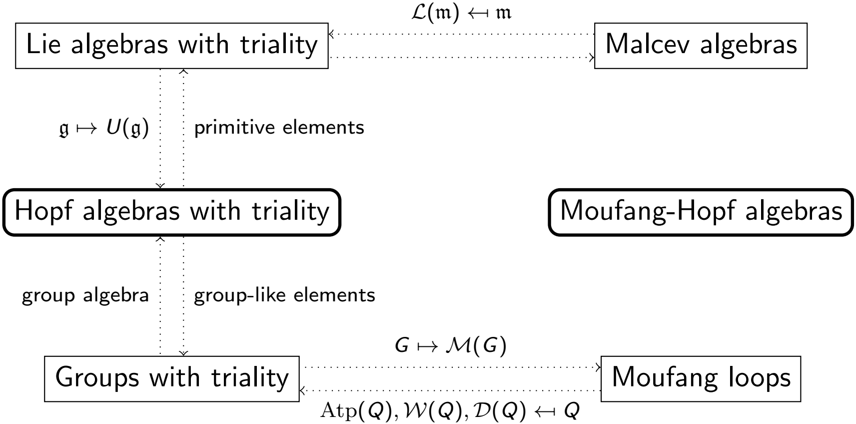


The last object

A **Moufang-Hopf algebra** is a (cocommutative) coassociative unital bialgebra $(U, \Delta, \epsilon, \cdot, 1)$ verifying the Moufang-Hopf identity

$$\sum u_{(1)}(v(u_{(2)}w)) = \sum((u_{(1)}v)u_{(2)})w$$

and such that there exists a map $S\colon U \to U$ (**antipode**) with

$$\sum S(u_{(1)})(u_{(2)}v) = \epsilon(u)v = \sum u_{(1)}(S(u_{(2)})v)$$

$$\sum(vu_{(1)})S(u_{(2)}) = \epsilon(u)v = \sum(vS(u_{(1)}))u_{(2)}$$

The universal enveloping algebra of a Malcev algebra

Given a nonassociative algebra A, the set

$$\mathrm{N}_{\mathrm{alt}}(A) = \{a \in A \mid (a, x, y) = -(x, a, y) = (x, y, a) \quad \forall x, y \in A\}$$

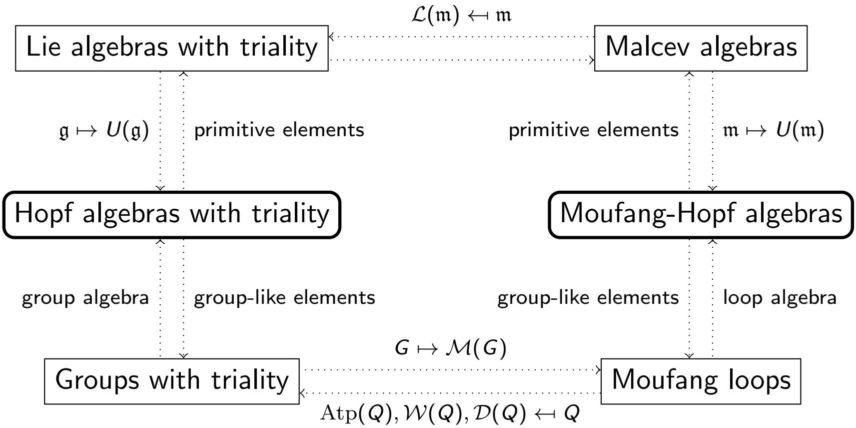
is a Malcev algebra with $[a, b] = ab - ba$.

Conversely, given a Malcev algebra $\mathfrak{m}$ over a field F $(\mathrm{char}\, F \neq 2, 3)$ it exists a Moufang-Hopf algebra $U(\mathfrak{m})$ and a monomorphism of Malcev algebras

$$\mathfrak{m} \hookrightarrow \mathrm{N}_{\mathrm{alt}}(U(\mathfrak{m}))$$

In case that $\mathfrak{m}$ is a Lie algebra then $U(\mathfrak{m})$ is the usual universal enveloping algebra of $\mathfrak{m}$ (Shestakov and Pérez-Izquierdo, 2004).

Objects appearing



Goal II

To relate Hopf algebras with triality with Moufang-Hopf algebras

This proves another way of constructing the universal enveloping algebras of Malcev algebras out of Hopf algebras with triality.

$\mathcal{MH}(H)$

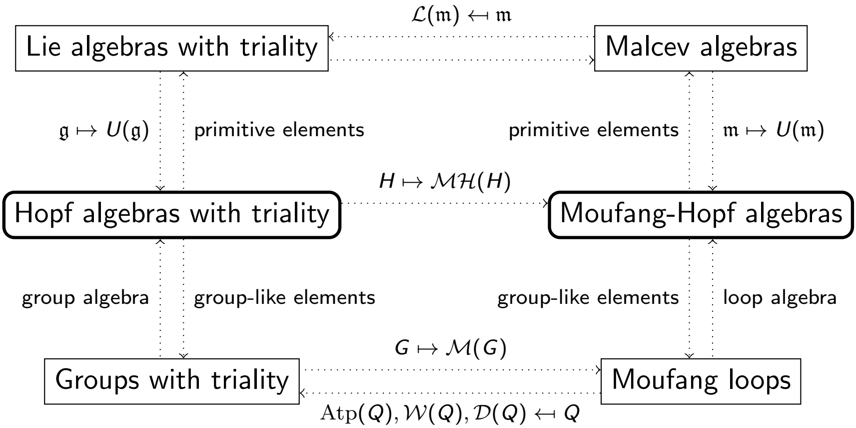
Given a Hopf algebra H with triality, define and

$$P(x) = \sum \sigma(x_{(1)})S(x_{(2)}) \quad \text{and} \quad \mathcal{MH}(H) = \{P(x) \mid x \in H\}\,.$$

Then $\mathcal{MH}(H)$ is a unital cocommutative Moufang-Hopf algebra with the coalgebra structure and antipode inherited from H, the same unit element and product defined by

$$u * v = \sum \rho^2(S(u_{(1)}))v\rho(S(u_{(2)})) = \sum \rho(S(v_{(1)}))u\rho^2(S(v_{(2)})).$$

Objects appearing



$\mathcal{D}(U)$

Definition

Given a cocommutative Moufang-Hopf algebra U, define $\mathcal{D}(U)$ as the unital associative algebra generated by $\{P_m, L_m, R_m \mid m \in U\}$ subject to the relations

$$P_1 = 1, \quad P_{\alpha m+\beta n} = \alpha P_m + \beta P_n, \quad \sum P_{m_{(1)}} P_n P_{m_{(2)}} = \sum P_{m_{(1)} n m_{(2)}}$$
$$\sum P_{m_{(1)}} L_{m_{(2)}} R_{m_{(3)}} = \epsilon(m)1, \quad \sum R_{m_{(1)}} P_n L_{m_{(2)}} = P_{S(m)n},$$
$$\sum L_{m_{(1)}} P_n R_{m_{(2)}} = P_{nS(m)},$$
$$\text{and cyclic permutations } P \to R \to L \to P \text{ of the previous}$$

for any $\alpha, \beta \in F$ and $m, n \in U$.

$\mathcal{D}(U)$

The maps

$$\begin{array}{lll} \Delta : P_m \mapsto \sum P_{m_{(1)}} \otimes P_{m_{(2)}} & \epsilon : P_m \mapsto \epsilon(m)1 & S : P_m \mapsto P_{S(m)} \\ L_m \mapsto \sum L_{m_{(1)}} \otimes L_{m_{(2)}} & L_m \mapsto \epsilon(m)1 & L_m \mapsto L_{S(m)} \\ R_m \mapsto \sum R_{m_{(1)}} \otimes R_{m_{(2)}} & R_m \mapsto \epsilon(m)1 & R_m \mapsto R_{S(m)} \end{array}$$

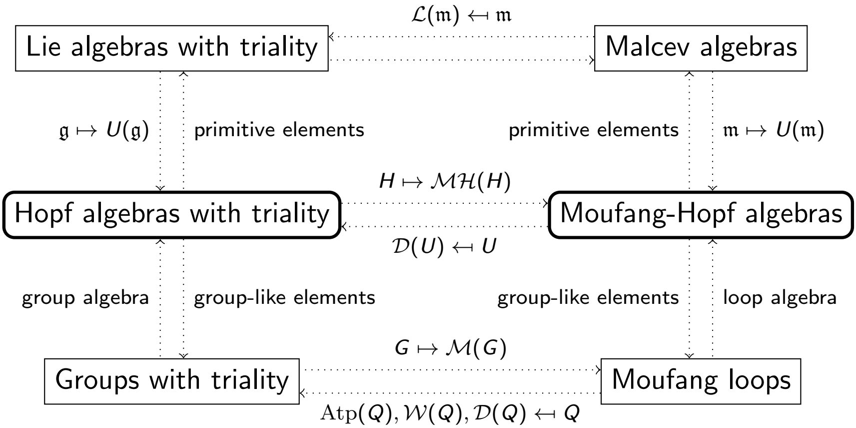
induce corresponding homomorphisms of algebras.

$\Delta\colon \mathcal{D}(U) \to \mathcal{D}(U) \otimes \mathcal{D}(U)$, $\epsilon\colon \mathcal{D}(U) \to F$ and $S\colon \mathcal{D}(U) \to \mathcal{D}(U)$ that make $\mathcal{D}(U)$ a cocommutative Hopf algebra. In fact, $\mathcal{D}(U)$ relative to the automorphisms induced by

$$\begin{array}{ccc} & \rho & \\ P_m & \mapsto & L_m \\ L_m & \mapsto & R_m \\ R_m & \mapsto & P_m \end{array} \qquad \begin{array}{ccc} & \sigma & \\ P_m & \mapsto & P_{S(m)} \\ L_m & \mapsto & R_{S(m)} \\ R_m & \mapsto & L_{S(m)} \end{array}$$

is a Hopf algebra with triality.

Objects appearing



$\mathcal{D}(U)$

Theorem

For any cocommutative Moufang-Hopf algebra U, the map

$$\begin{aligned} \iota\colon U &\to \mathcal{MH}(\mathcal{D}(U)) \\ m &\mapsto P_m \end{aligned}$$

is an isomorphism of Moufang-Hopf algebras. In particular,

$$U(\mathfrak{m}) \cong \mathcal{MH}(\mathcal{D}(\mathfrak{m})) \text{ for any Malcev algebra } \mathfrak{m}.$$

Universal property: given a Hopf algebra with triality H and a homomorphism $\varphi\colon U \to \mathcal{MH}(H)$ of Moufang-Hopf algebras, then φ extends to a homomorphism $\bar{\varphi}\colon \mathcal{D}(U) \to H$ of Hopf algebras with triality such that $\varphi = \bar{\varphi} \circ \iota$.

Nichols algebras

We consider the Nichols algebra

$$E(n) = F\langle g, x_1, \ldots, x_n \mid g^2 = 1, x_i^2 = 0, gx_i = x_i g, x_i x_j = -x_j x_i \rangle$$

with structural maps induced by

$$\Delta(g) = g \otimes g \qquad \epsilon(g) = 1 \qquad S(g) = g$$
$$\Delta(x_i) = 1 \otimes x_i + x_i \otimes g \qquad \epsilon(x_i) = 0 \qquad S(x_i) = -x_i g$$

We have that $\mathrm{Aut}(E(n)) \equiv GL_n(F)$: given $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \in GL_n(F)$, the map induced by

$$f_A(g) = g$$
$$f_A(x_i) = a_{i1}x_1 + \cdots + a_{in}x_n$$

defines an automorphism of $E(n)$.

Conversely, all automorphisms of $E(n)$ can be obtained in this way.

Nichols algebra automorphisms and representations of S_3

Consider the vector space $V = F\langle x_1, \cdots, x_n\rangle$; we have that $GL(V) \equiv Aut(E(n))$. If we

1. choose $\sigma, \rho \in GL(V)$ such that $\sigma^2 = Id = \rho^3$ and $\sigma\rho = \rho^2\sigma$ (a representation of the symmetric group S_3 in $GL(V)$ with $(12) \mapsto \sigma$ and $(123) \mapsto \rho$)
2. impose $\sigma(g) = g = \rho(g)$

then we obtain two automorphisms of the Nichols algebra $E(n)$ verifying $\sigma^2 = Id = \rho^3$ and $\sigma\rho = \rho^2\sigma$.

Representations of S_3

Representation theory says that, up to isomorphism, S_3 has three irreducible representations:

$$trivial : S_3 \to GL(\mathbb{C}), \qquad natural : S_3 \to GL(\mathbb{C}^2)$$

$$\tau \mapsto Id \qquad \sigma \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$signature : S_3 \to GL(\mathbb{C}) \qquad \rho \mapsto \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$$

$$\tau \mapsto sn(\tau)$$

where ω is a primitive cubic root of 1.

Changing basis

Any basis of V verifies the same properties as $\{x_1, \ldots, x_n\}$ in the definition. If $F = \mathbb{C}$ (algebraically closed field of zero characteristic), then there exists a basis of V such that

$$\rho = \begin{pmatrix} 1 & & & & & & & & & \\ & \ddots & & & & & & & & \\ & & 1 & & & & & & & \\ & & & 1 & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & 1 & & & & \\ & & & & & & \omega & & & \\ & & & & & & & \omega^2 & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & \omega \\ & & & & & & & & & & \omega^2 \end{pmatrix}, \sigma = \begin{pmatrix} 1 & & & & & & & & & \\ & \ddots & & & & & & & & \\ & & 1 & & & & & & & \\ & & & -1 & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & -1 & & & & \\ & & & & & & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & & \\ & & & & & & & \ddots & \\ & & & & & & & & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \end{pmatrix}$$

Triality condition

The triality condition can be written as

$$T(x) = T_{\sigma,\rho} = \sigma(x_{(1)})S(x_{(2)})\rho\sigma(x_{(3)})\rho(S(x_{(4)}))\rho^2\sigma(x_{(5)})\rho^2(S(x_{(6)})) = \epsilon(x)\cdot 1$$

Let's see whether it is satisfied by the basic elements:

$$\begin{aligned}
T(1) =& \sigma(1)S(1)\rho\sigma(1)\rho(1)\rho^2\sigma(1)\rho^2(1) = 1 \\
T(g) =& \sigma(g)S(g)\rho\sigma(g)\rho(g)\rho^2\sigma(g)\rho^2(g) = g^6 = 1 \\
T(x_i) =& \rho^2(S(x_i)) + \rho^2\sigma(x_i)\rho^2(S(g)) + \rho(S(x_i))\rho^2\sigma(g)\rho^2(S(g)) \\
&+ \rho\sigma(x_i)\rho(S(g))\rho^2\sigma(g)\rho^2(S(g)) + S(x_i)\rho\sigma(g)\rho(S(g))\rho^2\sigma(g)\rho^2(S(g)) \\
&+ \sigma(x_i)S(g)\rho\sigma(g)\rho(S(g))\rho^2\sigma(g)\rho^2(S(g)) \\
=& -\rho^2(x_i)g + \rho^2\sigma(x_i)g - \rho(x_i)g + \rho\sigma(x_i)g - x_i g + \sigma(x_i)g
\end{aligned}$$

Since $\epsilon(1) = 1 = \epsilon(g)$, 1 and g verify the triality condition. For x_i we need to do a case study.

x_i in a trivial module: $\sigma(x_i) = x_i = \rho(x_i)$: $T(x_i) = 0$

x_i in a signature module: $-\sigma(x_i) = x_i = \rho(x_i)$: $T(x_i) = -6x_i g$

x_i in a natural module: $\sigma(x_i) = x_j, \rho(x_i) = \xi x_i$: $T(x_i) = 0$

Since $\epsilon(x_i) = 0$ and we work in a field of zero caracteristic, basic elements x_i can't belong to a signature module. In an suitable basis, matrices of σ and ρ can be expressed as

$$\rho = \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & \omega & & & & \\ & & & & \omega^2 & & & \\ & & & & & \ddots & & \\ & & & & & & \omega & \\ & & & & & & & \omega^2 \end{pmatrix}, \sigma = \begin{pmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & 0 & 1 & & & \\ & & & 1 & 0 & & & \\ & & & & & \ddots & & \\ & & & & & & 0 & 1 \\ & & & & & & 1 & 0 \end{pmatrix}$$

Conclusion

By induction, we prove that if $x' = x_j x$ with x a product of different $x_i's$, then $T(x') = \epsilon(x') \cdot 1$ and $E(n)$ is a (non-cocommutative) Hopf algebra with triality. We compute $\mathcal{MH}(E(n)) = \{P(x) \mid x \in E(n)\}$:

$$P(g) = \sigma(g)S(g) = g^2 = 1$$
$$P(x_i) = (\sigma(x_i) - x_i)g$$
$$P(\prod_{i=1}^{k} x_i) = \prod_{i=1}^{k} P(x_i) = \prod_{i=1}^{k} \pm(\sigma(x_i) - x_i)g^k$$

Now we check whether the coalgebra structure of $E(n)$ is inheritated by $\mathcal{MH}(E(n))$:

$$\begin{aligned}\Delta(P(x_i)) &= \Delta((\sigma(x_i) - x_i)g) = \Delta(\sigma(x_i) - x_i)\Delta(g) \\ &= (1 \otimes (\sigma(x_i) - x_i) + (\sigma(x_i) - x_i) \otimes g)(g \otimes g) \\ &= g \otimes (\sigma(x_i) - x_i)g + (\sigma(x_i) - x_i)g \otimes 1\end{aligned}$$

Since $g \notin \mathcal{MH}(E(n))$, then $\Delta(P(x_i)) \notin \mathcal{MH}(E(n)) \otimes \mathcal{MH}(E(n))$, so $\mathcal{MH}(E(n))$ doesn't inherit the coalgebra structure of $E(n)$.

Taft algebras

We consider the Taft algebra

$$H = F\langle x_1, \dots, x_n, y \mid x_i^q = 1, x_i x_j = x_j x_i, y x_i = \omega x_i y, y^q = 0, \omega^q = 1 \text{ primitive root of } 1\rangle$$

with structural maps induced by

$$\begin{array}{lcr} \Delta(x_i) = x_i \otimes x_i & \epsilon(x_i) = 1 & S(x_i) = x_i^{q-1} \\ \Delta(y) = y \otimes x_1 + 1 \otimes y & \epsilon(y) = 0 & S(y) = -\omega^{-1} x_1^{q-1} y \end{array}$$

We know that

$$\begin{aligned} f \in Aut_{Hopf}(H) \Leftrightarrow & f|_C \in Aut_{Grupo}(C), \text{ where } C = \langle x_1, \dots, x_n\rangle \\ & f(x_1) = x_1 \\ & c_1^* = c_1^* \circ f, \text{ where } c_1^* : C \to F \\ & \qquad\qquad\qquad\qquad x_i \mapsto \omega \end{aligned}$$

Triality condition

The triality condition is trivially satisfied by y and x_1: for $f, g \in Aut_{Hopf}(H)$ they verify

$$\sum P(x_{(1)})g(P(x_{(2)}))g^2(P(x_{(3)})) = \epsilon(x)1, \text{ with } P(x) = \sum f(x_{(1)})S(x_{(2)})$$

We have that $P(y) = \sigma(y)S(x_1) + \sigma(1)S(y) = 0$, so for each element $x \in H$,

$$P(xy) = \sigma(x_{(1)})\sigma(y_{(1)})S(y_{(2)})S(x_{(2)}) = \sigma(x_{(1)})P(y)S(x_{(2)}) = 0 = \epsilon(x)\epsilon(y) \cdot 1$$

So H will have triality if $\langle x_2, \ldots, x_n \rangle \equiv \langle x_2 \rangle \times \langle x_n \rangle \equiv C_q \times \cdots \times C_q$ is a group with triality σ, ρ and $c_1^* = c_1^* \circ \sigma$, $c_1^* = c_1^* \circ \rho$.

Triality condition

If we consider C_q as an additive group, we can associate to each automorphism $f \in Aut_{Group}(C_q \times \cdots \times C_q)$ with $f : x_i \mapsto x_2^{e_{2i}} \cdot \cdots \cdot x_n^{e_{ni}}$, an additive invertible map $f' \in Aut_{Group}(\mathbb{Z}_q \times \cdots \times \mathbb{Z}_q)$, ie, a matrix

$$\begin{pmatrix} e_{22} & \cdots & e_{2n} \\ \vdots & \ddots & \vdots \\ e_{n2} & \cdots & e_{nn} \end{pmatrix}$$

with $e_{ij} \in \mathbb{Z}_q$ and invertible determinant in $\mathbb{Z}_q$.

The condition $c_1^* = c_1^* \circ f$ reads as $\sum_i e_{ij} \cong 1(mod\ q)\ \forall j = 2, \ldots, n$.

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